The Granular Partition Lattice of an Information Table

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Abstract

In this paper we study the lattice of all indiscernibility partitions induced from attribute subsets of a knowledge representation system (information table in the finite case). This lattice, that we here call granular partition lattice, is a very well studied order structure in granular computing and data base theory and it provides a complete hierarchical classification of the knowledge obtained from all possible choices of attribute subsets. We show that it has a lattice structure also in the infinite case and we provide several isomorphic characterizations for this lattice. We discuss the potentiality of this order structure from both a micro-granular and a macro-granular perspective. Furthermore, the sub-poset of all the *indiscernibility closures* needed to determine when an arbitrary partition is an indiscernibility one is studied. Finally, we show the monotonic behaviour of the granular partition lattice with respect to entropy of partitions and attribute dependency in decision tables.

Keywords: Set Partition, Rough sets, Granular Computing, Entropy

1. Introduction

Let us consider a basic structure to represent knowledge: an attribute-value system, i.e., a table whose rows represent the *objects* of a finite universe set U and the columns the *attributes* of another finite set Att. Usually, in rough set theory this structure is called *information table* $\mathcal{I}$ and it corresponds to a relation in first normal form in database theory [18].
If A is an attribute subset of Att, we can build the usual rough-set indiscernibility partition $\pi_{\mathcal{I}}(A)$ of U induced by A. The set $\Pi_{ind}(\mathcal{I})$ of all these indiscernibility partitions is partially ordered by the usual partial order $\preceq$ on the set $\Pi(U)$ of all (not necessarily indiscernibility) set partitions of U. The poset $\mathbb{P}_{ind}(\mathcal{I}) := (\Pi_{ind}(\mathcal{I}), \preceq)$ is a well known order structure in rough set theory and granular computing [76]. In [31] it has been proved that $\mathbb{P}_{ind}(\mathcal{I})$ is a complete lattice, but not necessarily a sub-lattice of the lattice $\mathbb{P}(U) := (\Pi(U), \preceq)$.
In this paper we call *indiscernibility partition lattice* the poset $\mathbb{P}_{ind}(\mathcal{I})$, and we will study this order structure when U and Att are arbitrary sets, i.e. not necessarily finite sets. For this general case we use the term *knowledge representation system* [41, 68], and we deserve the term *information table* to the case when U and Att are both finite sets. The important fact to note here is that, in order to prove the completeness of $\mathbb{P}_{ind}(\mathcal{I})$ also in the non-finite case, we cannot use a technique similar to that used in [31], therefore we introduce the isomorphic notion of *maximum partitioner poset* for a knowledge representation system $\mathcal{I}$ and prove it is a complete lattice. More in detail, we identify any indiscernibility partition $\pi \in \Pi_{ind}(\mathcal{I})$ with the greater attribute subset A such that $\pi_{\mathcal{I}}(A) = \pi$, and we call such a subset A the *maximum partitioner* of π. We then introduce the set $MAXP(\mathcal{I}) := \{Max(\pi) : \pi \in \Pi_{ind}(\mathcal{I})\}$ and the poset $\mathbb{M}(\mathcal{I}) := (MAXP(\mathcal{I}), \subseteq^*)$, where $\subseteq^*$ is the dual set inclusion order between attribute subsets. we will show that $\mathbb{M}(\mathcal{I})$ is a complete lattice that is isomorphic to the granular partition poset $\mathbb{P}_{ind}(\mathcal{I})$. As a consequence of this isomorphism, we carry out the structure of complete lattice from $\mathbb{M}(\mathcal{I})$ towards $\mathbb{P}_{ind}(\mathcal{I})$. Given this result, it is clear that the maximum partitioner notion is strictly related to a global view of the knowledge induced by a knowledge representation system and that the introduction of the order structure $\mathbb{M}(\mathcal{I})$ is fundamental for further detailed investigations of the indiscernibility partition lattice $\mathbb{P}_{ind}(\mathcal{I})$.

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Then, we will consider the set $Gran(\mathcal{I}) := \{(Max(\pi), \pi) : \pi \in \Pi_{ind}(\mathcal{I})\}$ and the poset $\mathbb{G}(\mathcal{I}) := (Gran(\mathcal{I}), \subseteq^* \times \preceq)$ and we will see that $\mathbb{G}(\mathcal{I})$ is a complete lattice isomorphic to both the lattices $\mathbb{P}_{ind}(\mathcal{I})$ and $\mathbb{M}(\mathcal{I})$. We call $\mathbb{G}(\mathcal{I})$ the *granular partition lattice* of $\mathcal{I}$. Hence we will obtain three distinct complete lattices that are isomorphic between them: the *indiscernibility partition lattice* $\mathbb{P}_{ind}(\mathcal{I})$, the *maximum partitioner lattice* $\mathbb{M}(\mathcal{I})$ and the *granular partition lattice* $\mathbb{G}(\mathcal{I})$. The advantage of this last lattice is a more complete view of the available knowledge. We can think $\mathbb{P}_{ind}(\mathcal{I})$ as the analogue of the extent lattice, $\mathbb{M}(\mathcal{I})$ as the analogue of the intent lattice and $\mathbb{G}(\mathcal{I})$ as the analogue of the concept lattice for a formal context [21]. More formally, a precise link with Formal Concept Analysis can be established. Indeed, we prove that the granular partition lattice $\mathbb{G}(\mathcal{I})$ is isomorphic to the partition pattern concept lattice (a generalization of a concept lattice) induced by $\mathcal{I}$ [1].

In this paper we also introduce a new link between set partitions and vector subspaces. In fact, we first associate to any set partition π of an arbitrary finite set X a vector subspace of V_π of $\mathbb{R}^{|X|}$. Next we prove a new result in set partition order theory (Theorem 5.6): the usual partial order between set partitions of X is equivalent to the corresponding inclusion relation among the above subspaces. This equivalence could provide new ideas for researches concerning the links among attribute subsets of any information table. In fact, by virtue of the above equivalence, for any attribute subset A we can identify its indiscernibility partition $\pi_{\mathcal{I}}(A)$ with the subspace $V_A := V_{\pi_{\mathcal{I}}(A)}$, and this identification provides an order isomorphism between the lattice $\mathbb{P}_{ind}(\mathcal{I})$ and a corresponding lattice of vector subspaces of $\mathbb{R}^{|U|}$.

In the above discussion, all the partitions were defined according to a given subset of attributes. On the other hand, we also take into account objects through the notion of generalized discernibility matrix [14]. Once fixed a set of objects Z, we will define a *lattice conditioned by* Z, which turns out to be a sub-poset (not necessarily a sub-lattice) of the granular partition lattice.

In order to exemplify the potentiality of these lattice structures, we take into account the cases of entropy and attribute dependency. To be more specific, we introduce the *evaluation dependency map* for a decision table and we prove several properties of this function in relation to positive regions generated by different attribute subsets. Furthermore, we show that both the entropy and the attribute dependency are anti-monotonic with respect to the lattice order of the maximum partitioner lattice $\mathbb{M}(\mathcal{I})$.

It should now be clear that the term *granular partition lattice* has been chosen to highlight the role of $\mathbb{G}(\mathcal{I})$ inside the granular computing paradigm, with explicit links to rough sets, formal concept analysis and potentially to other ways of granulating a universe. We recall that an information granule is a collection of entities arranged together due to their similarity, function of physical adjacency, indistinguishability and so on. The term granular computing, introduced by Zadeh [78, 79], indicates the way to represent and process knowledge in terms of granules and it founds links and applications with rough set theory [33, 34, 72, 40, 62], machine learning [74], interactive computing [54], interval analysis [30], formal concept analysis [21, 22, 29, 60, 69], database theory [23, 24, 48], data mining [25, 36, 37, 38, 26, 75], fuzzy set theory [39, 42, 79], etc.

In this paper, we focused our attention mainly on the finite case. However, the tools we introduce can be easily adapted to the infinite case, i.e. whenever both U and Att are infinite. Moreover, in literature, several papers generalize the tools of rough set theory to the infinite context (see for instance[46, 43, 44, 45, 71, 70]). Therefore, the mathematical analysis of the granular partition lattice turns out to be another good starting point for a formal investigation of the infinite case. For example, Polkowski [45] introduces a new approach to analyze the case in which Att is countable. He defines *chains of indiscernibility relations* that generate a decreasing sequence of partitions of U. Any element of this sequence is then associated to a topology whose closure operator coincides with the upper approximation of a rough set $X \subseteq U$. This enables him to link the notion of fractal dimension to a knowledge representation system. A similar investigation could be undertaken also in our framework by considering the maximal chains of the granular partition lattice. As a further example, we introduced a closure operator $M : \mathcal{P}(Att) \to \mathcal{P}(Att)$, which could be useful to establish a link with the classical theory of *Scott information systems* (see [19, 50]). We plan to follow these links in forthcoming papers.

Furthermore, we highlight the role of the granular partition lattice related to the investigation of several types of discrete mathematical structures. Recently, several researches devoted to the study of the links among paradigms derived from Granular Computing and Rough Set Theory and several types of discrete structures: graphs [7, 9, 11, 12, 13, 15, 20, 49, 56, 57], hypergraphs [6, 8, 58, 59, 60, 61], matroids [27, 28, 63, 65, 66, 80]. Then, in discrete mathematics a useful help for the study of any structure, that is object of investigation, is to associate to it a different type of structure that preserves some properties of the original structure. As it is well known, this is the basic idea of category theory [51]. From this perspective, both the concept lattice of a formal context and the granular partition lattice of an information table, can be considered two useful ways that enable us to associate two different complete

lattices to any discrete mathematical structure that can be described by means of a data table. Let us notice here that it is not so usual to associate to a graph a lattice having good order properties. There are in literature several types of interesting associations between graphs and order structures, but often these structures are only posets and not lattices [64]. Moreover, we note that any node of the granular partition lattice can be identified with an equivalence class $[A]_{\approx}$, where A is an attribute subset of $\mathcal{I}$ and $[A]_{\approx}$ is a *union-closed* family (i.e. a subset family that is closed with respect to the set union), and these families are actually well studied in discrete mathematics [2]. This means that we can see the granular lattice as a *macro-lattice structure* whose nodes are *micro-granules* that are union-closed set families. From a mathematical point of view, we obtain a set partition of the power set $\mathcal{P}(Att)$ whose blocks are union-closed micro-granules that are interrelated among them by means of a complete lattice structure. Therefore we introduce these two different points of view explicitly in Definition 4.5: the notions of *macro* and *micro* granular lattice. In a forthcoming paper we will investigate in more detail the properties of macro and micro granularity of this order structure.
We conclude this introductory part with a content description of the various sections of this paper.
In Section 2 we recall some basic notions concerning posets, set partitions and rough sets.
In Section 3, we introduce the maximum partitioner poset $\mathbb{M}(\mathcal{I})$, show it has a lattice structure also in the infinite case and prove its equivalence with the partition lattice generated by the rough-set indiscernibility relation.
In Section 4, the granular partition lattice is introduced and it is shown to be isomorphic to the indiscernibility partition lattice and equivalent to the pattern concept lattice, an interesting generalization of a formal concept lattice. In this way we can connect the study of indiscernibility partitions in a knowledge representation system to the study of a generalized form of the concept lattice.
In Section 5, we treat the above discussed links between ordering of set partitions and corresponding inclusion relations of vector subspaces.
In Section 6, new lattice structures based on the generalized discernibility relation and conditioned to a set of objects are introduced and studied. They will turn out to be sub-posets and in general not sub-lattices of the granular partition lattice.
In Section 7, entropy and attribute dependency in decision tables are studied in the granular partition lattice (more precisely, in the isomorphic maximum partitioner lattice) and their monotonicity with respect to the lattice order proved.

