DEPENDENCY AND ACCURACY MEASURES FOR DIRECTED GRAPHS

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Abstract. In this paper we use finite directed graphs (digraphs) as mathematical models to study two basic notions widely analyzed in granular computing: the attribute dependency and the approximation accuracy. To be more specific, at first we interpret any digraph as a Boolean information table, next we study the approximation accuracy for three fundamentals digraph families: the directed path, the directed cycle and the transitive tournament. We also introduce a new global average for the attribute dependency in any information table and we determine such number for any directed path. For the transitive tournament we provide a lower bound.

1. Introduction

In database theory it is very frequent the need to study finite tables having a very large quantity of data, therefore many researches have been directed towards the purpose of reducing and simplifying the interpretation of these data. With such an aim, Pawlak [45] developed the so called *rough set theory* (abbreviated RST). RST is an elegant and powerful methodology in extracting and minimizing rules from data tables (*information systems* or, also, *information tables* in Pawlak terminology) which has been further developed and generalized in several recent works [35, 38, 43, 44, 54, 63, 64, 79].
Next, RST has been considered as a part of the more general emerging methodological paradigm named *granular computing* (abbreviated GrC) [49, 50, 52, 62].
GrC deals with representing and processing information in the form of some type of aggregates. These aggregates are generally called *information granules* or simply granules and they arise in the process of data abstraction and knowledge derivation from data. The scope of GrC covers various fields of study related to knowledge representation and data extraction.
In 1979 the concept of *information granularity* was introduced by Zadeh [77] and it was related to the research on fuzzy sets. Next, the term *granular computing* was introduced again by Zadeh in 1997 (see [78]). Roughly speaking, information granules are collections of entities arranged together due to their similarity, functional or physical adjacency, indistinguishability, and so on.
Since 1979, granular computing has become a very developed area of research in the scope of both applied and theoretical information science [50, 62]. From a methodological perspective, GrC can be considered as an important attempt to investigate several research fields by means of the unifying granularity paradigm: rough set theory [16, 46, 47, 48, 65, 75] and its generalizations [14, 74, 76], information tables [13, 30, 28, 72], operative research [35], machine learning [73], interval analysis [40], formal concept analysis [39, 67], database theory [36, 53], data mining [37, 41, 42, 69], fuzzy set theory [51, 78], fuzzy-rough set theory [38], interactive computing [55, 56].
The unifying perspective of GrC provides the useful interpretation tool which permit us to assign a same name for several notions used in different research fields. This is, for example, the case of two very important notions investigated (often with slightly different names) in data mining, database theory, rough set theory and fuzzy set theory: the *attribute approximation accuracy* [46, 47, 48, 63, 70, 79, 80] and the *attribute dependency* [68] for data tables. The utility of the above notions is so important in analyzing the data from a bi-dimensional table, that it goes beyond the scope of the specific disciplines that study them. Therefore, more generally, we prefer to speak of attribute approximation accuracy and attribute dependency in GrC by means of the unifying terminology derived from GrC itself.

In this paper, we study the two above notions in the digraph context. To be more specific, we interpret any finite digraph D (without loops and parallel arcs) as a particular type of Boolean information system. We use the more natural way to interpret a digraph D in terms of information system: its adjacency matrix $Adj(D)$. We apply the classical notion of *indiscernibility relation* [45] for any information system,

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in order to compute the basic rough approximation functions and the positive regions for three basic digraph families: the n-directed cycle $\vec{C}_n$, the n-directed path $\vec{P}_n$ and the n-transitive tournament $\vec{T}_n$. We associate to any information system two new data tables: the *approximation accuracy table* and the *attribute dependency table*, which contain, respectively, the approximation accuracies and the attribute dependencies with respect to any possible pair of attribute subset of the information system. We can note that if one user needs to know *all* the approximation accuracies or *all* the attribute dependencies of one only specific information system, he must build the corresponding approximation accuracy table or the corresponding attribute dependency table of its information system. Nevertheless, if the examined information system has n attributes, the dimension of the above tables is $2^n \times 2^n$, therefore in practice it is very difficult to build information system families with increasing attribute number in order to evaluate how these tables change when n becomes very large. From this perspective, some basic types of digraphs are excellent mathematical models to study the variability of the attribute dependency and the approximation accuracy when the number of attributes grows indefinitely. In the case of the attribute dependency, for any information system we also introduce an average which globally measures a type of dependency degree associated to the examined information system. We completely determine this new numerical parameter for any finite directed path: in this case we provide both a closed formula and an exact asymptotic estimate. For the transitive tournament we also provide a lower estimate for the aforementioned dependency average.

This work can be considered a continuation of a research project started in [18, 22, 27, 29, 31] and further developed in [23, 24, 25, 26], where the incidence matrix of a hypergraph and the adjacency matrix of a simple undirected graph have been studied as particular types of information tables (for investigations concerning graphs and hypergraphs with RST and related methods in the scope of the mathematical morphology and spatial information systems see [57, 58, 59, 60, 61]). As a matter of fact, in [22, 27, 31] has been showed how a detailed study of finite undirected simple graphs by means of the investigation tools derived from GrC, leads to discover new and unexplored mathematical properties of these discrete structures.
Let us note that in any finite discrete structure which is formalizable by means of the notion of information system, we obtain two basic advantages in studying such a structure from the standpoint of the information systems: (C1) to isolate *only* the specific information of our interest, neglecting all not relevant notions; (C2) to provide a temporal dynamic interpretation to an apparently static situation.
In reference to (C1), to many discrete structure (graphs, digraphs, groups, posets) we can associate several types of information systems, depending on our study necessity. When we choose a specific type of information system for our structure, this means that we are selecting a way to reduce the structure study complexity towards a more particular point of view. Then, after making this choice, we can use all RST investigation tools to study our structure from the simplified observatory induced from our decision.
In reference to (C2), the papers [8, 9, 15, 33] can be methodologically considered some of the more recents and interesting tempts of studying dynamically an information table. In this perspective, the main advantage of interpretying a discrete structure as an information system is that to provide a global and hierarchical classification for the knowledge obtained from *all* possible choices of attribute subsets. The key tool to visualize dynamically the study of the indiscernibility relations in an information system is the *granular partition lattice* (see [71]). This order structure of all the indiscernibility partitions is the formal tool allowing us to visualize hierarchically and globally the knowledge evolution in any information system. We can consider the study of the granular partition lattice of any discrete structure having a corresponding associated information system as a natural way to provide a dynamic interpretation to an apparently static situation. In fact, we can imagine an initial indistinct knowledge, obtained by choosing an attribute empty subset, which next evolves through various temporal ramifications whenever one adjoins further attributes to its available knowledge. In this temporal process, the complete knowledge is reached when one can obtain the availability of *all* attributes in its information system. For example, the graphs can be studied by means of sequential or parallel dynamics (see [1, 2, 4, 5, 6, 7]), or also for their analogies with both sequential and parallel dynamics on order structures (see [10, 11, 12, 17, 19, 20, 21]).

We conclude this introductory section with a brief description of the content of each section of this paper.
In Section 2 we give the necessary definitions and introduces the notations which we will use in the sequel.
In Section 3 we introduce the basic granular computation on digraphs. In particular, we provide a geometrical characterization of the indiscernibility relation in digraph context.
Section 4 is devoted to the study of the Pawlak exactness in our context. To illustrate our analysis we use three simple families of digraphs: the n-path $\vec{P}_n$, the n-cycle $\vec{C}_n$ and the n-transitive tournament $\vec{T}_n$.

In Section 5 we introduce the concept of attribute dependency average for an information system $\mathcal{I}$. This statistical concept is introduced in order to measure the average capacity to transmit dependency of a generic attribute subset of $\mathcal{I}$.
In Section 6 we determine the attribute dependency average for the information system associates to the digraph $\vec{P}_n$ and we provide a lower bound for the digraph $\vec{T}_n$.

2. Recalls and Notations.

If X is a set (in this paper we consider only finite sets), we denote by $\mathcal{P}(X)$ the power set of X. If $A \in \mathcal{P}(X)$ and X is clear from the context, we write A^c instead of $X \setminus A$. We denote by $|X|$ the number of elements in the set X. We usually denote by $\pi = \{B_i : i \in I\}$ a set partition of X whose blocks are the subsets B_i.
If $x \in X$, we denote by $\pi(x)$ the (unique) block of π which contains the element x. In particular, when X is a finite set, a set partition π of X has only a finite number of blocks $B_1, \ldots, B_M$, in this case we use the standard notations $\pi := B_1|\ldots|B_M$ to denote that π is a set partition having blocks $B_1, \ldots, B_M$. If π is a set partition having M distinct blocks, we set $|\pi| := M$.
We denote by $\Pi(X)$ the set of all set-partitions of X. It is well known that on the set $\Pi(X)$ we can consider a partial order $\preceq$ defined as follows: if $\pi, \pi' \in \Pi(X)$, then

$$\pi \preceq \pi' :\Longleftrightarrow (\forall B \in \pi)\,(\exists B' \in \pi')\ :\ B \subseteq B' \tag{1}$$

A useful and immediate characterization of the partial order in (1) is the following:

$$\pi \preceq \pi' \Longleftrightarrow (\forall x \in X)\,(\pi(x) \subseteq \pi'(x)) \tag{2}$$

The pair $\mathbb{P}(X) = (\Pi(X), \preceq)$ is a complete lattice which is called *partition lattice* of the set X.