2. Basics

In this section, we recall some basic notions concerning Rough Set theory (for details we refer the reader to the Pawlak monograph [41]) and some basic notions on partitions.

2.1. Posets

For general references to lattices and order theory we refer to [19, 55], here we briefly recall the definitions used in the paper. A *partially ordered set* (abbreviated *poset*) is a pair $P(X, \leq)$, where X is a set and $\leq$ is a binary relation on X that is reflexive, antisymmetric and transitive. If $P = (X, \leq)$ is a partially ordered set and $x, y \in X$, we also write $x < y$ if $x \leq y$ and $x \neq y$. If x, y are two distinct elements of X, we say that y *covers* x, denoted by $x \lessdot y$ (or, equivalently, by $y \gtrdot x$), if $x \leq y$ and there is no element $z \in X$ such that $x < z < y$. An element $x \in X$ is called *minimal* in P if $z \leq x$ implies $z = x$, and in a similar way one defines a *maximal* element in P. If there is an element $\hat{0} \in X$ such that $\hat{0} \leq x$ then $\hat{0}$ is unique and it is called the *minimum* of P. The *maximum* of P (if there exists) is defined analogously and usually denoted by $\hat{1}$.
A *chain* C of P is a subset $C \subset X$ such that for all $x, y \in C$ we have $x \leq y$ or $y \leq x$. If a chain C has $n+1$ elements $x_0, \ldots, x_n$ such that $x_0 < \cdots < x_n$, it is said that C is an $n+1$-*chain* and has *length* n; in this case often we write $C = \{x_0 < \cdots < x_n\}$. If X itself is a chain the poset P is said *linearly ordered.* If P is not linearly ordered, a *maximal chain* of P is a chain C of P which is not properly contained in any other chain of P. If $C = \{x_0 < \cdots < x_n\}$ is an $n+1$-chain, we say that C is *saturated* if $x_i \lessdot x_{i+1}$ for $i = 0, 1, \ldots, n-1$. Let us note that $C = \{x_0 < \cdots < x_n\}$ is maximal iff C is saturated, x_0 is minimal and x_n is maximal. A subset $A \subset X$ is called an *anti-chain* of P if for all pair of distinct elements $x, y \in A$ we have that $x \nleq y$ and $y \nleq x$. We denote by $\mathcal{A}(P)$ the family of all the anti-chains of P.
A *weighted poset* is a pair (P, w), where $P = (X, \leq)$ is a poset and w is a map (called *weight function* on P) from X into $[0, \infty)$. If $Y \subset X$, the *weight* of Y is the number $w(Y) := \sum_{y \in Y} w(y)$. The number $\xi(P, w) := \max\{w(A) : A \in \mathcal{A}(P)\}$ is called the *width* of (P, w).

A poset $P = (X_1, \leq_1)$ is said *isomorphic* to another poset $P_2 = (X_2, \leq_2)$ if there exists a bijective map $\phi : X_1 \to X_2$ such that $x \leq_1 y \iff \phi(x) \leq_2 \phi(y)$, for all $x, y \in X_1$. The *dual poset* of P is the poset $P^* := (X, \leq^*)$, where $\leq^*$ is the partial order on X defined by $x \leq^* y :\iff y \leq x$, for all $x, y \in X$. A poset P is called *self-dual* if P is isomorphic to its dual poset P^*.
We recall now some basic facts concerning the graded posets (see for example [55], cap.3). A finite poset $P = (X, \leq)$ having a minimum $\hat{0}$ is said *graded* of *rank* l if all the maximal chains in P have length l, in this case the non negative integer l is called *rank* of P and we denote it by $rank(P)$. It can be easily proved that P is a graded poset of rank l iff there exists a unique function $\rho : X \to \mathbb{N}$ (called *rank function* of P) such that $\rho(\hat{0}) = 0$, $\rho(x) = l$ for any maximal element x of P and $\rho(y) = \rho(x) + 1$ if $x, y \in X$ and y covers x. We recall that any finite distributive lattice is also a graded poset (see [55], cap.3).

2.2. Set Partitions

If X is an arbitrary set and π is a set partition on X, we usually denote by $\{B_i : i \in I\}$ the block family of π. If $x \in X$, we denote by $\pi(x)$ the block of π which contains the element x. If $Y \subseteq X$ and $Y \subseteq B_i$, for some index $i \in I$, we say that Y is a *sub-block* of π and we write $Y \preccurlyeq \pi$. When X is finite we use the standard notation $\pi := B_1 | \dots | B_{|\pi|}$, where $|\pi|$ is the number of distinct blocks of π. We denote by $\Pi(X)$ the set of all set-partitions of X. It is well known that on the set $\Pi(X)$ we can consider a partial order $\preceq$ defined as follows: if $\pi, \pi' \in \Pi(X)$, then

$$\pi \preceq \pi' :\iff (\forall B \in \pi)\,(\exists B' \in \pi') \,:\, B \subseteq B' \tag{1}$$

Two useful and equivalent characterizations of the partial order given in (1) are the following:

$$\pi \preceq \pi' \iff (\forall x \in X)\,(\pi(x) \subseteq \pi'(x)) \tag{2}$$

The pair $\mathbb{P}(X) = (\Pi(X), \preceq)$ is a complete lattice which is called *partition lattice* of the set X. We now recall the basic facts about the meet and the join of this lattice.
Let $\pi_1 = A_1 | \dots | A_m$ and $\pi_2 = B_1 | \dots | B_n$ be two partitions on the same finite universe X, i.e., $\pi_1, \pi_2 \in \Pi(X)$, we firstly set

$$\mathcal{S}_{\pi_1, \pi_2} := \{C \subseteq X : \text{ if } x \in C, \text{ then } \pi_1(x) \subseteq C \text{ and } \pi_2(x) \subseteq C\}$$

Then the meet of π_1 and π_2, denoted by $\pi_1 \wedge \pi_2$, is the set partition of X whose blocks are given by

$$\pi_1 \wedge \pi_2 := \{A_i \cap B_j : i = 1, \dots m; j = 1, \dots, n\}. \tag{3}$$

On the other hand, the more simple way to describe the join of π_1 and π_2, denoted by $\pi_1 \vee \pi_2$, is the following:

$$\pi_1 \vee \pi_2 := C_1 | \dots | C_k, \tag{4}$$

where $C_1, \dots, C_k$ are the minimal elements of the set family $\mathcal{S}_{\pi_1, \pi_2}$ with respect to the inclusion.

Example 2.1. Let us consider $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and let $\pi_1 = \{x_1, x_2\}|\{x_3\}|\{x_4, x_5\}|\{x_6\}$ and $\pi_2 = \{x_1, x_3\}|\{x_2\}|\{x_4\}|\{x_5\}|\{x_6\}$ be two set partitions of X. Then $\pi_1 \wedge \pi_2$ is the partition

$$\pi_1 \wedge \pi_2 = \{x_1\}|\{x_2\}|\{x_3\}|\{x_4\}|\{x_5\}|\{x_6\}.$$

The family $\mathcal{S}_{\pi_1, \pi_2}$ is equal to:

$$\mathcal{S}_{\pi_1, \pi_2} = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_6\}, \{x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_6\}\}.$$

Then the meet of π_1 and π_2 is the partition

$$\pi_1 \vee \pi_2 = \{x_1, x_2, x_3\}|\{x_4, x_5\}|\{x_6\}.$$

2.3. Knowledge Representation Systems and Indiscernibility

The following definition is based on [41]

Definition 2.2. *A* knowledge representation system *is a structure* $\mathcal{I} = \langle U, Att, Val, F \rangle$*, where* U *is a nonempty set called* universe set*,* Att *is a non empty set called* attribute set *and* $F : U \times Att \to Val$*, called* information map*, is an application from the direct product* $U \times Att$ *into the* value set Val*. The elements of* U *are called* objects*, the elements of* Att *are called* attributes *and those of* Val *are called* values*. In particular, if* $Val = \{0, 1\}$ *we say that* $\mathcal{I}$ *is a* Boolean knowledge representation system*. When* $U = \{u_1, \dots, u_m\}$ *and* $Att = \{a_1, \dots, a_n\}$ *are both finite sets, we also define the so called* information table $Tab[\mathcal{I}]$ *of* $\mathcal{I}$*, which is a rectangular table* $m \times n$ *whose rows are indexed with all objects in* U*, whose columns are indexed with all attributes in* Att *and which contains the value* $F(u_i, a_j)$ *in the place* (i, j)*.*

When U and Att are both finite sets we can (and we do) identify the knowledge representation system $\mathcal{I}$ with its information table $T[\mathcal{I}]$. In the following, with the symbol $\mathcal{I} = \langle U, Att, Val, F \rangle$ we will denote a knowledge representation system where the sets U and Att have both arbitrary cardinality. On the other hand, when we use the term *information table* we implicitly assume that U and Att are both finite sets.

If $A \subseteq Att$, it is usual to consider the binary relation $\equiv_A$ on the universe set U defined as follows: if $u, u' \in U$ then

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a), \forall a \in A. \tag{5}$$

The binary relation $\equiv_A$ is an equivalence relation on U and it is called A-*indiscernibility relation*. If $u \in U$, we denote by $[u]_A$ the equivalence class of u with respect to $\equiv_A$. We also set

$$\pi_{\mathcal{I}}(A) := \{[u]_A : u \in U\}. \tag{6}$$

If $B \subseteq U$ is such that $B = [u]_A$, for some $u \in U$, we say that B is an A-*granule* of $\mathcal{I}$.