In GrC a data table is interpreted as an information system in the following way: an *information system* [45] is a structure $\mathcal{I} = \langle U, Att, Val, F\rangle$, where $U = \{u_1, u_2, \ldots, u_m\}$ is a non-empty finite set called *universe set*, $Att = \{a_1, a_2, \ldots, a_n\}$ is a non empty finite set called *attribute set*, Val is a non-empty finite set whose elements are called *values* and $F : U \times Att \to Val$ is an application called *information map*. The elements of U are called *objects* and the elements of Att are called *attributes*. In particular, if $Val = \{0, 1\}$ then $\mathcal{I}$ is called a *Boolean information system*.
The basic mathematical notion that one usually uses to reduce the redundant information when an attribute subset A has been fixed is the so-called *indiscernibility relation* [45]. To be more specific, If $\mathcal{I}$ is an information system and $A \subseteq Att$, it is usual to consider the binary relation $\equiv_A$ on the universe set U defined as follows: if $u, u' \in U$ then

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a), \forall a \in A. \tag{3}$$

The binary relation $\equiv_A$ is an equivalence relation on U which is called *A-indiscernibility relation* of $\mathcal{I}$. If $u \in U$, the equivalence class $[u]_A$ of u with respect to $\equiv_A$ is called the A-*indiscernibility class* of u, moreover a subset $C \subseteq U$ is called *A-granule* if $C = [u]_A$ for some $u \in U$. If $C_1, \ldots, C_n$ are all the distinct A-granules in U, the set partition of U

$$\pi_{\mathcal{I}}(A) := \{[u]_A : u \in U\} = C_1|\ldots|C_n \tag{4}$$

is called the *A-indiscernibility partition* of $\mathcal{I}$. By means of the previous indiscernibility notion, sophisticated techniques have been developed for the characterization of object sets in terms of values assigned from the attributes, the determination of (total or partial) dependencies between attribute subsets, the reduction of superfluous data (attributes), the determination of most characteristic and significant attribute subsets for the tabular data interpretation, the generation of appropriate operational rules of decision.

For all general notions concerning digraphs we refer the reader to [3]. Here we recall only some basic notions which we will use in the sequel. Let $D = (V(D), Arc(D))$ be a digraph (in what follows the letter D will be used to denote always a digraph without loops and parallel arcs), where $V(D) = \{v_1, \ldots, v_n\}$ and $Arc(D) = \{e_1, \ldots, e_m\}$ are respectively the vertex set and the arc set of D. For each arc $e \in Arc(D)$ there are exactly two distinct vertices $v_i := t(e)$ (called *tail* of e) and $v_j := h(e)$ (called *head* of e) in $V(D)$ such that $e = (v_i, v_j)$. If (v, w) is an arc of D, we also write $v \to w$. As it is well known, a digraph D is uniquely determined by its *adjacency matrix* $Adj(D)$, that is the $n \times n$ matrix (a_{ij}) defined by $a_{ij} := 1$ if $(v_i, v_j) \in Arc(D)$ and $a_{ij} := 0$ otherwise. The *converse digraph* of a digraph D, denoted by D^*, is defined by $V(D^*) = V(D)$ and $(v_i, v_j) \in Arc(D^*)$ iff $(v_j, v_i) \in Arc(D)$.

If n is a positive integer, we denote by $\vec{P_n}$ the direct path on n vertices , i.e. the digraph having vertex set $V(\vec{P_n}) = \{v_1, \ldots, v_n\}$ and arc set $Arc(\vec{P_n}) = \{(v_i, v_{i+1}) : i = 1, \ldots, n-1\}$. Let $n \geq 2$ we denote by $\vec{C_n}$ the direct cycle on n vertices , i.e. the digraph having vertex set $V(\vec{C_n}) = \{v_1, \ldots, v_n\}$ and arc set $Arc(\vec{C_n}) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. A *n-transitive tournament* is the digraph $\vec{T_n}$ such that

$$V(\vec{T_n}) = \{v_1, \ldots, v_n\}, \qquad Arc(\vec{T_n}) = \{(v_i, v_j) : 1 \leq i \leq n-1,\, j = i+1, \ldots, n\}.$$

If $v \in V(D)$, we set

$$N_D^+(v) := \{z \in V(D) : (v, z) \in Arc(D)\},\ N_D^-(v) := \{u \in V(D) : (u, v) \in Arc(D)\}. \tag{5}$$

Usually $N_D^+(v)$ is called the *out-neighborhood* of v in D and $N_D^-(v)$ is called the *in-neighborhood* of v in D.

There are two immediate and natural ways to interpret a digraph D as a Boolean information system. We call *out-information system* of the digraph D the Boolean information system denoted by $[D]^+$ and defined as follows. Both the universe set and the attribute set of $[D]^+$ are $V(D) = \{v_1, \ldots, v_n\}$. The information map of $[D]^+$ is $F_D^+(v_i, v_j) := 1$ if $v_i \to v_j$ and $F_D^+(v_i, v_j) := 0$ otherwise. Analogously, we call *in-information system* of D the Boolean information system $[D]^-$ having both objects and attributes identical to the previous case, but information map $F_D^-(v_i, v_j) := 1$ if $v_j \to v_i$ and $F_D^-(v_i, v_j) := 0$ otherwise. Then, if we decide to study D by means of the information system $[D]^+$, this choice is equivalent to say that we want investigate the digraph D only from the arrow outgoing point of view. There are obviously many others information systems that we can associate to the digraph D. For example, we can define the Boolean information system $[D]_2^+ := \langle V(D), V(D) \times V(D), F, \{0, 1\}\rangle$ such that $F(v_i, (v_j, v_k)) := 1$ iff $v_i \to v_j$ and $v_i \to v_k$. In any case, the choice of an information system for D corresponds to a specific point of view on D. We can immediately see what is the geometrical meaning of the indiscernibility relation in the information system $[D]^+$.
In $[D]^+$ an attribute subset is a vertex subset $A \subseteq V(D)$. If $v \in V(D)$, we denote respectively by $\equiv_A^+$, $[v]_A^+$ and $\pi_D^+(A)$ the A-indiscernibility relation, the A-indiscernibility class of v and the A-indiscernibility partition in the information systems $[D]^+$. Then, if $v \in V(D)$, it results that $v' \in [v]_A^+$ iff v and v' *"out-see"* all vertices $z \in A$ in the same way, that is, $v \to z$ iff $v' \to z$ for all $z \in A$. So that an A-granule for the information system $[D]^+$, abbreviated an A^+*-granule* of D, is exactly a vertex subset of D whose vertices have all the same out-adjacency relation with respect all vertices in A. Therefore, if we fix a vertex subset A of D, we can interpret A as if it were a type of symmetry block against which to examine the out behavior of all vertices in D. Hence, from a geometric point of view, the choice of a fixed vertex subset A can be thought of as the choice of an "out symmetry axis" in the digraph D and the blocks of $\pi_D^+(A)$ as the "out symmetry point subsets" with respect to A. After the above discussion, it is also natural to call A^+*-symmetry block* of D any A^+-granule $C \in \pi_D^+(A)$.

Remark 2.1. *(i) In RST terminology, the* information table $T(\mathcal{I})$ *of an information system* $\mathcal{I}$ *is the* $m \times n$ *matrix whose rows are indexed with all objects in* U*, whose columns are indexed with all attributes in Att and which contains the value* $F(u_i, a_j)$ *in the place* (i, j)*. Therefore the information table of* $[D]^+$ *is exactly* $Adj(D)$*.*
(ii) It is immediate to observe that for any digraph D *we have*

$$[D]^- = [D^*]^+ \tag{6}$$

and

$$T([D]^-) = T([D]^+)^t = Adj(D)^t \tag{7}$$

By virtue of the identities (6) *and* (7)*, in this paper we study only the out information system* $[D]^+$*. Generally, when we introduce some new concept relatively to* $[D]^+$ *we use the preposition* out*, and we will leave it for granted (without defining it explicitly) that an analogue concept is defined also for the information system* $[D]^-$*, in association with the preposition* in*.*

3. Indiscernibility in Digraphs

In this section, we describe the first elementary characterization for the granular geometry on a generic digraph D. The next result allows us to establish a basic link between the indiscernibility relation and its equivalent geometric characterization.

Proposition 3.1. *Let $v, v' \in V(D)$ and $A \subseteq V(D)$. Then the following conditions are equivalent:*
(i) $v \equiv_A^+ v'$;
(ii) for all $z \in A$ it results that $v \to z$ if and only if $v' \to z$;
(iii) $N_D^+(v) \cap A = N_D^+(v') \cap A$.

Proof. It is a simple verify that follows directly by (3). □

Remark 3.2. *(i): As a direct consequence of previous theorem, it can be shown that if $v \in V(D)$ and $A \subseteq V(D)$, then $[v]_A^+ = \{v' : N_D^+(v) \cap A = N_D^+(v') \cap A\}$.*
(ii): Proposition 3.1 also provides a sufficient condition for two vertices of the digraph have no common arcs. To be more detailed, if $v \equiv_A^+ v'$ and $v' \in A$, then $v \not\to v'$, since there are no loops into D.

In next results we determine the form of any A^+-indiscernibility partition of $\vec{P}_n$, $\vec{C}_n$ and $\vec{T}_n$. In order to simplify the exposure form of all statements of this sub-section we will use the following convention.

Remark 3.3. *Let D be one of the three digraphs $\vec{P}_n$, $\vec{C}_n$ or $\vec{T}_n$, so that $V(D) = \{v_1, \ldots, v_n\}$. If $A \subseteq V(D)$ is such that $|A| = k$, we implicitly assume that $1 \leq k < n$. Therefore, if $|A| = k$, we always have exactly one of the following two cases:*
(i) $A = \{v_{i_1}, \ldots, v_{i_k}\}$,
(ii) $A = \{v_1, v_{i_2}, \ldots, v_{i_k}\}$,
where $1 < i_1 < i_2 < \cdots < i_k \leq n$.
Then in the case (i) we will set $A_{-1} := \{v_{i_1-1}, \ldots, v_{i_k-1}\}$. In the case (ii) we will set $A_{-1} := \{v_{i_2-1}, \ldots, v_{i_k-1}\}$ if $D = \vec{P}_n$, whereas $A_{-1} := \{v_n, v_{i_2-1}, \ldots, v_{i_k-1}\}$ if $D = \vec{C}_n$.
Finally, only when $D = \vec{P}_n$ or $D = \vec{T}_n$, if we take a subset $A \subseteq V(D)$ such that $|A| = 1$ we implicitly assume that $A \neq \{v_1\}$.

We begin to describe the A^+-indiscernibility partitions of $\vec{P}_n$.

Proposition 3.4. *Let $n \geq 2$, $V = V(\vec{P}_n) = \{v_1, \ldots, v_n\}$. Then*

$$\pi^+_{\vec{P}_n}(A) = \begin{cases} V & \text{if } A = \emptyset \vee A = \{v_1\} \\ v_{i_1-1}|\ldots|v_{i_k-1}|(A_{-1})^c & \text{if } A = \{v_{i_1}, \ldots, v_{i_k}\} \wedge 1 \leq |A| < n-1 \\ v_{i_2-1}|\ldots|v_{i_k-1}|(A_{-1})^c & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \wedge 1 < |A| < n \\ v_1|\ldots|v_n & \text{if } A = V \vee A = \{v_2, \ldots, v_n\} \end{cases}$$

Proof. We must compute $N^+_{\vec{P}_n}(v) \cap A$ for every vertex $v \in V$ and for any vertex subset $A \subseteq V$. Let $A = \{v_1\}$, no neighbourhood $N^+_{\vec{P}_n}(v)$ contains the element v_1, so every intersection $N^+_{\vec{P}_n}(v) \cap A$ is empty and $v_1, \ldots, v_n$ form a single block. Moreover, if $A = \emptyset$ or $A = V(\vec{P}_n)$, the result is trivial. Now, let $|A| = k$ of the form $A = \{v_{i_1}, \ldots, v_{i_k}\}$ where $1 < i_1 < \cdots < i_k \leq n$. It's clear that the neighbourhoods intersecting A are of the form $N^+_{\vec{P}_n}(v_{i_j-1})$, for $j = 1, \ldots, k$. Since $N^+_{\vec{P}_n}(v_{i_j-1}) \neq N^+_{\vec{P}_n}(v_{i_l-1})$ if $j \neq l$, every vertex $v_{i_1-1}, \ldots, v_{i_k-1}$ form a single block and the remaining vertices form the last block, hence $\pi^+_{\vec{P}_n}(A) = v_{i_1-1}|\ldots|v_{i_k-1}|(A_{-1})^c$. In particular, if $A = \{v_2, \ldots, v_n\}$, the set $(A_{-1})^c$ consists of a single element, namely v_n, thus $\pi^+_{\vec{P}_n}(A) = v_1|\ldots|v_n$. Finally, suppose $A = \{v_1, v_{i_2}, \ldots, v_{i_k}\}$, where $1 < i_2 < i_k \leq n$. In this case, the neighbourhoods intersecting A are $N^+_{\vec{P}_n}(v_{i_j-1})$, for $j = 2, \ldots, k$, because v_1 does not belong to any neighbourhood. So $\pi^+_{\vec{P}_n}(A) = v_{i_2-1}|\ldots|v_{i_k-1}|(A_{-1})^c$. □

We examine now the case of $\vec{C}_n$.

Proposition 3.5. *Let $n \geq 2$ and let $A = \{v_{i_1}, \ldots, v_{i_k}\}$ be a generic subset of $V = V(\vec{C}_n) = \{v_1, \ldots, v_n\}$. Then*

$$\pi^+_{\vec{C}_n}(A) = \begin{cases} V & \text{if } A = \emptyset \\ v_{i_1-1}|\ldots|v_{i_k-1}|(A_{-1})^c & \text{if } A = \{v_{i_1}, \ldots, v_{i_k}\} \wedge 1 \leq |A| < n-1 \\ v_n|v_{i_2-1}|\ldots|v_{i_k-1}|(A_{-1})^c & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \wedge 1 \leq |A| < n-1 \\ v_1|\ldots|v_n & \text{if } A = V \vee |A| = n-1 \end{cases}$$

Proof. If $A = \emptyset$ or if $A = V(\vec{C}_n)$ computations are obvious. Now, let $A = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V$ not containing v_1. Then the neighbourhoods intersecting A are $N^+_{\vec{C}_n}(v_{i_j-1})$, with $j = 1, \ldots, k$; furthermore, $N^+_{\vec{C}_n}(v_{i_j-1}) \neq N^+_{\vec{C}_n}(v_{i_l-1})$ if $j \neq l$, so every vertex $v_{i_1-1}, \ldots, v_{i_k-1}$ form a single block and the remaining vertices form the last block, since the intersections of their neighbourhoods with A are empty. Let $A = \{v_1, v_{i_2} \ldots, v_{i_k}\} \subseteq V$, where $1 < i_2 < \cdots < i_k \leq n$. As before, $N^+_{\vec{C}_n}(v_{i_j-1})$, with $j = 2, \ldots, k$ and $N^+_{\vec{C}_n}(v_n)$

are the neighbourhoods intersecting A. Since $N^+_{\vec{C}_n}(v_j) \neq N^+_{\vec{C}_n}(v_l)$ if $j \neq l$ and $j, l \in \{i_2-1, \dots, i_k-1, n\}$, we deduce that the elements $v_{i_2-1}, \dots, v_{i_k-1}, v_n$ form single blocks, while the other vertices stay in another unique block. Finally, by the previous argument, if $|A| = n-1$, we observe that $(A_{-1})^c$ consists of a single element, therefore $\pi^+_{\vec{C}_n}(A) = v_1|\dots|v_n$. □

Finally, also for the n-transitive tournament $\vec{T}_n$ we can determine the general form of any partition $\pi^+_{\vec{T}_n}(A)$.