Definition 2.3. *We call* $\pi_{\mathcal{I}}(A)$ *the* A-indiscernibility partition*, or* A-granularity partition*, of* $\mathcal{I}$*.*

The inclusion relation between any two attribute subsets A and B of $\mathcal{I}$ is related to the partial order $\preceq$ between the corresponding indiscernibility partitions as follows:

$$A \subseteq B \Longrightarrow \pi_{\mathcal{I}}(B) \preceq \pi_{\mathcal{I}}(A) \tag{7}$$

Definition 2.4. *We call* partition lattice *of* $\mathcal{I}$ *the complete lattice*

$$\mathbb{P}(\mathcal{I}) := (\Pi(U), \preceq).$$

We denote respectively by $\hat{0}_{\mathcal{I}}$ *and* $\hat{1}_{\mathcal{I}}$ *the minimum and the maximum of the lattice* $\mathbb{P}(\mathcal{I})$*.*

Let us note that $\hat{0}_{\mathcal{I}}$ is the set partition of U such that each single element of U is a block, whereas $\hat{1}_{\mathcal{I}}$ is the set partition of U having the unique block $B = \{U\}$.
We set now

$$\Pi_{ind}(\mathcal{I}) := \{\pi_{\mathcal{I}}(A) : A \subseteq Att\}.$$

Since $\Pi_{ind}(\mathcal{I})$ is a subset of $\Pi(U)$, we can consider on $\Pi_{ind}(\mathcal{I})$ the induced partial order $\preceq$ from the previous partition lattice of $\mathcal{I}$. We set therefore

$$\mathbb{P}_{ind}(\mathcal{I}) := (\Pi_{ind}(\mathcal{I}), \preceq) \tag{8}$$

In this way $\mathbb{P}_{ind}(\mathcal{I})$ becomes a sub-poset of $\mathbb{P}(\mathcal{I})$. According to Yao [76], the order structure $\mathbb{P}_{ind}(\mathcal{I})$ "can be used to develop a partition model of databases".

Remark 2.5. *In [31] it has been proved that the partial order* $\preceq$ *induces a lattice structure on* $\Pi_{ind}(\mathcal{I})$ *when* $\mathcal{I}$ *is an information table. However, also in the finite case,* $\mathbb{P}_{ind}(\mathcal{I})$ *is* not *always a sub-lattice of* $\mathbb{P}(\mathcal{I})$*. In fact, the join of two elements* $\pi_{\mathcal{I}}(A)$ *and* $\pi_{\mathcal{I}}(B)$ *in* $\mathbb{P}_{ind}(\mathcal{I})$ *can be different from their join in* $\mathbb{P}(\mathcal{I})$ *(see [31]).*

Definition 2.6. *We call* indiscernibility partition poset *of the knowledge representation system* $\mathcal{I}$ *the partially ordered set* $\mathbb{P}_{ind}(\mathcal{I})$*. We also denote respectively by* $\hat{1}_{gr\mathcal{I}}$ *and* $\hat{0}_{gr\mathcal{I}}$ *the maximum and the minimum element of this lattice.*

Remark 2.7. *Let us note here that* $\hat{1}_{gr\mathcal{I}}$ *always coincides with the maximum* $\hat{1}_{\mathcal{I}}$ *of the partition lattice* $\Pi(\mathcal{I})$*, since* $\pi_{\mathcal{I}}(\emptyset) = U = \hat{1}_{\mathcal{I}}$*, whereas (in general)* $\hat{0}_{gr\mathcal{I}}$ *can be different with respect to the minimum* $\hat{0}_{\mathcal{I}}$ *of* $\Pi(\mathcal{I})$*.*

2.3.1. Dependency among attributes

The Partition Lattice of a knowledge representation system is strictly linked to some rough set notions, namely, the positive region and the quality of approximations, as we are going to show.

Definition 2.8. *[41] Let A and B be two fixed subsets of Att. The* positive region *of B relatively to A is defined as*

$$POS_A(B) := \{x \in U : \pi_{\mathcal{I}}(A)(x) \subseteq \pi_{\mathcal{I}}(A)(x)\} = \{x \in U : [x]_A \subseteq [x]_B\}. \tag{9}$$

The A-positive region of B can be considered as the set of all objects in the universe set U that can be properly classified by means of blocks of $\pi_{\mathcal{I}}(B)$ with the restriction of using only the knowledge expressed by the granulation $\pi_{\mathcal{I}}(A)$.

We notice that it is possible to characterize the partial order $\preceq$ among indiscernibility partitions in terms of the positive region. In fact, if A and B are any two attribute subsets of $\mathcal{I}$ then

$$\pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(B) \iff POS_A(B) = U \tag{10}$$

Moreover, the following result holds.

Proposition 2.9. *If $A, B, C \subseteq Att$ are three subsets of attributes and $A \subseteq B$ (i.e., $\pi_{\mathcal{I}}(B) \preceq \pi_{\mathcal{I}}(A)$) then $POS_A(C) \subseteq POS_B(C)$.*

Proof. Let $y \in POS_A(C)$, then $\pi_{\mathcal{I}}(A)(y) \subseteq \pi_{\mathcal{I}}(C)(y)$. Since $\pi_{\mathcal{I}}(B) \preceq \pi_{\mathcal{I}}(A)$, by (2) we have that $\pi_{\mathcal{I}}(B)(x) \subseteq \pi_{\mathcal{I}}(A)(x)$ for all $x \in X$, hence also $\pi_{\mathcal{I}}(B)(y) \subseteq \pi_{\mathcal{I}}(A)(y) \subseteq \pi_{\mathcal{I}}(C)(y)$. Therefore $y \in POS_B(C)$. □

When $\mathcal{I}$ is an information table it is also possible to define the *quality of approximation* of B relatively to A:

$$\gamma_A(B) := \frac{|POS_A(B)|}{|U|} \tag{11}$$

Let us note that if $A' \subseteq A$ then

$$POS_{A'}(B) \subseteq POS_A(B), \tag{12}$$

therefore

$$\gamma_{A'}(B) \leq \gamma_A(B). \tag{13}$$

If $k = \gamma_A(B)$, then it results that $0 \leq k \leq 1$ and it is usual to write $A \Rightarrow_k B$. In particular, if $k = 1$, it is said that B *totally depends* on A; if $0 < k < 1$, it is said that B *partially depends* on A, finally, if $k = 0$ it is said that B is *totally independent* from A. In particular, it is usual to write $A \Rightarrow B$ instead of $A \Rightarrow_1 B$. The quality of approximation of B by A is a measure of the representability degree of the knowledge provided by the attribute subset B in terms of the knowledge provided by the other attribute subset A.

For an information table $\mathcal{I}$ we can also refine (10) in the following way:

$$\pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(B) \iff POS_A(B) = U \iff (A \Rightarrow_1 B) \iff \gamma_A(B) = 1 \tag{14}$$

2.3.2. Discernibility Matrix

By negation of the indiscernibility relation we obtain the discernibility relation, at the basis of the so-called *discernibility matrix*.

Definition 2.10. *[53] Let u and u' be two objects of an information table $\mathcal{I}$. The attribute subset defined by*

$$\Delta_{\mathcal{I}}(u, u') := \{a \in Att : F(u, a) \neq F(u', a)\}, \tag{15}$$

is the (u, u')-entry in the discernibility matrix $\Delta[\mathcal{I}]$ of $\mathcal{I}$.

The following result relates the entries of the discernibility matrix to the indiscernibility relation in an information table.

Proposition 2.11. *Let $D \subseteq Att$ and $v, w \in U$. Then:*
(i) $D = \Delta_{\mathcal{I}}(v, w) \Longrightarrow v \equiv_{Att \setminus D} w$;
(ii) $v \equiv_{Att \setminus D} w \Longrightarrow \Delta_{\mathcal{I}}(v, w) \subseteq D$;
(iii) Let $C \subseteq Att$. Then $\Delta_{\mathcal{I}}(v, w) \cap C = \emptyset \iff v \equiv_C w$.

The discernibility matrix is at the basis of the *core* and the *reducts* computation so it has a special role in rough set theory (see [41]). A reduct of an information table $\mathcal{I}$ can be considered as a subset of all attributes of $\mathcal{I}$ sufficient to provide the same knowledge of the whole attribute set. The core of $\mathcal{I}$ is the subset of all attributes of $\mathcal{I}$ whose elimination causes a substantial change in the knowledge induced from $\mathcal{I}$. More formally:

Definition 2.12. *[41] An attribute $c \in Att$ is said* indispensable *if $\pi_{\mathcal{I}}(Att) \neq \pi_{\mathcal{I}}(Att \setminus \{c\})$. The subset of all indispensable attributes of Att is called* core *of $\mathcal{I}$ and it is denoted by $CORE(\mathcal{I})$. A subset $C \subseteq Att$ is said a* reduct *of $\mathcal{I}$ if:*
(i) $\pi_{\mathcal{I}}(Att) = \pi_{\mathcal{I}}(C)$;
(ii) $\pi_{\mathcal{I}}(Att) \neq \pi_{\mathcal{I}}(C \setminus \{c\})$ for all $c \in C$.
We denote by $RED(\mathcal{I})$ the family of all reducts of $\mathcal{I}$.

3. Maximum partitioners of a knowledge representation system

We now introduce an equivalence relation on attributes of a knowledge representation system and show its relationship to the partition lattice through the notion of maximum partitioner.

Definition 3.1. *If A and B are two attribute subsets of $\mathcal{I}$ we set*

$$A \approx B :\Longleftrightarrow \pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(B). \tag{16}$$

Let us note that the binary relation $\approx$ is an equivalence relation on $\mathcal{P}(Att)$. We denote by $\pi_{\approx}(\mathcal{I})$ the set partition on $\mathcal{P}(Att)$ induced by $\approx$, and we also set

$$[A]_{\approx} := \{B \subseteq Att : A \approx B\}. \tag{17}$$

In the next result, we show that each equivalence class with respect to the relation $\approx$ is uniquely determined by means of one of its maximal members[1]. Moreover, we also show that the set inclusion relation among these maximal members completely characterizes the partial order in the indiscernibility partition lattice of $\mathcal{I}$.