Proposition 3.6. *Let $\vec{T}_n$ be the transitive $n-$tournament, where $n \geq 3$ and $V = V(\vec{T}_n)$. Then*

$$\pi^+_{\vec{T}_n}(A) = \begin{cases} V & \text{if } A = \emptyset \vee A = \{v_1\} \\ v_1 \dots v_{i_1-1}|v_{i_1} \dots v_{i_2-1}| \dots |v_{i_k} \dots v_n & \text{if } A = \{v_{i_1}, \dots, v_{i_k}\} \wedge 1 \leq |A| < n-1 \\ v_1 \dots v_{i_2-1}| \dots |v_{i_k} \dots v_n & \text{if } A = \{v_1, v_{i_2}, \dots, v_{i_k}\} \wedge 2 \leq |A| < n \\ v_1| \dots |v_n & \text{if } A = V \vee A = \{v_2, \dots, v_n\} \end{cases}$$

Proof. Computations for $A = \emptyset$ and for $A = V$ are obvious. Moreover, if $A = \{v_1\}$, we have that $N^+_{\vec{T}_n}(v_i)$ doesn't contain v_1 for all indices i, so $A \cap N^+_{\vec{T}_n}(v_i) = \emptyset$. Let $A = \{v_i\}$, where $i \neq 1$. Since $N^+_{\vec{T}_n}(v_j) = \{v_{j+1}, \dots, v_n\}$, clearly $N^+_{\vec{T}_n}(v_j) \cap A = \{v_i\}$ if and only if $A \subseteq N^+_{\vec{T}_n}(v_j)$. This happens if and only if $j = 1, \dots, i-1$, otherwise this intersection is empty. Now, we can provide the proof for the general case. Let $A = \{v_{i_1}, \dots, v_{i_k}\}$ where $1 < i_1 < \dots < i_k \leq n$ and $2 \leq |A| = k < n-1$. Then the first block consists of $v_1, \dots, v_{i_1-1}$; the second block contains all vertices v_j such that $N^+_{\vec{T}_n}(v_j) \cap (A \setminus \{v_{i_1}\}) \neq \emptyset$, i.e. it contains exactly the $i_2 - i_1$ elements $v_{i_1}, \dots, v_{i_2-1}$ and so on. In particular, if $A = \{v_2, \dots, v_n\}$, $(A_{-1})^c$ consists of a single element, therefore $\pi^+_{\vec{T}_n}(A) = v_1|\dots|v_n$. The same argument, however starting from the index i_2, works if $A = \{v_1, v_{i_2}, \dots, v_{i_k}\}$ where $1 < i_2 < \dots < i_k \leq n$ and $2 \leq |A| < n$. Thus, in this case, $\pi^+_{\vec{T}_n}(A) = v_1 \dots v_{i_2-1}| \dots |v_{i_k} \dots v_n$ and the thesis follows. □

Remark 3.7. *Let $D = \vec{P}_n$ or $D = \vec{T}_n$ and $A \subseteq V(D)$ of the form $A = \{v_{i_1}, \dots, v_{i_k}\}$, where $1 < i_1 < \dots < i_k \leq n$ and $A' = A \cup \{v_1\}$. Then $\pi^+_D(A) = \pi^+_D(A')$.*

4. Approximation Accuracy in Some Basic Digraphs

In this section we recall some basic notion on the Pawlak exactness and then we adapt it to our digraph model. Furthermore, we will compute the A^+-lower and the A^+-upper approximations for the basic digraph families $\vec{P}_n$, $\vec{C}_n$ and $\vec{T}_n$.

Definition 4.1. *Let $\mathcal{I} = \langle U, Att, Val, F\rangle$ be an information system, $A \subseteq Att$ and $Y \subseteq U$. The* A-lower approximation *of Y is the following subset of U:*

$$\mathbf{l}_A(Y) := \{x \in U : [x]_A \subseteq Y\} = \bigcup\{C \in \pi_A(\mathcal{I}) : C \subseteq Y\}.$$

The A-upper approximation of Y is defined as:

$$\mathbf{u}_A(Y) := \{x \in U : [x]_A \cap Y \neq \emptyset\} = \bigcup\{C \in \pi_A(\mathcal{I}) : C \cap Y \neq \emptyset\}.$$

The subset Y is called A-exact *if and only if $\mathbf{l}_A(Y) = \mathbf{u}_A(Y)$ and* A-rough *otherwise.*

Roughly speaking, the lower approximation represents the set of elements *certainly* belonging to Y, while the upper approximation is the set of objects that *surely or possibly* belong to Y (in both cases w.r.t. our knowledge expressed by A).

A classical concept in RST [46, 47, 48], recently investigated in [63, 70, 79], is the *approximation accuracy* of an attribute subset with respect to another attribute subset, which now we recall.

Definition 4.2. *Let $\mathcal{I}$ be an information system and let A and Y be two attribute subsets. We call* A-approximation accuracy *of Y the quantity $\alpha_A(Y) := \frac{|\mathbf{l}_A(Y)|}{|\mathbf{u}_A(Y)|}$.*

Exactness with respect to an attribute subset can be characterized by means of the approximation accuracy, in fact an object subset Y is A-exact if and only if $\alpha_A(Y) = 1$. Moreover when $A = \emptyset$, we can not define the approximation accuracy.
We can put the values $\alpha_A(Y)$ in a $2^n - 1 \times 2^m - 1$ table in the following way:

Definition 4.3. *Let $\mathcal{I}$ be an information system such that $|U| = m$ and $|Att| = n$. We call* approximation accuracy table *of $\mathcal{I}$, denoted by $T_{ac}(\mathcal{I})$, the $2^n - 1 \times 2^m - 1$ table having as rows all non-empty subsets A of Att, as columns all non-empty subsets Y of U and in the place corresponding to row A and column Y the value $\alpha_A(Y)$.*

	$\{v_1\}$	$\{v_2\}$	$\{v_3\}$	$\{v_1,v_2\}$	$\{v_1,v_3\}$	$\{v_2,v_3\}$	V
$\{v_1\}$	0	0	0	0	0	0	1
$\{v_2\}$	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1	1
$\{v_3\}$	0	1	0	$\frac{1}{3}$	1	$\frac{1}{3}$	1
$\{v_1,v_2\}$	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1	1
$\{v_1,v_3\}$	0	1	0	$\frac{1}{3}$	1	$\frac{1}{3}$	1
$\{v_2,v_3\}$	1	1	1	1	1	1	1
V	1	1	1	1	1	1	1

TABLE 1. Approximation Accuracy Table of $[\vec{P_3}]^+$.

In next results we completely determine the approximation accuracy table for the three basic digraph families $\vec{P_n}$, $\vec{C_n}$ and $\vec{T_n}$. We begin to examine the digraph $\vec{P_n}$.

Theorem 4.4. *Let A and Y be two vertex subsets of $\vec{P_n} = \{v_1, \ldots, v_n\}$, where $A \neq \emptyset$ and $Y \neq \emptyset$. The A^+-accuracy of Y is:*

$$\alpha_A^+(Y) = \begin{cases} 0 & \text{if } A = \{v_1\} \wedge Y \neq V \\ 1 & \text{if } \{v_2, \ldots, v_n\} \subseteq A \vee (|A| = k \wedge (Y \supseteq (A_{-1})^c \vee Y \subseteq A_{-1})) \\ \frac{|Y \cap A_{-1}|}{|Y \cup (A_{-1})^c|} & \text{otherwise.} \end{cases}$$

Proof. We will compute the A^+-lower and the A^+-upper approximation of Y as A and Y run over the vertex subsets. At first, we compute the A^+-lower approximation. By Proposition 3.4, if $A = V$ or $A = \{v_2, \ldots, v_n\}$, then $\pi^+_{\vec{P_n}}(A) = v_1|\ldots|v_n$ and therefore $\mathbf{l}_A^+(Y) = Y$ for any choice of the vertex subset Y. The case $Y = V$ is obvious. Note that in this case $Y \supseteq (A_{-1})^c$ and $\alpha_A^+(Y) = 1$. Moreover, when $A = \{v_1\}$, we obtain $\mathbf{l}_A^+(Y) = \emptyset$ for any $Y \neq V$. Let $|A| = k$; then by Proposition 3.4 $\pi^+_{\vec{P_n}}(A) = v_{i_1-1}|\ldots|v_{i_k-1}|(A_{-1})^c$ or $\pi^+_{\vec{P_n}}(A) = v_{i_2-1}|\ldots|v_{i_k-1}|(A_{-1})^c$ depending on whether $v_1 \notin A$ or $v_1 \in A$. In both cases, if $Y \supseteq (A_{-1})^c$, then $[v]_A^+ \subseteq Y$ if and only if $v \in (A_{-1})^c$ or $v \in Y \cap A_{-1}$, hence $\mathbf{l}_A^+(Y) = (A_{-1})^c \cup (Y \cap A_{-1}) = Y$. If $|A| = k$ and $Y \not\supseteq (A_{-1})^c$, then $[v]_A^+ \subseteq Y$ if and only if $v \in Y \cap A_{-1}$, so $\mathbf{l}_A^+(Y) = Y \cap A_{-1}$. Summarizing, we have proved that

$$\mathbf{l}_A^+(Y) = \begin{cases} \emptyset & \text{if } A = \{v_1\} \\ Y & \text{if } \{v_2, \ldots, v_n\} \subseteq A \vee (|A| = k \wedge Y \supseteq (A_{-1})^c) \\ Y \cap A_{-1} & \text{otherwise.} \end{cases} \tag{8}$$

We can now compute the upper approximation. By Proposition 3.4, if $A = \{v_1\}$, then $\pi^+_{\vec{P_n}}(A) = V$ and therefore $\mathbf{u}_A^+(Y) = V$. The case $Y = V$ is obvious. Let $|A| = k$ and $Y \cap (A_{-1})^c \neq \emptyset$, then $[v]_A^+ \cap Y \neq \emptyset$ if and only if $v \in (A_{-1})^c$ or $v \in Y \cap A_{-1}$ and thus in this case $\mathbf{u}_A^+(Y) = Y \cup (A_{-1})^c$. If $|A| = k$ and $Y \subseteq A_{-1}$, then $[v]_A^+ \cap Y \neq \emptyset$ if and only if $v \in Y \cap A_{-1} = Y$, so $\mathbf{u}_A^+(Y) = Y$. If $A = V$ or $A = \{v_2, \ldots, v_n\}$, by Proposition 3.4, it is easy to see that $[v]_A^+ \cap Y \neq \emptyset$ if and only if $v \in Y$. Summarizing, we have:

$$\mathbf{u}_A^+(Y) = \begin{cases} V & \text{if } A = \{v_1\} \vee Y = V \\ Y \cup (A_{-1})^c & \text{if } |A| = k \wedge Y \cap (A_{-1})^c \neq \emptyset \\ Y & \text{otherwise.} \end{cases} \tag{9}$$

We can conclude by means of Definition 4.2. In particular, we observe that if $A = V$ or $A = \{v_2, \ldots, v_n\}$ or $Y = V$ then Y is always A^+-exact. In the other cases, Y is A^+-exact if and only if $A \neq \emptyset \wedge A \neq \{v_1\} \wedge (|A| = k \wedge (Y \supseteq (A_{-1})^c \vee Y \subseteq A_{-1}))$. The theorem is thus proved. □

Example 4.5. In Table 1, we represent the approximation accuracy table of $[\vec{P_3}]^+$.

We determine now the accuracy table for the n-directed cycle.

Theorem 4.6. *Let A and Y be two vertex subsets of $V(\vec{C_n}) = \{v_1, \ldots, v_n\}$, where $A \neq \emptyset$ and $Y \neq \emptyset$. The A^+-approximation accuracy of Y is:*

$$\alpha_A^+(Y) = \begin{cases} 1 & \text{if } |A| \geq n-1 \vee (1 \leq |A| \leq n-2 \wedge (Y \supseteq (A_{-1})^c \vee Y \subseteq A_{-1})) \\ \frac{|Y \cap A_{-1}|}{|Y \cup (A_{-1})^c|} & \text{otherwise.} \end{cases}$$

Proof. We have to compute the A^+-lower and the A^+-upper approximations of $\vec{C_n}$. By Proposition 3.5, if $Y = V$, we have that $\mathbf{l}_A^+(Y) = V$. Suppose that $|A| \geq n-1$, then $\pi^+_{\vec{C_n}}(A) = v_1|\ldots|v_n$, therefore,

	$\{v_1\}$	$\{v_2\}$	$\{v_3\}$	$\{v_1, v_2\}$	$\{v_1, v_3\}$	$\{v_2, v_3\}$	V
$\{v_1\}$	0	0	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1
$\{v_2\}$	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1	1
$\{v_3\}$	0	1	0	$\frac{1}{3}$	1	$\frac{1}{3}$	1
$\{v_1, v_2\}$	1	1	1	1	1	1	1
$\{v_1, v_3\}$	1	1	1	1	1	1	1
$\{v_2, v_3\}$	1	1	1	1	1	1	1
V	1	1	1	1	1	1	1

TABLE 2. Approximation Accuracy Table of $[\vec{C}_3]^+$.

for any choice of the vertex subset Y, it results $\mathbf{l}_A^+(Y) = Y$. Let $1 \leq |A| = k \leq n-2$ and let $Y \supseteq (A_{-1})^c$. By Proposition 3.5 we deduce $[v]_A^+ \subseteq Y$ if and only if $v \in (A_{-1})^c$ or $v \in Y \cap A_{-1}$ and $\mathbf{l}_A^+(Y) = (A_{-1})^c \cup (Y \cap A_{-1}) = Y$. If $1 \leq |A| \leq n-2$ and $Y \supsetneqq (A_{-1})^c$, then $[v]_A^+ \subseteq Y$ if and only if $v \in Y \cap A_{-1}$, so $\mathbf{l}_A^+(Y) = Y \cap A_{-1}$. Hence, we have shown the following claim:

$$\mathbf{l}_A^+(Y) = \begin{cases} Y & \text{if } |A| \geq n-1 \vee (1 \leq |A| \leq n-2 \wedge Y \supseteq (A_{-1})^c) \\ A_{-1} \cap Y & \text{otherwise} \end{cases} \tag{10}$$

Now, we can compute the upper approximation. By Proposition 3.5, if $|A| \geq n-1$, it results that $\pi^+_{\vec{C}_n}(A) = v_1|\dots|v_n$, therefore it's clear that $\mathbf{u}_A^+(Y) = Y$ for any choice of the vertex subset Y. The case $Y = V$ is obvious. Let $1 \leq |A| = k \leq n-2$ and $Y \cap (A_{-1})^c \neq \emptyset$, then $[v]_A^+ \cap Y \neq \emptyset$ if and only if $v \in (A_{-1})^c$ or $v \in Y \cap A_{-1}$, therefore $\mathbf{u}_A^+(Y) = Y \cup (A_{-1})^c$. Finally, if $1 \leq |A| \leq n-2$ and $Y \subseteq A_{-1}$, then $[v]_A^+ \cap Y \neq \emptyset$ if and only if $v \in Y \cap A_{-1} = Y$, hence we conclude that $\mathbf{u}_A^+(Y) = Y$. In this way, we have proved that:

$$\mathbf{u}_A^+(Y) = \begin{cases} V & \text{if } Y = V \\ Y \cup (A_{-1})^c & \text{if } 1 \leq |A| \leq n-2 \wedge Y \cap (A_{-1})^c \neq \emptyset \\ Y & \text{otherwise} \end{cases} \tag{11}$$

We complete the proof by means of Definition 4.2. In particular, we observe that if $A = V$ or $|A| = n-1$ or $Y = V$, then Y is always A^+-exact. In the other cases, Y is A^+-exact if and only if $1 \leq |A| \leq n-2 \wedge (Y \supseteq (A_{-1})^c \vee Y \subseteq A_{-1})$. □

Example 4.7. In Table 2, we represent the approximation accuracy table of $[\vec{C}_3]^+$.

Finally, we determine the accuracy table for the n-transitive tournament $\vec{T}_n$. Let $A \subseteq V(\vec{T}_n)$ such that $|A| = k$. We set then

$$A' := \begin{cases} N^+_{\vec{T}_n}(v_{i_1-1})^c & \text{if } A = \{v_{i_1}, \dots, v_{i_k}\} \wedge 1 < i_1 < \dots < i_k \leq n \\ N^+_{\vec{T}_n}(v_{i_2-1})^c & \text{if } A = \{v_1, v_{i_2}, \dots, v_{i_k}\} \wedge 1 < i_2 < \dots < i_k \leq n \end{cases}$$

and

$$C_A^j := \begin{cases} N^+_{\vec{T}_n}(v_{i_j-1})^c \setminus N^+_{\vec{T}_n}(v_{i_{j-1}})^c & \text{if } A = \{v_{i_1}, \dots, v_{i_k}\} \wedge 1 < i_1 < \dots < i_k \leq n \\ N^+_{\vec{T}_n}(v_{i_j-1})^c \setminus N^+_{\vec{T}_n}(v_{i_{j-1}})^c & \text{if } A = \{v_1, v_{i_2}, \dots, v_{i_k}\} \wedge 1 < i_2 < \dots < i_k \leq n \end{cases}$$

where, respectively, $1 < j < k$ and $2 < j < k$. In both cases, we also set $A'' := N^+_{\vec{T}_n}(v_{i_k-1})$. Let Y be another vertex subset. Then, in both cases, we set:

$$Q_A(Y) := \bigcup_{v_{i_j} \in A} \{C_A^j : C_A^j \cap Y \neq \emptyset\},$$

and

$$T_A(Y) := \bigcup_{v_{i_j} \in A} \{C_A^j : C_A^j \subseteq Y\}.$$

We have then the following result.