Proposition 3.2. *Let $A \subseteq Att$. Then:*
(i) if $\{A_j : j \in J\} \subseteq [A]_{\approx}$, also $\bigcup_{j \in J} A_j \in [A]_{\approx}$.
(ii) There exists a unique subset $M(A) \in [A]_{\approx}$ such that $B \subseteq M(A)$ for all $B \in [A]_{\approx}$ and we have that $M(A) = \bigcup\{B : B \in [A]_{\approx}\}$.
(iii) We have

$$M(A) = \{a \in Att : (u, u' \in U \wedge u \equiv_A u') \Longrightarrow F(u, a) = F(u', a)\}, \tag{18}$$

that is equivalent to

$$M(A) = \{a \in Att : (u, u' \in U \wedge a \in \Delta(u, u')) \Longrightarrow u \not\equiv_A u'\}. \tag{19}$$

(iv) If $A' \subseteq Att$ then

$$\pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(A') \Longleftrightarrow M(A') \subseteq M(A) \tag{20}$$

and

$$\pi_{\mathcal{I}}(A) \prec \pi_{\mathcal{I}}(A') \Longleftrightarrow M(A') \subsetneq M(A). \tag{21}$$

(v) If $B \subseteq Att$ then $A \approx B \Longleftrightarrow M(A) = M(B)$.
(vi) $A \approx Att \Longleftrightarrow M(A) = Att$.
(vii) Let $A' \subseteq Att$ such that $A \subseteq A'$. Then $M(A) \subseteq M(A')$.

Proof. (i) : Let $u, u' \in U$ such that $u \equiv_A u'$ and let $\{A_j : j \in J\} \subseteq [A]_{\approx}$. Then $u \equiv_{A_j} u'$ for all $j \in J$ by definition of the relation $\approx$. If $z \in \bigcup_{j \in J} A_j$ there exists some index $j \in J$ such that $z \in A_j$, so that $F(u, z) = F(u', z)$ because $u \equiv_{A_j} u'$. Hence $u \equiv_{\bigcup_{j \in J} A_j} u'$. This implies that $\pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(\bigcup_{j \in J} A_j)$. On the other hand, by (7) we have that $\pi_{\mathcal{I}}(\bigcup_{j \in J} A_j) \preceq \pi_{\mathcal{I}}(A_j) = \pi_{\mathcal{I}}(A)$ because $A_j \approx A$. Since $\preceq$ is a partial order on $\Pi_{ind}(\mathcal{I})$ we deduce that $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(\bigcup_{j \in J} A_j)$, that is equivalent to the condition $\bigcup_{j \in J} A_j \in [A]_{\approx}$.

[1]The results given in propositions 3.2 and 3.4 also appear in [14]. For the sake of completeness, the entire proofs are reported also here.

(ii) : Let $M(A) := \bigcup\{B : B \in [A]_{\approx}\}$. By part (i) it follows that $M(A) \in [A]_{\approx}$, moreover we also have that $B \subseteq M(A)$ for all $B \in [A]_{\approx}$. Uniqueness is obvious.
(iii) : Let $B := \{b \in Att : (u, u' \in U \wedge u \equiv_A u') \Longrightarrow F(u, b) = F(u', b)\}$. We show that $B \approx A$. In fact, let $u, u' \in U$ such that $u \equiv_A u'$ and let $b \in B$. By definition of B we have that $F(u, b) = F(u', b)$, so that $u \equiv_B u'$. Hence $\pi_{\mathfrak{I}}(A) \preceq \pi_{\mathfrak{I}}(B)$. Let now $C \in [A]_{\approx}$ and $c \in C$. Then for all $u, u' \in U$ such that $u \equiv_A u'$, by definition of $\approx$ we have that $u \equiv_C u'$, so that $F(u, c) = F(u', c)$. It follows that $C \subseteq B$ for all $C \in [A]_{\approx}$, in particular $A \subseteq M(A) \subseteq B$. By (7) we obtain then $\pi_{\mathfrak{I}}(B) \preceq \pi_{\mathfrak{I}}(A)$. Therefore $\pi_{\mathfrak{I}}(B) = \pi_{\mathfrak{I}}(A)$, i.e. $B \in [A]_{\approx}$. By part (ii) we deduce then that $B \subseteq M(A)$. Hence $B = M(A)$.
(iv) : If A and A' are two attribute subsets of $\mathfrak{I}$ such that $M(A') \subseteq M(A)$ then $\pi_{\mathfrak{I}}(M(A)) \preceq \pi_{\mathfrak{I}}(M(A'))$ by virtue of (7), which is equivalent to $\pi_{\mathfrak{I}}(A) \preceq \pi_{\mathfrak{I}}(A')$ by virtue of part (ii). We assume now that $\pi_{\mathfrak{I}}(M(A)) \preceq \pi_{\mathfrak{I}}(M(A'))$. Let $v \in M(A')$ and let us suppose by contradiction that $v \notin M(A)$, so that $M(A) \subsetneq M(a) \cup \{v\}$. By (7) we obtain then $\pi_{\mathfrak{I}}(M_A \cup \{v\}) \preceq \pi_{\mathfrak{I}}(M(A))$. On the other hand, by maximality of $M(A)$ proved in part (ii) we also deduce that $M(A) \not\approx M(A) \cup \{v\}$, i.e. $\pi_{\mathfrak{I}}(M(A)) \neq \pi_{\mathfrak{I}}(M(A) \cup \{v\})$. Then there exist two objects $u, u' \in U$ such that $u \equiv_{M(A)} u'$ and $u \not\equiv_{M(A) \cup \{v\}} u'$, and this is possible only if $F(u, v) \neq F(u', v)$. Now, since $\pi_{\mathfrak{I}}(M(A)) \preceq \pi_{\mathfrak{I}}(M(A'))$ and $u \equiv_{M(A)} u'$, we obtain $u \equiv_{M(A')} u'$, therefore $F(u, v) = F(u', v)$ because $v \in M(A')$. This shows the contradiction and conclude the proof of (20). Finally, (21) follows immediately by (20) and part (ii).
(v) : If $A \approx B$ by the previous part (ii) it follows that $M(A) = M(B)$. On the other hand, let $M(A) = M(B)$. Again by (ii) we have that $A \approx M(A)$ and $B \approx M(B)$, therefore $A \approx B$ by virtue of the transitive property of the equivalence relation $\approx$.
(vi) : The result is a direct consequence of previous part (v) because $Att = M(Att)$.
(vii) : Let $a \in M(A)$ and let u, $u' \in U$ such that $u \equiv_{A'} u'$. Since $A \subseteq A'$ it holds $u \equiv_A u'$. By part (iii), since $a \in M(A)$, $F(u, a) = F(u', a)$. Thus $a \in M(A')$ and the thesis holds. □

We introduce now an appropriate terminology for the attribute subsets $M(A)$, described in Proposition 3.2.

Definition 3.3. • *If $A \subseteq Att$, we call the attribute subset $M(A)$ the* co-maximal *of A in $\mathfrak{I}$.*

- *If $\pi \in \Pi_{ind}(\mathfrak{I})$ is such that $\pi = \pi_{\mathfrak{I}}(A)$, for some $A \subseteq Att$, we say that $M(A)$ is the* maximum partitioner *of π, and we set $M(A) := Max(\pi)$. Therefore, with this notation, we have that $M(A) = Max(\pi_{\mathfrak{I}}(A))$.*

- *We set*
$$MAXP(\mathfrak{I}) := \{M(A) : A \subseteq Att\} = \{Max(\pi) : \pi \in \Pi_{ind}(\mathfrak{I})\}, \tag{22}$$
and we also introduce the following poset
$$\mathbb{M}(\mathfrak{I}) := (MAXP(\mathfrak{I}), \subseteq^*), \tag{23}$$
where $\subseteq^*$ is the dual inclusion order.

- *We call* maximum partitioners *of $\mathfrak{I}$ the elements of $MAXP(\mathfrak{I})$.*

The following characterization of $MAXP(\mathfrak{I})$ is simple but useful.

Proposition 3.4. *$MAXP(\mathfrak{I}) = \{B \subseteq Att : B = M(B)\}$.*

Proof. If $B \in MAXP(\mathfrak{I})$ there exists $A \subseteq Att$ such that $B = M(A)$. By (ii) of Proposition 3.2 it follows that $B \in [A]_{\approx}$, therefore $B = M(A) = M(B)$ by unicity of the maximum partitioner. On the other hand, if $B = M(B)$ then $B \in MAXP(\mathfrak{I})$ by definition of $MAXP(\mathfrak{I})$. □

We show now that the posets $\mathbb{M}(\mathfrak{I})$ and $\mathbb{P}_{ind}(\mathfrak{I})$ are isomorphic.

Theorem 3.5. *The poset $\mathbb{M}(\mathfrak{I})$ is isomorphic to the indiscernibility partition poset $\mathbb{P}_{ind}(\mathfrak{I})$.*

Proof. The map $f : \Pi_{ind}(\mathfrak{I}) \longrightarrow MAXP(\mathfrak{I})$ given by $f(\pi) := Max(\pi)$ is clearly bijective. Let us prove that f is an isomorphism between posets. If A and A' are two subsets of Att such that $M(A') \subseteq M(A)$ then $\pi_{\mathfrak{I}}(M(A)) \preceq \pi_{\mathfrak{I}}(M(A'))$ by virtue of (7), which is equivalent to $\pi_{\mathfrak{I}}(A) \preceq \pi_{\mathfrak{I}}(A')$. We assume now that $\pi_{\mathfrak{I}}(M(A)) \preceq \pi_{\mathfrak{I}}(M(A'))$. Let $v \in M(A')$ and let us suppose by contradiction that $v \notin M(A)$, so that $M(A) \subsetneq M(a) \cup \{v\}$. By (7) we obtain then $\pi_{\mathfrak{I}}(M_A \cup \{v\}) \preceq \pi_{\mathfrak{I}}(M(A))$. On the other hand, by maximality of $M(A)$, $M(A) \not\approx M(A) \cup \{v\}$, i.e. $\pi_{\mathfrak{I}}(M(A)) \neq \pi_{\mathfrak{I}}(M(A) \cup \{v\})$. Then there exist two elements $u, u' \in U$ such that $u \equiv_{M(A)} u'$ and $u \not\equiv_{M(A) \cup \{v\}} u'$, and this is possible only if $F(u, v) \neq F(u', v)$. Now, since $\pi_{\mathfrak{I}}(M(A)) \preceq \pi_{\mathfrak{I}}(M(A'))$ and $u \equiv_{M(A)} u'$, we obtain $u \equiv_{M(A')} u'$, therefore $F(u, v) = F(u', v)$ since $v \in M(A')$. This shows the contradiction and conclude the proof of the theorem. □

Let us prove now that $\mathbb{M}(\mathfrak{I})$ is a complete lattice.