Theorem 4.8. *Let A and Y be two vertex subsets of $\vec{T_n} = \{v_1, \ldots, v_n\}$, where $y \neq \emptyset$. The A^+-accuracy of Y is:*

$$\alpha_A^+(Y) = \begin{cases} 0 & \text{if } A = \{v_1\} \vee (|A| = k \wedge A' \not\subseteq Y \wedge A'' \not\subseteq Y \wedge T_A(Y) = \emptyset) \\ 1 & \text{if } \{v_2, \ldots, v_n\} \subseteq A \vee Y = V \vee \left(\exists j_1, \ldots, j_l : Y = \bigcup_{k=1}^{l} C_A^{j_k} \cup A' \cup A''\right) \\ \frac{|T_A(Y) \cup A' \cup A''|}{|Q_A(Y) \cup A' \cup A''|} & \text{otherwise.} \end{cases}$$

Proof. We have to compute both the A^+-lower and the A^+-upper approximations. We start computing the A^+-lower approximation. The case $Y = V$ is obvious. Let $A = V$, then $\pi^+_{\vec{T_n}}(A) = v_1|\ldots|v_n$ therefore it results that $\mathbf{l}_A^+(Y) = Y$. By Proposition 3.6, the same holds when $A = \{v_2, \ldots, v_n\}$. Computations are obvious also when $A = \{v_1\}$.
Let A and Y be two vertex subsets such that $|A| = k \wedge A' \not\subseteq Y \wedge A'' \not\subseteq Y \wedge T_A(Y) = \emptyset$. This means that no A^+-indiscernibility block is contained in Y and hence $\mathbf{l}_A^+(Y) = \emptyset$.
Let A and Y be two vertex subsets such that $|A| = k$. In general, it is immediate to see that $[v]_A \subseteq Y$ if and only if $v \in A' \wedge A' \subseteq Y$ or $v \in A'' \wedge A'' \subseteq Y$ or $\exists C_A^j$ containing v and such that $C_A^j \subseteq Y$. Thus, in general, $\mathbf{l}_A^+(Y) = T_A(Y) \cup A' \cup A''$. Hence, we have proved that

$$\mathbf{l}_A^+(Y) = \begin{cases} Y & \text{if } \{v_2, \ldots, v_n\} \subseteq A \vee Y = V \\ \emptyset & \text{if } A = \{v_1\} \vee (|A| = k \wedge A' \not\subseteq Y \wedge A'' \not\subseteq Y \wedge T_A(Y) = \emptyset) \\ T_A(Y) \cup A' \cup A'' & \text{otherwise} \end{cases} \tag{12}$$

Now, we can compute the A^+-upper approximation. Computations are obvious when $Y = V$, $A = \{v_1\}$, $A = V$ or $A = \{v_2, \ldots, v_n\}$.
Let $|A| = k$ and Y be a vertex subset. It's easy to see that, in general, $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A'$ or $v \in A''$ or $v \in C_A^j$ for some j, clearly depending on how Y and A are. This means that $\mathbf{u}_A(Y) = Q_A(Y) \cup A' \cup A''$. Hence,

$$\mathbf{u}_A^+(Y) = \begin{cases} Y & \text{if } \{v_2, \ldots, v_n\} \subseteq A \\ V & \text{if } A = \{v_1\} \vee Y = V \\ Q_A(Y) \cup A' \cup A'' & \text{otherwise} \end{cases} \tag{13}$$

By means of Definition 4.2, we complete the proof. In particular, we observe that if $A = V$ or $A = \{v_2, \ldots, v_n\}$ or $Y = V$ or if $\exists j_1, \ldots, j_l : Y = \bigcup_{k=1}^{l} C_A^{j_k} \cup A' \cup A''$, then Y is A^+-exact. □

5. Attribute Dependency Averages

If $\mathfrak{I} = \langle U, Att, Val, F\rangle$ is an information system and A and B are two subsets of Att, two important notions (see [45]) in RST are the *A-positive region* of B

$$POS_A(B) = \{u \in U : [u]_A \subseteq [u]_B\}, \tag{14}$$

i.e. the set of all objects in the universe set U properly classifiable by means of blocks of $\pi_{\mathfrak{I}}(B)$ using only the knowledge expressed by classification $\pi_{\mathfrak{I}}(A)$, and the *A-attribute dependency function* (or *quality of approximation* of B by A)

$$\gamma_A : X \in \mathcal{P}(Att) \mapsto \gamma_A(X) := \frac{|POS_A(X)|}{|U|}. \tag{15}$$

It is clear then that, if A and B are two any attribute subsets of $\mathfrak{I}$, we have

$$B \subseteq A \Longrightarrow \pi_{\mathfrak{I}}(A) \preceq \pi_{\mathfrak{I}}(B) \tag{16}$$

and

$$\pi_{\mathfrak{I}}(A) \preceq \pi_{\mathfrak{I}}(B) \iff POS_A(B) = U \iff \gamma_A(B) = 1 \tag{17}$$

The quality of approximation of B by A is a measure of the representability degree of the knowledge provided from the attribute subset B in terms of the knowledge provided from the other attribute subset A. We can also interpret the number $\gamma_A(B)$ as the "probability" which whenever two objects are A-indiscernible then they also are B-indiscernible.

We can use the attribute dependency function in order to introduce a global attribute dependency measure for the information system $\mathfrak{I}$.

Definition 5.1. *Let $\mathcal{I}$ be an information system such that $|Att| = n$. We call* attribute dependency table *of $\mathcal{I}$, denoted by $T_{dep}(\mathcal{I})$, the $2^n \times 2^n$ table having as rows and columns all subsets of Att and in the place corresponding to row A and column B the value $\gamma_A(B)$.*

Let us note that the attribute dependency table is a source of very useful information when one studies the properties of $\mathcal{I}$. In fact, by (17) we can observe that all order relations between indiscernibility relations can be deduced from the table $T_{dep}(\mathcal{I})$. In other terms, a complete knowledge of the attribute dependency table permits to build the granular partition lattice [71] of the information system $\mathcal{I}$. We can also think the table $T_{dep}(\mathcal{I})$ as a type of numerical completion of the granular partition lattice of $\mathcal{I}$. The study of average numbers is very frequent both in digraph context (see [3]) and in several sub-fields of GrC (see [32, 34, 66]), therefore, due to the importance of the attribute dependency table for any information system, it is natural to introduce the following new statistical-type concepts relatively to any information system.

Definition 5.2. *Let $\mathcal{I}$ be an information system such that $|Att| = n$ and $A \subseteq Att$. We call* attribute dependency average from A *in $\mathcal{I}$ the number $\lambda_r(A) := \frac{1}{2^n}\sum_{B\subseteq Att}\gamma_A(B)$. Analogously, we call* attribute dependency average towards A *in $\mathcal{I}$ the number $\lambda_c(A) := \frac{1}{2^n}\sum_{B\subseteq Att}\gamma_B(A)$.*

Let us note that the number $\lambda_r(A)$ $[\lambda_c(A)]$ can be considered, by using a statistical terminology, a *marginal average* on the row [column] corresponding to the attribute subset A in the dependency table of $\mathcal{I}$. At this point, we introduce one of the more relevant global measure studied in this paper.

Definition 5.3. *We call* attribute dependency average *of $\mathcal{I}$ the number*

$$\lambda(\mathcal{I}) := \frac{1}{2^n}\sum_{A\subseteq Att}\lambda_r(A) = \frac{1}{2^n}\sum_{A\subseteq Att}\lambda_c(A).$$

We can interpret the number $\lambda(\mathcal{I})$ as the average capacity to transmit (or to receive) dependency of a generic (non determinate) attribute subset in the information system $\mathcal{I}$.
We can adapt all previous concepts in the case of the information system $[D]^+$. In particular we call *out symmetry transmission quality table* of D, the table $T^+_{sym}(D) := T_{dep}([D]^+)$. We call *out-average symmetry transmission from A* in D the number

$$\lambda^+_r(A) := \frac{1}{2^n}\sum_{B\subseteq V}\gamma^+_A(B).$$

Analogously, we call *out-average symmetry transmission towards A* in D the number

$$\lambda^+_c(A) := \frac{1}{2^n}\sum_{B\subseteq V}\gamma^+_B(A).$$

Finally, we call *out-average transmission symmetry* of D the number

$$\lambda^+(D) := \frac{1}{2^n}\sum_{A\subseteq V}\lambda^+_r(A) = \frac{1}{2^n}\sum_{A\subseteq V}\lambda^+_c(A).$$

6. The cases of $\vec{P}_n$ and $\vec{T}_n$

As in the previous section, our first case of study is the digraph $\vec{P}_n$.

Lemma 6.1. *Let A and B be two distinct vertex subsets of $V = V(\vec{P}_n) = \{v_1, \dots, v_n\}$. Then*

$$\gamma^+_A(B) = \begin{cases} 1 & \text{if } B \subseteq A \vee B = \{v_1\} \vee \{v_2, \dots, v_n\} \subseteq A \\ 0 & \text{if } (A = \emptyset \vee A = \{v_1\}) \wedge (B \neq \emptyset \wedge B \neq \{v_1\}) \\ \frac{k}{n} & \text{if } |A| = k \wedge v_1 \notin A \\ \frac{k-1}{n} & \text{if } |A| = k \wedge v_1 \in A \end{cases} \tag{18}$$

Proof. By Proposition 3.4, we have that if $B = \emptyset$ or $B = \{v_1\}$, then $\pi^+_{\vec{P}_n}(B) = V$. Therefore if A is any subset of V and $v \in V$ is any vertex, it follows that $[v]^+_A \subseteq [v]^+_B = V$. Similarly, if $A = V$ or if $A = \{v_2, \dots, v_n\}$, we have $\pi^+_{\vec{P}_n}(A) = v_1|\dots|v_n$. Moreover, by (16), if $B \subseteq A$, then $\pi^+_A \preceq \pi^+_B$. In other words, in these cases $POS^+_A(B) = V$. A fortiori, even if $A = \emptyset$ or $A = \{v_1\}$, then $POS^+_A(B) = V$. This is false when $B \neq \emptyset$ or $B \neq \{v_1\}$, in fact in this case we would have $[v]^+_A = V$ contained in one of its

	$\emptyset$	$\{v_1\}$	$\{v_2\}$	$\{v_3\}$	$\{v_1,v_2\}$	$\{v_1,v_3\}$	$\{v_2,v_3\}$	V
$\emptyset$	1	1	0	0	0	0	0	0
$\{v_1\}$	1	1	0	0	0	0	0	0
$\{v_2\}$	1	1	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{v_3\}$	1	1	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{v_1,v_2\}$	1	1	1	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{v_1,v_3\}$	1	1	$\frac{1}{3}$	1	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$
$\{v_2,v_3\}$	1	1	1	1	1	1	1	1
V	1	1	1	1	1	1	1	1

TABLE 3. Out Symmetry Transmission Quality Table of $[\vec{P_3}]^+$.

subsets, absurd. Let A and B be two generic vertex subsets distinct by the previous. It's clear that the classes of vertices which form single blocks with respect to A are contained in their classes with respect to B, so $A_{-1} \subseteq POS_A^+(B)$. Let $v \in POS_A^+(B) \setminus A_{-1}$. Then $[v]_A^+ = (A_{-1})^c$. We have two cases: $v \in B$ or $v \notin B$. In the first case, $[v]_B^+$ has a single element, so $[v]_A^+$ can't be contained in $[v]_B^+$ unless $(A_{-1})^c$ consists of a single element, but this happens if $A = \{v_2, \ldots, v_n\}$ but this situation has been excluded. In the second case, we have $[v]_A^+ \subseteq [v]_B^+$ if and only if $A_{-1}^c \subseteq B_{-1}^c$, i.e. if and only if $B_{-1} \subseteq A_{-1}$, i.e. $B \subseteq A$, but it cannot happen. Thus $POS_A^+(B) = \{v_{i_1-1}, \ldots, v_{i_k-1}\}$. In this way, we have shown that

$$POS_A^+(B) = \begin{cases} V & \text{if } B \subseteq A \vee B = \{v_1\} \vee \{v_2, \ldots, v_n\} \subseteq A \\ \emptyset & \text{if } (A = \emptyset \vee A = \{v_1\}) \wedge (B \neq \emptyset \wedge B \neq \{v_1\}) \\ A_{-1} & \text{otherwise} \end{cases} \tag{19}$$

Finally, (18) follows directly from (19). □

Example 6.2. In Table 3, we represent $T_{sym}^+(P_3)$.
Let us examine now, for example, the values on the row corresponding to the vertex subset $\{v_1, v_2\}$. By means of our previous interpretation, we can say that $\{v_1, v_2\}$ transmits out-symmetry towards all other vertex subsets with sequence $1, 1, 1, \frac{1}{3}, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. Hence it is also natural to say that $\{v_1, v_2\}$ has a *transmission out-symmetry average capacity* in $\vec{P_3}$ equal to the average of the above sequence, that is $\frac{2}{3}$.

We now compute the out-average symmetry transmission from a vertex subset A to $\vec{P_n}$.

Lemma 6.3. *Let $A \subseteq V = V(\vec{P_n})$. Then we have*

$$\lambda_r^+(A) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } A = \emptyset \vee A = \{v_1\} \\ 1 & \text{if } \{v_2, \ldots, v_n\} \subseteq A \\ \frac{1}{n} + \frac{3}{2^n}(1 - \frac{1}{n}) & \text{if } |A| = 1 \\ \frac{1}{2^{n-k}}(1 + \frac{k-1}{n}(2^{n-k} - 1)) & \text{if } A = \{v_1, v_{i_2}, \ldots, v_{i_k}\} \\ \frac{1+2^k+\frac{k}{n}(2^n-2^k-1)}{2^n} & \text{if } |A| = k \neq n-1 \wedge v_1 \notin A \end{cases}$$

Proof. By (18) of Lemma 6.1, in the rows corresponding to $A = \emptyset$ and $A = \{v_1\}$ we have only two entries 1 and the others are all 0, so $\lambda_r^+(\emptyset) = \lambda_r^+(\{v_1\}) = \frac{1}{2^{n-1}}$. When $A = V$, we have that $\gamma_A^+(B) = 1$ for every $B \subseteq V = V(\vec{P_n})$, therefore $\lambda_r^+(V) = 1$. If A is singleton $\{v_i\} \neq \{v_1\}$, by (18) only in three entries we have 1, in the other entries we have $\frac{1}{n}$. Therefore we have $\lambda_r^+(A) = \frac{1}{2^n}\left(\frac{2^n}{n} - \frac{3}{n} + 3\right)$, i.e. $\frac{1}{n} + \frac{3}{2^n}(1 - \frac{1}{n})$. Let $A = \{v_1, v_{i_2}, \ldots, v_{i_k}\}$. By (18), the entries are 1 when the sets in the columns of the dependency table are contained in A, therefore there are exactly $\binom{k}{0} + \cdots + \binom{k}{k} = 2^k$ entries equal to 1, i.e. the number of all vertex subsets of a set of cardinality k, while the other entries are equal to $\frac{k-1}{n}$. If A doesn't contain the vertex v_1 and has cardinalitiy $|A| = k \neq n-1$, there are $2^k + 1$ entries equal to 1, i.e. those corresponding to all its subsets and one corresponding to the second column. By (18), we deduce that the remaining entries have value $\frac{k}{n}$. □

In next result, we will compute the out-average transmission symmetry of $\vec{P_n}$.

Theorem 6.4. *The quantity $\lambda^+(\vec{P}_n)$ is equal to:*

$$\frac{1}{4^n}\left\{4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+\sum_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1+\frac{k-1}{n}(2^{n-k}-1)\right]\right\}+$$

$$\frac{1}{4^n}\left\{\sum_{k=2}^{n-2}\binom{n-1}{k}\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]\right\}.$$

and $\lim_{n\to\infty}\lambda^+(\vec{P}_n)=\frac{1}{2}$.

Proof. We must add the numbers $\lambda_r^+(A)$ determined in Lemma 6.3, when A varies among the subsets of $V(\vec{P}_n)$. So, if $A=\emptyset$ or $A=\{v_1\}$, we have to add 2, while if $A=\{v_2,\dots,v_n\}$ or $A=v$ we have to add 2^n. If A is a singleton v_i, then we have exactly three entries equal to 1, while the others take the value $\frac{1}{n}$. Since there are $n-2$ of these singletons, we can add the quantity $(n-2)\left(3+\frac{2^n-3}{n}\right)$. Since, for $k=2,\dots,n-1$ the number of sets of cardinality k containing the vertex v_1 is $\binom{n-1}{k-1}$, we have to add the quantity $\sum_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1+\frac{k-1}{n}(2^{n-k}-1)\right]$. Moreover, since for $k=2,\dots,n-2$ the number of sets of cardinality k not containing the vertex v_1 is $\binom{n-1}{k}$, we also have to add the quantity $\sum_{k=2}^{n-2}\binom{n-1}{k}2^k\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]$. Thus, we conclude

$$\lambda^+(\vec{P}_n)=\frac{1}{4^n}\left\{4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+\sum_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1+\frac{k-1}{n}(2^{n-k}-1)\right]\right\}+$$

$$\frac{1}{4^n}\left\{\sum_{k=2}^{n-2}\binom{n-1}{k}\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]\right\}.$$