Theorem 3.6. *The poset* $\mathbb{M}(\mathfrak{I})$ *is a complete lattice.*

Proof. Let $\{A_j : j \in J\}$ be a family of maximum partitioners of $\mathfrak{I}$. We first show that $\bigcap_{j\in J} A_j$ is a maximum partitioner of $\mathfrak{I}$, that is $M(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} A_j$.
Since $\bigcap_{j\in J} A_j \subseteq M(\bigcap_{j\in J} A_j)$, in order to prove the thesis we only must show the reverse inclusion. Let us assume by contradiction that there exists an element $c \in M(\bigcap_{j\in J} A_j)$ such that $c \notin \bigcap_{j\in J} A_j$. So that there exists some index $j \in J$ such that $c \notin A_j$. Since $A_j = M(A_j)$, from the condition $c \notin M(A_j)$ and by (18) we deduce that there exist two objects $u, u' \in U$ such that $u \equiv_{A_j} u'$ and $F_{\mathfrak{I}}(u, c) \neq F_{\mathfrak{I}}(u', c)$. Then, since $u \equiv_{A_j} u'$, we also have $u \equiv_{\bigcap_{j\in J} A_j} u'$, that is equivalent to $u \equiv_{M(\bigcap_{j\in J} A_j)} u'$, and since by hypothesis $c \in M(\bigcap_{j\in J} A_j)$, we deduce that $F_{\mathfrak{I}}(u, c) = F_{\mathfrak{I}}(u', c)$. This contradiction proves the assertion.
Therefore, since $A_j \subseteq^* \bigcap_{j\in J} A_j \subseteq A_j$ for all $j \in J$ we have that A is an upper bound of $\{A_j : j \in J\}$. Let now $C \subseteq \Lambda_{\mathfrak{I}}$ such that $A_j \subseteq^* C$, for all $j \in J$. Then obviously $\bigcap_{j\in J} A_j \subseteq^* C$, so C is the join of $\{A_j : j \in J\}$.
Let us prove now that the meet of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathfrak{I})$ is equal to $M(\bigcup_{j\in J} A_j)$. Let $\{A_j : j \in J\}$ be a family of maximum partitioners of $\mathfrak{I}$ as before. Since $A_j \subseteq \bigcup_{j\in J} A_j$, by (vii) of Proposition 3.2, $A_j = M(A_j) \subseteq M(\bigcup_{j\in J} A_j)$, so $B := M(\bigcup_{j\in J} A_j)$ is a lower bound of the family in $\mathbb{M}(\mathfrak{I})$. Let now $C \in MAXP(\mathfrak{I})$ be such that, for all $j \in J$, $A_j \subseteq C$. Thus $\bigcup_{j\in J} A_j \subseteq C$ and again by (vii) of Proposition 3.2, $B = M(\bigcup_{j\in J} A_j) \subseteq M(C) = C$. This proves that the meet of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathfrak{I})$ does exist and it is equal to B. □

Corollary 3.7. *Let* $\{A_j : j \in J\} \subseteq MAXP(\mathfrak{I})$ *be a family of maximum partitioners of* $\mathfrak{I}$*. Then:*

1. *the join of* $\{A_j : j \in J\}$ *in* $\mathbb{M}(\mathfrak{I})$ *is* $A := \bigcap_{j\in J} A_j$.
2. *the meet of* $\{A_j : j \in J\}$ *in* $\mathbb{M}(\mathfrak{I})$ *is* $B := M(\bigcup_{j\in J} A_j)$.

Definition 3.8. *We call* maximum partitioner lattice *of* $\mathfrak{I}$ *the lattice* $\mathbb{M}(\mathfrak{I})$.

At this point, from the previous results we can deduce that $\mathbb{P}_{ind}(\mathfrak{I})$ is a complete lattice also in the case U and Att are not necessarily finite sets.

Corollary 3.9. $\mathbb{P}_{ind}(\mathfrak{I})$ *is a complete lattice that is isomorphic to the maximum partitioner lattice* $\mathbb{M}(\mathfrak{I})$.

Proof. The proof follows directly by Theorem 3.5 and Theorem 3.6, since a poset isomorphism between a complete lattice and a poset induces a natural complete lattice structure on the poset such that the poset isomorphism is a complete lattice isomorphism. □

Corollary 3.10. *The join and the meet in the lattice* $\mathbb{P}_{ind}(\mathfrak{I})$ *are obtained as follows. Let* $\{\pi_j : j \in J\} \subseteq \Pi_{ind}(\mathfrak{I})$ *and set* $A_j := f(\pi_j) = Max(\pi_j)$*, for all* $j \in J$*. Then:*

1. *the join of* $\{\pi_j : j \in J\}$ *in* $\mathbb{P}_{ind}(\mathfrak{I})$ *is the partition* $\pi_{\mathfrak{I}}(A)$*, where* $A := \bigcap_{j\in J} A_j$.
2. *the meet of* $\{\pi_j : j \in J\}$ *in* $\mathbb{P}_{ind}(\mathfrak{I})$ *is the partition* $\pi_{\mathfrak{I}}(B)$*, where* $B := M(\bigcup_{j\in J} A_j)$.

Definition 3.11. *We call* indiscernibility partition lattice *of* $\mathfrak{I}$ *the lattice* $\mathbb{P}_{ind}(\mathfrak{I})$.

Remark 3.12. *Let us note that in general the union of a family of maximum partitioner is not a maximum partitioner, as we illustrate in the next example.*

Example 3.13. Let us consider the underlying information table $\mathfrak{I}$ having universe set $U = \{u_1, u_2, u_3, u_4, u_5\}$, attribute set $Att = \{1, 2, 3, 4\}$:

	1	2	3	4
u_1	0	1	1	0
u_2	1	0	0	0
u_3	1	0	1	1
u_4	1	1	1	0
u_5	0	1	1	1

Then the diagram of the lattice $\mathbb{M}(\mathfrak{I})$ is drawn in Figure 2.

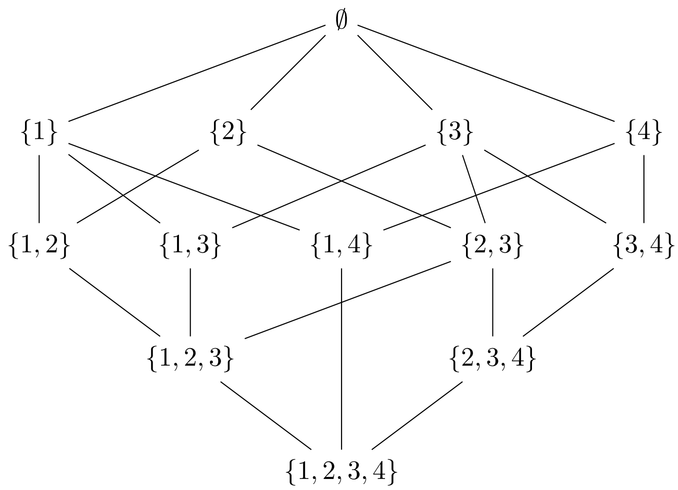


Figure 1: The lattice $\mathbb{M}(\mathfrak{I})$

The union of the two maximum partitioner $A_1 = \{1, 2\}$ and $A_2 = \{1, 4\}$ is the attribute set $\{1, 2, 4\}$ which is not a maximum partitioner. The meet of A_1 and A_2 in $\mathbb{M}(\mathfrak{I})$ is the attribute set $M(A_1 \cup A_2)$. This holds in general (see Theorem 3.6). Moreover we can observe also that the lattice in Figure 2 is not graded since the two chains $\{\emptyset, \{1\}, \{1,4\}, \{1,2,3,4\}\}$ and $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$ are both maximal chains, but they have different length.

4. Granular Partition Lattice

In order to obtain a more explicit analogy of the indiscernibility partition lattice of a knowledge representation system with the concept lattice of a formal context [21], we also introduce the following set:

$$Gran(\mathfrak{I}) := \{(M(A), \pi_{\mathfrak{I}}(A)) : A \subseteq Att\} = \{(Max(\pi), \pi) : \pi \in \Pi_{ind}(\mathfrak{I})\}. \tag{24}$$

On the previous set we consider the following order structure:

$$\mathbb{G}(\mathfrak{I}) := (Gran(\mathfrak{I}), \subseteq^* \times \preceq) \tag{25}$$

where $\subseteq^*$ is the order in $\mathbb{G}(\mathfrak{I})$ and $\subseteq^* \times \preceq$ is the direct product order of $\subseteq^*$ and $\preceq$.
We obtain then the following result.

Theorem 4.1. *The poset $\mathbb{G}(\mathfrak{I})$ is a lattice and it is isomorphic to both the lattices $\mathbb{M}(\mathfrak{I})$ and $\mathbb{P}_{ind}(\mathfrak{I})$.*

Proof. By the previous Proposition 3.2 (*ii*), we can identify each granular partition $\pi \in \Pi_{ind}(\mathfrak{I})$ with the pair $(Max(\pi), \pi)$, or simply with the attribute subset $Max(\pi)$. Moreover, again by Proposition 3.2 (*iv*), the partial order $\preceq$ becomes equivalent to the following two partial orders $\subseteq^* \times \preceq$ and $\subseteq^*$:

$$(Max(\pi), \pi) \preceq' (Max(\pi'), \pi') \;:\Longleftrightarrow\; Max(\pi) \subseteq^* Max(\pi') \;\wedge\; \pi \preceq \pi' \tag{26}$$

and

$$Max(\pi) \subseteq^* Max(\pi') :\Longleftrightarrow Max(\pi) \supseteq Max(\pi') \tag{27}$$

This is sufficient to conclude the proof. □

Definition 4.2. *We call $\mathbb{G}(\mathfrak{I})$ the* granular partition lattice *of $\mathfrak{I}$.*

Remark 4.3. *It is convenient to use the lattice $\mathbb{G}(\mathfrak{I})$ when we need to simultaneously represent both the lattices $\mathbb{P}_{ind}(\mathfrak{I})$ and $\mathbb{M}(\mathfrak{I})$. However, by virtue of Theorem 3.9 and Theorem 4.1 we can use interchangeably the previous lattices $\mathbb{P}_{ind}(\mathfrak{I})$, $\mathbb{M}(\mathfrak{I})$ and $\mathbb{G}(\mathfrak{I})$.*

Example 4.4. If we consider the information table given in Example 3.13, the diagram of the granular lattice $\mathbb{G}(\mathfrak{I})$ is the following:

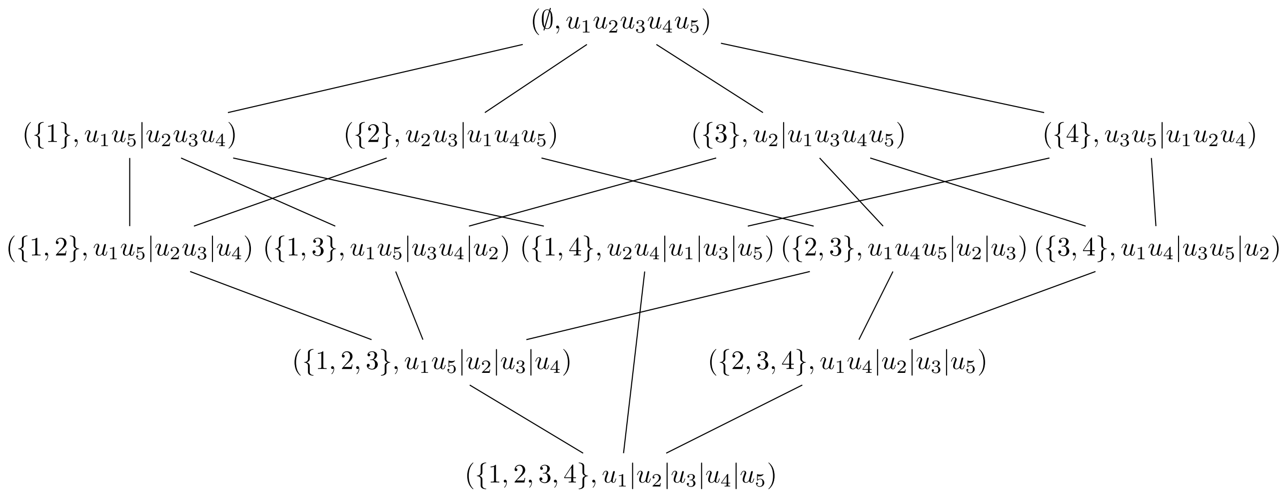


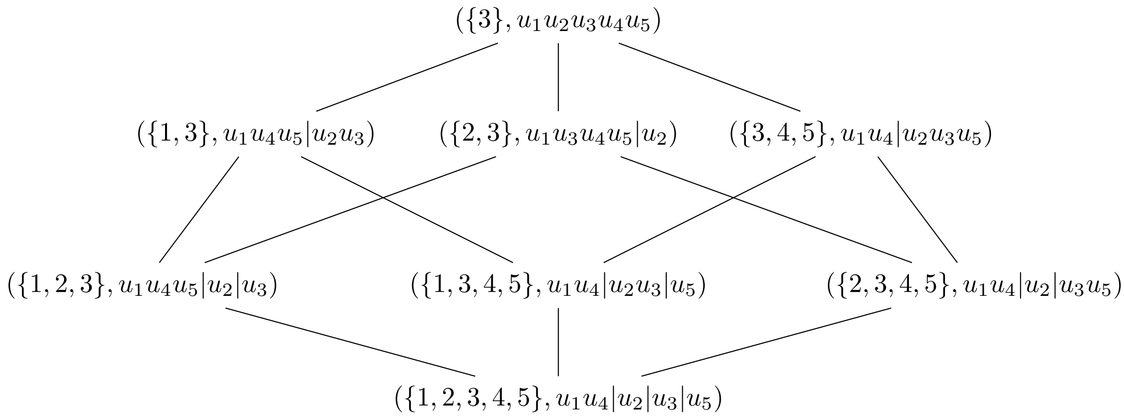
Figure 2: The Lattice $\mathbb{G}(\mathcal{I})$

Definition 4.5. *We call* macro-granular representation *of* $\mathbb{G}(\mathcal{I})$ *the usual Hasse diagram of the lattice* $\mathbb{G}(\mathcal{I})$. *We call* micro-granular representation *of* $\mathbb{G}(\mathcal{I})$ *the Hasse diagram of* $\mathbb{G}(\mathcal{I})$, *where any node is represented by the ordered pair* $([B]_{\approx}, \pi_{\mathcal{I}}(B))$, *for* $B \in MAXP(\mathcal{I})$. *A similar definition is given also for* $\mathbb{M}(\mathcal{I})$, *and in this case we represent simply* B *or* $[B]_{\approx}$ *without the corresponding set partition* $\pi_{\mathcal{I}}(B)$.

Example 4.6. Let us consider the following information table $\mathcal{I}$:

	1	2	3	4	5
u_1	1	0	0	1	0
u_2	0	1	0	0	1
u_3	0	0	0	0	1
u_4	1	0	0	1	0
u_5	1	0	0	0	1

Then the macro-granular representation of $\mathbb{G}(\mathcal{I})$ is given by



On the other hand, the micro-granular representation of $\mathbb{M}(\mathcal{I})$ is the drawn in Figure 3 (we have ordered any micro-granule by means of the usual inclusion relation; let us also notice that any micro-granule is a union-closed family of sets by part (i) of Proposition 3.2).

4.1. Indiscernibility Partition Lattice and Pattern Structures

A pattern structure can be considered a generalization of the formal context notion. For more details concerning the pattern structures and related arguments we refer the reader to [1], here we briefly recall its definition.

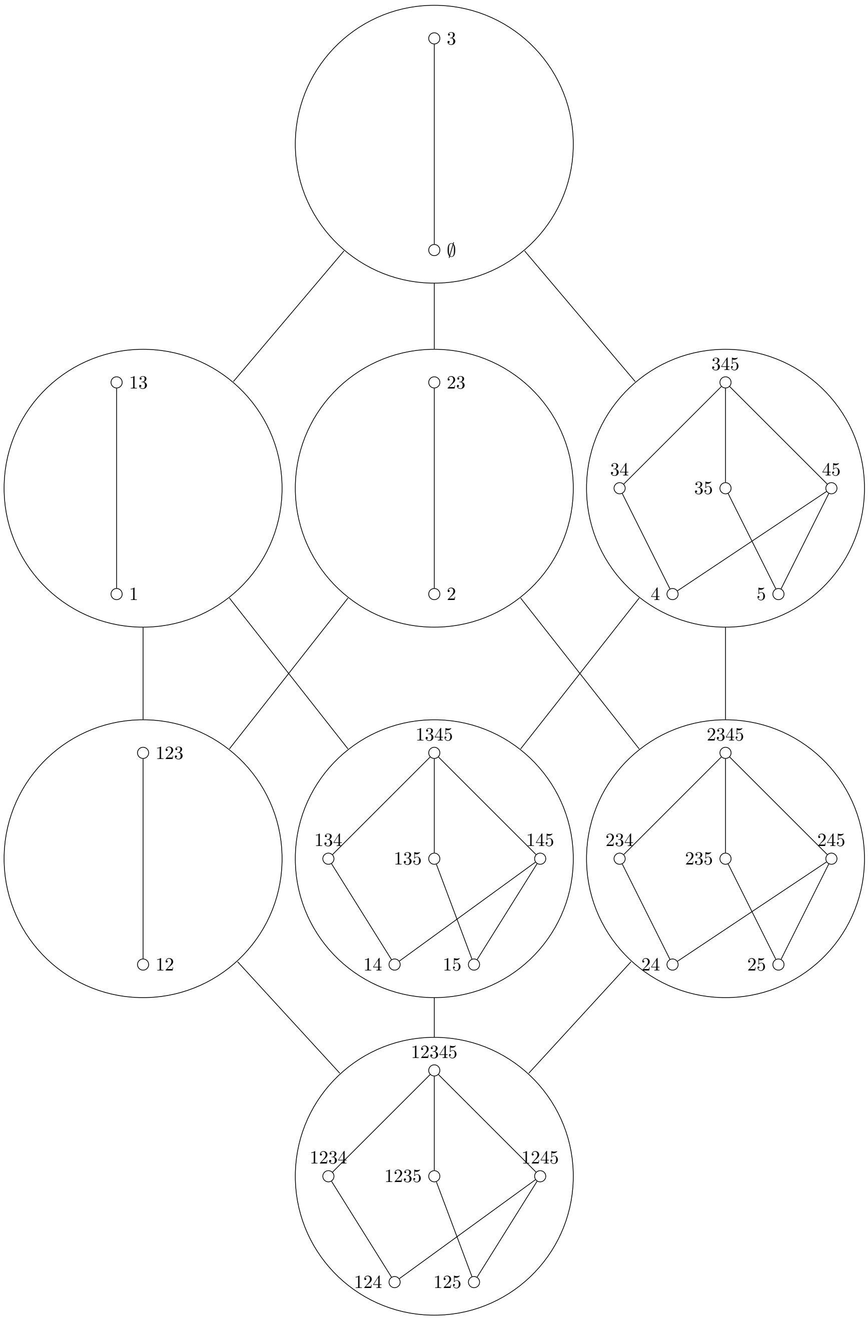


Figure 3: Micro granular representation of $\mathbb{M}(\mathfrak{I})$.

Definition 4.7. *A* pattern structure *is a triple* $\mathcal{S} = (G, (D, \sqcap), \delta)$*, where* G *is a subset whose elements are called* objects*,* $(D, \sqsubseteq)$ *is a meet-semilattice whose inf-operation is denoted by* $\sqcap$ *and* $\delta : G \to D$ *is a function called* description map*. The elements of* D *are called* patterns*. If* $g \in G$*, the pattern* $\delta(g)$ *is called the* description *of the object* g *in* $(D, \sqsubseteq)$*.*

If $\mathcal{S} = (G, (D, \sqcap), \delta)$ is a pattern structure, one can consider two operators:

$$A \in \mathcal{P}(G) \mapsto A^{\square} := \bigsqcap_{g \in A} \delta(g) \in D \tag{28}$$

and

$$d \in D \mapsto d^{\square} := \{g \in G : d \sqsubseteq \delta(g)\} \in \mathcal{P}(G) \tag{29}$$

These operators form a Galois connection between the partially ordered sets $(\mathcal{P}(G), \subseteq)$ and $(D, \sqsubseteq)$.

Definition 4.8. *Let* $\mathcal{S} = (G, (D, \sqcap), \delta)$ *be a pattern structure. A* pattern concept *of* $\mathcal{S}$ *is an ordered pair* $(A, d) \in \mathcal{P}(G) \times D$ *such that* $A^{\square} = d$ *and* $d^{\square} = A$*.*

We denote by $PCO(\mathcal{S})$ the set of all pattern concepts of $\mathcal{S}$. In [1], the following partial order $\leqslant$ on $PCO(\mathcal{S})$ is considered . If (A_1, d_1) and (A_2, d_2) are two pattern concepts of $\mathcal{S}$ then

$$(A_1, d_1) \leqslant (A_2, d_2) :\iff A_1 \subseteq A_2 (\iff d_2 \sqsubseteq d_1) \tag{30}$$

It results then (see [1]) that the partially ordered set

$$\mathbb{P}_{co}(\mathcal{S}) := (PCO(\mathcal{S}), \leqslant)$$

is a complete lattice, which is called *pattern concept lattice* of the pattern structure $\mathcal{S}$.
In [6], we associated a pattern structure to any information table in a very natural way.

Definition 4.9. *Let* $\mathcal{I} = \langle U, Att, F, Val \rangle$ *be an information table. We denote by* $\mathcal{I}_{ps}$ *the pattern structure* $(G, (D, \sqsubseteq), \delta)$ *such that* $G := Att$*,* $(D, \sqsubseteq) := \mathbb{P}(\mathcal{I})$ *(that is,* $D = \Pi(U)$ *and* $\sqsubseteq = \preceq$*) and* $\delta(a) := \pi_{\mathcal{I}}(\{a\})$ *for all* $a \in Att$*. The pattern structure* $\mathcal{I}_{ps}$ *is called* partition pattern structure *of* $\mathcal{I}$*.*

In the next result we show that the dual lattice $\mathbb{P}_{co}(\mathcal{I}_{ps})^*$ of $\mathbb{P}_{co}(\mathcal{I}_{ps})$ coincides exactly with the lattice $\mathbb{G}(\mathcal{I})$ introduced in (25).

Theorem 4.10. $\mathbb{P}_{co}(\mathcal{I}_{ps})^* = \mathbb{G}(\mathcal{I})$.

Proof. By definition of the pattern structure $\mathcal{I}_{ps}$ we have that

$$PCO(\mathcal{I}_{ps}) = \{(A, \pi) : A \subseteq Att, \pi \in \Pi(U), A^{\square} = \pi \text{ and } \pi^{\square} = A\}. \tag{31}$$

We will prove that

$$PCO(\mathcal{I}_{ps}) = Gran(\mathcal{I}). \tag{32}$$

Let $A \subseteq Att$. Since $A = \bigcup_{a \in A} \{a\}$, by Corollary 3.10 we have that $\bigwedge_{a \in A} \pi_{\mathcal{I}}(\{a\}) = \pi_{\mathcal{I}}(A)$, therefore by (28) and by definition of the description map in $\mathcal{I}_{ps}$ we obtain

$$A^{\square} = \bigwedge_{a \in A} \delta(a) = \bigwedge_{a \in A} \pi_{\mathcal{I}}(\{a\}) = \pi_{\mathcal{I}}(A). \tag{33}$$

Analogously, if $\pi \in \Pi(U)$, by (29) and by definition of description map in $\mathcal{I}_{ps}$ we obtain

$$\pi^{\square} = \{a \in Att : \pi \preceq \pi_{\mathcal{I}}(\{a\})\}. \tag{34}$$

Now, by (34) we deduce that

$$\pi_{\mathcal{I}}(A)^{\square} = \{a \in Att : \pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(\{a\})\} = \{a \in Att : (u, u' \in U \wedge u \equiv_A u') \Longrightarrow u \equiv_{\{a\}} u'\},$$

i.e.

$$\pi_{\mathcal{I}}(A)^{\square} = \{a \in Att : (u, u' \in U \wedge u \equiv_A u') \Longrightarrow F(u, a) = F(u', a)\},$$

therefore by (18) we deduce that

$$\pi_{\mathcal{I}}(A)^{\square} = M(A). \tag{35}$$

At this point, by (24), (31), (33) and (35) we obtain the identity (32) :

$$PCO(\mathcal{I}_{ps}) = \{(M(A), \pi_{\mathcal{I}}(A)) : A \subseteq Att\} = Gran(\mathcal{I}).$$

Finally, if $(M(A), \pi_{\mathcal{I}}(A))$ and $(M(A'), \pi_{\mathcal{I}}(A')$ are two pattern concepts of $PCO(\mathcal{I}_{ps})$, by definition of dual order and by (30) we deduce that

$$(M(A), \pi_{\mathcal{I}}(A)) \leqslant^* (M(A'), \pi_{\mathcal{I}}(A')) :\Longleftrightarrow (M(A), \pi_{\mathcal{I}}(A)) \subseteq^* \times \preceq (M(A'), \pi_{\mathcal{I}}(A')). \tag{36}$$

Hence, by (25), (32) and (36) we obtain the thesis:

$$\mathbb{P}_{co}(\mathcal{I}_{ps})^* := (PCO(\mathcal{I}_{ps}), \leqslant^*) = (Gran(\mathcal{I}), \subseteq^* \times \preceq) := \mathbb{G}(\mathcal{I}).$$

□

By the previous Theorem we also immediately deduce the following result.

Corollary 4.11. *[6] The indiscernibility partition lattice $\mathbb{P}_{ind}(\mathcal{I})$ is order-isomorphic to $\mathbb{P}_{co}(\mathcal{I}_{ps})^*$.*

5. The Partition Lattice as a Vector Subspace Lattice

Let X be a finite set having m elements $x_1, \dots, x_m$. In this section we show that the study of the partition lattice $\mathbb{P}(X)$ is equivalent to the study of a lattice of vector subspaces of $\mathbb{R}^m$. We begin with the following definition.

Definition 5.1. *Let $\pi = B_1|\cdots|B_N$ be a set partition of X. We call* relation matrix *of π the symmetric $m \times m$ matrix $R_\pi := (R_{ij})$ such that*

$$R_{ij} := \begin{cases} 1 & \text{if } \exists k \in \{1, \dots, N\} \text{ such that } \{x_i, x_j\} \subseteq B_k \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

We denote by V_π the vector subspace of $\mathbb{R}^m$ generated by the rows (or, equivalently, by the columns) of the relation matrix R_π.

Definition 5.2. *Let $\pi = B_1|\cdots|B_N$ be a set partition of X. We denote by $E_k(\pi)$ the m-tuple $(\epsilon_{k1}, \dots, \epsilon_{km})$, where $\epsilon_{ki} := 1$ if $x_i \in B_k$ and $\epsilon_{ki} := 0$ otherwise, for $k = 1, \dots, N$.*

Proposition 5.3. *If $\pi = B_1|\cdots|B_N \in \Pi(X)$ then $\mathcal{B}_\pi := \{E_1(\pi), \dots, E_N(\pi)\}$ is a basis of V_π. Hence* $\dim(V_\pi) = |\pi| = N$.

Proof. The N vectors $E_1(\pi), \dots, E_N(\pi)$ are linearly independent because $B_1, \dots, B_N$ are all disjoint subsets, moreover they are exactly the distinct rows of the matrix R_π. Hence $\mathcal{B}_\pi$ is a basis of V_π. □

Example 5.4. Let $X = \{1, 2, 3, 4, 5\}$ and $\pi = 13|25|4$, where $B_1 = \{1, 3\}$, $B_2 = \{2, 5\}$ and $B_3 = \{4\}$. Then

$$R_\pi = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and $E_1(\pi) = (1, 0, 1, 0, 0)$, $E_2(\pi) = (0, 1, 0, 0, 1)$, $E_3(\pi) = (0, 0, 0, 1, 0)$. So that

$$V_\pi = span\{(1, 0, 1, 0, 0), (0, 1, 0, 0, 1), (0, 0, 0, 1, 0)\} = \{(a, b, a, c, b) : a, b, c \in \mathbb{R}\}.$$

Let us note that the structure of any vector subspace V_π depends only from the number $m = |X|$ and from the implicit order $x_1, \dots, x_m$ of the elements of X. Therefore we can identify the set $\{V_\pi : \pi \in \Pi(X)\}$ with the following

$$S[m] := \{V_\pi : \pi \in \Pi(\{1, \dots, m\})\}. \tag{38}$$

In general, if W_1 and W_2 are two vector subspaces of a vector space W, we write $W_1 \subseteq_{ssv} W_2$ if W_1 is a vector subspace of W_2. If W is a vector space on any field K, we denote by $SS(W)$ the family of all K-vector subspaces of W. Then the binary relation $\subseteq_{ssv}$ is a partial order on $SS(W)$, and we set $\mathbb{SS}(W) := (SS(W), \subseteq_{ssv})$. The following result is standard in vector space theory.

Theorem 5.5. *If W is a vector space on any field K then $\mathbb{SS}(W)$ is a complete lattice.*

Proof. Let us note first that, if W_1 and W_2 are two vector subspaces of W, then $W_1 \subseteq_{ssv} W_2$ if and only if $W_1 \subseteq W_2$. If $\{W'_j : j \in J\} \subseteq SS(W)$ then $\bigwedge_{j\in J} W'_j := \bigcap_{j\in J} W'_j$ and $\bigvee_{j\in J} W'_j := span\{\bigcup_{j\in J} W'_j\}$ are respectively the meet and the join in $\mathbb{SS}(W)$. In fact $\bigcap_{j\in J} W'_j$ is clearly the biggest subset of W contained in all the W_j's and since it is a linear subset of W it is the meet of the family $\{W'_j : j \in J\}$. Moreover $span\{\bigcup_{j\in J} W'_j\}$ is the smallest subspace of W containing all the W_j's, by definition of linear span. So $\mathbb{SS}(W)$ is a complete lattice. □

Usually, $\mathbb{SS}(W)$ is called the *subspace vector lattice* of the vector space W. In the next result we show that the partial order $\preceq$ between two set partitions on a finite set X is equivalent to the dual order of $\subseteq_{ssv}$ on the set $S[m]$.

Theorem 5.6. *Let $\pi = B_1|\cdots|B_N$ and $\pi' = B'_1|\cdots|B'_M$ be two set partitions of X. Then the following conditions are equivalent:*
(i) $V_{\pi'} \subseteq_{ssv} V_\pi$.
(ii) For any $l \in \{1,\dots,M\}$ there exist $l_1,\dots,l_s \in \{1,\dots,N\}$ such that $E_l(\pi') = E_{l_1}(\pi) + \cdots E_{l_s}(\pi)$.
(iii) $\pi \preceq \pi'$.