We set $R=\sum_{k=2}^{n-1}\binom{n-1}{k-1}2^k\left[1+\frac{k-1}{n}(2^{n-k}-1)\right]$ and $S=\sum_{k=2}^{n-2}\binom{n-1}{k}\left[1+2^k+\frac{k}{n}(2^n-2^k-1)\right]$. Then, by means of both the Stifel formula and the identities $\sum_{k=0}^{n}\binom{n}{k}2^k=3^n$, $\sum_{k=0}^{n}\binom{n}{k}k=n2^{n-1}$, $\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$, with algebraic manipulations we obtain

$$R+S\leq 3^n+2^{2n-1}-2-3\cdot 2^n-3n-\frac{n-1}{n}2^n.$$

Then we have

$$\lambda^+(\vec{P}_n)=\frac{1}{4^n}\left[4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+R+S\right]\leq$$

$$\leq\frac{1}{4^n}\left[4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)+3^n+2^{2n-1}-2-3\cdot 2^n-3n-\frac{n-1}{n}2^n\right],$$

hence

$$\lim_{n\to\infty}\lambda^+(\vec{P}_n)\leq\frac{1}{2}.$$

In a similar way, again by means of both the Stifel formula and the identities $\sum_{k=0}^{n}\binom{n}{k}k=n2^{n-1}$, $\sum_{k=0}^{n}\binom{n}{k}k2^k=2n\cdot 3^{n-1}$, $\sum_{k=0}^{n-1}\binom{n-1}{k}=2^{n-1}$, $\sum_{k=0}^{n-1}\binom{n-1}{k}2^k=3^{n-1}$, $\sum_{k=0}^{n-1}\binom{n-1}{k}k=(n-1)2^{n-2}$, we deduce that

$$R+S\geq\left(1-\frac{1}{n}\right)2^{2n-1}-(2\cdot 3^{n-1}+2^{n-1})\left(1-\frac{1}{n}\right)+\frac{2^{n+1}-2^n+2^{n-1}(n-1)^2+2n-2-2(n-1)(2^{n-3}+1)}{n}.$$

Adding the term $Q=4+2^{n+1}+(n-2)\left(3+\frac{2^n-3}{n}\right)$, dividing by 4^n and passing to the limit as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\lambda^+(\vec{P}_n)\geq\frac{1}{2}.$$

Hence $\lim_{n\to\infty}\lambda^+(\vec{P}_n)=\frac{1}{2}$. □

6.1. **The $\vec{T_n}$ case.** Regarding to the out-average transmission symmetry for the n-tournament $\vec{T_n}$, actually we are able to provide the following lower estimate.

Theorem 6.5.

$$\lambda^+(\vec{T_n}) \geq \left(\frac{3}{4}\right)^{n-1}. \tag{20}$$

Proof. By Proposition 3.6, if $A = V$ or $A = \{v_2, \ldots, v_n\}$, we have $\pi^+_{\vec{T_n}}(A) = v_1|\ldots|v_n$. Moreover, if $B = \emptyset$ or $B = \{v_1\}$, then $\pi^+_{\vec{T_n}}(B) = V$. Therefore if A is any subset of V and $v \in V$ is any vertex, it follows that $[v]^+_A \subseteq [v]^+_B = V$. In other words, $POS^+_A(B) = V$. A fortiori, even if $A = \emptyset$ or $A = \{v_1\}$, then $POS^+_A(B) = V$. This argument leads to a contradiction when $B \neq \emptyset$ or $B = \{v_1\}$, because we would have $[v]^+_A = V$ contained in one of its subsets, so $POS^+_A(B) = \emptyset$. Furthermore, it's clear that if $B \subseteq A$, then, by (16), $POS^+_A(B) = V$. Hence, we have shown that

$$POS^+_A(B) = \begin{cases} V & \text{if } B \subseteq A \vee B = \{v_1\} \vee \{v_2, \ldots, v_n\} \subseteq A \vee (|A| = k \wedge B = A \cup \{v_1\}) \\ \emptyset & \text{if } (A = \emptyset \vee A = \{v_1\}) \wedge (B \neq \emptyset \wedge B \neq \{v_1\}) \end{cases} \tag{21}$$

It is immediate to deduce by (21) that

$$\gamma^+_A(B) = \begin{cases} 1 & \text{if } B \subseteq A \vee B = \{v_1\} \vee \{v_2, \ldots, v_n\} \subseteq A \vee (|A| = k \wedge B = A \cup \{v_1\}) \\ 0 & \text{if } (A = \emptyset \vee A = \{v_1\}) \wedge (B \neq \emptyset \wedge B \neq \{v_1\}) \end{cases} \tag{22}$$

Now, let $A \subseteq V = V(\vec{T_n})$ such that $|A| = k$ and $A \neq \{v_2, \ldots, v_n\}$. Then:

$$\lambda^+_r(A) \geq \begin{cases} \frac{1}{2^{n-k}} & \text{if } v_1 \in A \\ \\ \frac{1}{2^{n-k-1}} & \text{if } v_1 \notin A \end{cases} \tag{23}$$

In order to prove the claim, we must use the numbers $\gamma^+_A(B)$ in (22). Let $A \subseteq V = V(\vec{T_n})$ and $|A| = k$. In order to prove (23), we have to show that in the corresponding row, the number of entries equal to 1 is exactly

$$\begin{cases} 2^k & \text{if } v_1 \in A \\ 2^{k+1} & \text{if } v_1 \notin A \end{cases} \tag{24}$$

If A contains v_1, then it has exactly 2^{k-1} subsets B with first element v_1, so by (22), $\gamma^+_A(B) = 1$. By Remark 3.7, we also have that $\pi^+_{\vec{T_n}}(B) = \pi^+_{\vec{T_n}}(B')$ when $B' = B \cup \{v_1\}$. Therefore, it results $\gamma_A(B') = 1$ when $B' = B \cup \{v_1\}$ and B is one of the previous. This means that in the row corresponding to A there are 2^k entries equal to 1. By the same argument, it's easy to deduce that if A doesn't contain v_1, then its corresponding row has exactly 2^{k+1} entries equal to 1. In this way, (23) is an immediate consequence of (24).
By (22), the total number of entries equal to 1 is exactly the sum of all such entries row by row. Therefore we have 2 of such entries in the first two rows and 2^n in the last two. Now, again by (23), if $|A| = k$ and A contains v_1, then we have to add 2^k. Since there are $\binom{n-1}{k-1}$ of such rows for every k, i.e. the number of $k-$subsets containing a fixed element, then we have to add the term $\sum_{k=2}^{n-1} 2^k \binom{n-1}{k-1}$. Furthermore, again by (23), if $|A| = k$ and A doesn't contain v_1, then we have to add 2^{k+1}. Since there are $\binom{n-1}{k}$ of such rows for every k, i.e. the number of $k-$subsets not containing a fixed element, then we have to add the term $\sum_{k=1}^{n-2} 2^{k+1} \binom{n-1}{k}$. Thus, after reindexing the sums, the number of entries equal to 1 is exactly $4 + 2^{n+1} + 4\sum_{k=1}^{n-2} 2^k \binom{n-1}{k}$. Using the relation $\sum_{k=0}^{n-1} 2^k \binom{n-1}{k} = 3^{n-1}$, we obtain, after dividing by 4^n, the quantity $\left(\frac{3}{4}\right)^n$. Since, by (22), in the dependency table of $\vec{T_n}$ all other values are 0 or strictly positive rational numbers, we obtain the lower estimate

$$\lambda^+(\vec{T_n}) \geq \left(\frac{3}{4}\right)^{n-1}.$$

□

7. Conclusions

Granular Computing (GrC) is an emerging paradigm in computing and applied mathematics to process data and information. In several recent researches this paradigm, in combination with rough set theory (RST), has been used to discover new properties of graphs and digraphs. Actually there is a wide literature concerning the connections between GrC, RST and graph theory. GrC and RST can be used in several ways to investigate a given graph or digraph.
In this paper, we studied digraphs as they were mathematical models for GrC and RST. To be more specific, we examined the out-adjacency matrix of a digraph as a Boolean information table and we investigated the attribute dependency and the approximation accuracy for some basic digraph families. Moreover, we also defined a new quantity expressing a global average for the attribute dependency. A similar number can be introduced also in order to give a global average for approximation accuracy, but we will analyze it in forthcoming papers.
From an interpretative perspective, the introduction of a global average for the attribute dependency can be useful also in relation to data tables which are not adjacency matrices of directed graphs. However, in order to better understand the potentiality of such a new measure, the study of digraphs as specific types of binary data tables is the first step to take.

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