Proof. $(i) \implies (ii)$: Let $j \in \{1,\dots,M\}$. Since $E_j(\pi') \in V_{\pi'} \subseteq_{ssv} V_\pi = span\{E_1(\pi),\dots,E_N(\pi)\}$, it follows that $E_j(\pi') = a_1E_1(\pi) + \cdots + a_NE_N(\pi)$, for some (uniquely determined) $a_1,\dots,a_N \in \mathbb{R}$. On the other hand, the n-tuple $E_j(\pi')$ contains only 1 or 0 in its entries and any two distinct vectors among $E_1(\pi),\dots,E_N(\pi)$ contain the number 1 in different places. This implies that $a_i = 1$ or $a_i = 0$, for $i = 1,\dots,N$, i.e. $E_j(\pi')$ is the sum of some vectors $E_1(\pi),\dots,E_N(\pi)$.
$(ii) \implies (i)$: It follows immediately by Proposition 5.3.
$(ii) \implies (iii)$: Let x_i and x_j be any two distinct elements of X and let us assume that $\pi(x_i) = \pi(x_j)$. In order to have the thesis we must show that also $\pi'(x_i) = \pi'(x_j)$. The identity $\pi(x_i) = \pi(x_j)$ is equivalent to saying that there is a block B_k of π such that $\{x_i, x_j\} \subseteq B_k$, therefore the n-vector $E_k(\pi)$ contains 1 in both places i and j. We assume now (by contradiction) that $\pi'(x_i) \neq \pi'(x_j)$. Then there exists a block B'_l of π' such that $x_i \in B_l$ and $x_j \notin B_l$, and this implies that the n-vector $E_l(\pi')$ contains 1 in the place i and 0 in the place j. On the other hand, by hypothesis, $E_l(\pi')$ is the sum of some vectors $E_1(\pi),\dots,E_N(\pi)$. Now, the unique vector among $E_1(\pi),\dots,E_N(\pi)$ which contains 1 in the place i is $E_k(\pi)$, therefore the vector $E_k(\pi)$ must be necessarily a summand of the previous sum. However $E_k(\pi)$ also contains 1 in the place j, whereas $E_l(\pi')$ contains 0 in the place j, and this is a contradiction. This proves (iii).
$(iii) \implies (ii)$: Let $l \in \{1,\dots,M\}$. Since $\pi \preceq \pi'$, the block B'_l is a disjoint union of some blocks $B_{l_1},\dots,B_{l_s}$ of π, therefore we have $E_l(\pi') = E_{l_1}(\pi) + \cdots + E_{l_s}(\pi)$. □

We consider now the poset

$$\mathbb{S}[m] := (S[m], \subseteq^*_{ssv}). \tag{39}$$

Then we obtain the following result.

Theorem 5.7. *If X is a set having m elements, the map $\phi : \pi \in \Pi(X) \mapsto V_\pi \in S[m]$ is an order isomorphism between the set partition lattice $\mathbb{P}(X)$ and the poset $\mathbb{S}[m]$.*

Proof. The thesis is a direct consequence of the equivalence of (i) and (iii) in Theorem 5.6. □

We consider now an information table $\mathcal{I} = \langle U, Att, Val, F\rangle$ such that $U = \{u_1,\dots,u_m\}$ is a universe set having m objects and $Att = \{a_1,\dots,a_n\}$ has n attributes. If $A \subseteq Att$ we set

$$V_A := V_{\pi_{\mathcal{I}}(A)} \tag{40}$$

Therefore we can canonically associate to any attribute subset A a vector subspace V_A of $\mathbb{R}^m$, which is the vector subspace induced from the indiscernibility partition $\pi_{\mathcal{I}}(A)$ of the universe m-set U.

Example 5.8. Let us consider the information table $\mathcal{I}$ given in Example 3.13:

	1	2	3	4
u_1	0	1	1	0
u_2	1	0	0	0
u_3	1	0	1	1
u_4	1	1	1	0
u_5	0	1	1	1

If $A = \{1, 2\} \subseteq Att$, then $\pi_{\mathcal{I}}(A) = u_1u_5|u_2u_3|u_4$ and so V_A is the linear subspace of $\mathbb{R}^n$ given by:

$$V_A := V_{\pi_{\mathcal{I}}(A)} = span\{(1,0,0,0,1), (0,1,1,0,0), (0,0,0,1,0)\} = \{(a,b,b,c,a) : a,b,c \in \mathbb{R}\}.$$

Definition 5.9. *We call* granular partition vector representation *of $\mathcal{I}$ the following subset of $S[m]$:*

$$S_{ind}(\mathcal{I}) := \{V_A : A \subseteq Att\}.$$

We also set $\mathbb{S}_{ind}(\mathcal{I}) := (S_{ind}(\mathcal{I}), \subseteq^*_{ssv})$.

We obtain then the following result.

Theorem 5.10. *The map $\psi : A \in MAXP(\mathcal{I}) \mapsto V_A \in S_{ind}(\mathcal{I})$ is an order isomorphism between the lattice $\mathbb{M}(\mathcal{I})$ and the poset $\mathbb{S}_{ind}(\mathcal{I})$. Therefore $\mathbb{S}_{ind}(\mathcal{I})$ is a complete lattice isomorphic to the indiscernibility partition lattice $\mathbb{P}_{ind}(\mathcal{I})$.*

Proof. Let us consider the map $\xi : A \in MAXP(\mathcal{I}) \mapsto \pi_{\mathcal{I}}(A) \in \Pi_{ind}(\mathcal{I})$. By Theorem 3.9, the map ξ induces an order isomorphism between the lattice $\mathbb{M}(\mathcal{I})$ and the lattice $\mathbb{P}_{ind}(\mathcal{I})$. On the other hand, the order isomorphism ϕ in the statement of Theorem 5.7 can be restricted to the sub-lattice $\mathbb{P}_{ind}(\mathcal{I})$ of $\mathbb{P}(\mathcal{I})$, and by (40) it is clear that the image of this restricted map is exactly the subset $S_{ind}(\mathcal{I})$ of $S[m]$. Hence $\psi = \phi \circ \xi$ and the thesis follows. □

6. Generalized Indiscernibility Relation and Two Related Posets

The generalized indiscernibility relation is a notion inspired by the application of rough set ideas to graphs, where it represents a new kind of symmetry [10]. In the general case, it can be useful to understand when a given partition can be obtained from an indiscernibility relation and to relate rough sets with formal concept analysis [14]. Here, it will be used in order to define two sub-posets of the indiscernibility partition lattice as a sort of indiscenibility partition lattices conditioned to a given subset of a knowledge representation system.

Definition 6.1. *[14] Let $\mathcal{I} = \langle U, Att, Val, F\rangle$ be an information table. Given a set of objects $Z \subseteq U$ we define as* generalized indiscernibility relation *the attribute set*

$$\Gamma(Z) := \{a \in Att \ : \ \forall\, z, z' \in Z, F(z,a) = F(z',a)\}. \tag{41}$$

with the agreement that $\Gamma(\emptyset) = \emptyset$, and as generalized discernibility relation *the set*

$$\Delta(Z) := Att \setminus \Gamma(Z) = \{a \in Att \ : \ \exists\, z, z' \in Z : F(z,a) \neq F(z',a)\}. \tag{42}$$

In particular, we also set $\Gamma(z) := \Gamma(\{z\})$ and $\Gamma(z,z') := \Gamma(\{z,z'\})$.

Clearly, Δ is a generalization of the discernibility matrix since $\Delta(u,u')$, as defined by equation (15), coincides with $\Delta(\{u,u'\})$ defined in equation (42).
The following result is immediate.

Proposition 6.2. *$\Gamma(Z)$ is the unique attribute subset C of $\mathcal{I}$ such that :*
(i) $Z \preccurlyeq \pi_{\mathcal{I}}(C)$;
(ii) if $A \subseteq Att$ and $Z \preccurlyeq \pi_{\mathcal{I}}(A)$, then $A \subseteq C$.

Let us observe that the reverse implication of the property *(ii)* holds, namely if $A \subseteq \Gamma(Z)$, then $Z \preccurlyeq \pi_{\mathcal{I}}(A)$. Moreover, if $Z_1, Z_2 \subseteq U$ then

$$Z_1 \subseteq Z_2 \Longrightarrow \Gamma(Z_2) \subseteq \Gamma(Z_1), \tag{43}$$

so that the operator $\Gamma : Z \in (\mathcal{P}(U), \subseteq) \mapsto \Gamma(Z) \in (\mathcal{P}(Att), \subseteq)$ is order reversing. Analogously, we have

$$Z_1 \subseteq Z_2 \Longrightarrow \Delta(Z_1) \subseteq \Delta(Z_2), \tag{44}$$

so that the operator $\Delta : Z \in (\mathcal{P}(U), \subseteq) \mapsto \Delta(Z) \in (\mathcal{P}(Att), \subseteq)$ is an order preserving set operator.
We discuss now the link between the generalized indiscernibility relation and the indiscernibility partition lattice. We firstly introduce new posets, sub-poset of $\mathbb{P}_{ind}(\mathcal{I})$, $\mathbb{M}(\mathcal{I})$ and $\mathbb{G}(\mathcal{I})$ respectively.
If $Z \subseteq U$ we set

$$\Pi_{ind}(\mathcal{I}|Z) := \{\pi \in \Pi_{ind}(\mathcal{I}) : Z \preccurlyeq \pi\} \text{ and } \mathbb{P}_{ind}(\mathcal{I}|Z) := (\Pi_{ind}(\mathcal{I}|Z), \preceq), \tag{45}$$

$$MAXP(\mathcal{I}|Z) := \{B \in MAXP(\mathcal{I}) : Z \preccurlyeq \pi_{\mathcal{I}}(B)\} \text{ and } \mathbb{M}(\mathcal{I}|Z) := (MAXP(\mathcal{I}|Z), \subseteq^*), \tag{46}$$

$$G(\mathcal{I}|Z) := \{(M(A), \pi_{\mathcal{I}}(A)) \in Gran(\mathcal{I}) : Z \preccurlyeq \pi_{\mathcal{I}}(A)\} \text{ and } \mathbb{G}(\mathcal{I}|Z) := (T(\mathcal{I}|Z), \preceq'). \tag{47}$$

In the next result we show that the generalized indiscernibility relation is also a specific type of maximum partitioner.

Proposition 6.3. *Let $Z \subseteq U$. Then $\Gamma(Z) \in MAXP(\mathcal{I}|Z)$.*

Proof. Let $C := \Gamma(Z)$ and $D := M(C)$. If $z, z' \in Z$, by definition of $\Gamma(Z)$ we have $z \equiv_C z'$, therefore we also obtain $z \equiv_D z'$, because $\pi_{\mathcal{I}}(D) = \pi_{\mathcal{I}}(C)$. Hence $Z \preccurlyeq \pi_{\mathcal{I}}(D)$. By (ii) of Proposition 6.2 we deduce then that $D \subseteq C$, i.e. $C = M(C)$. This shows that $C \in MAXP(\mathcal{I})$. By part (i) of Proposition 6.2 we also have $Z \preccurlyeq \pi_{\mathcal{I}}(C)$, so that $C \in MAXP(\mathcal{I}|Z)$. □

Corollary 6.4. *If $Z \subseteq U$ then $\Gamma(Z)$ is a maximum partitioner of $\mathcal{I}$.*

Moreover, we can also use the generalized indiscernibility relation in order to completely characterize the relative maximum partitioners, as we show in the following result.

Proposition 6.5. *Let $Z \subseteq U$ and let $A \subseteq Att$. Then $A \in MAXP(\mathcal{I}|Z)$ if and only if $A \in MAXP(\mathcal{I})$ and $A \subseteq \Gamma(Z)$.*

Proof. Let $A \in MAXP(\mathcal{I}|Z)$. Then $A \in MAXP(\mathcal{I})$ by definition of $MAXP(\mathcal{I}|Z)$. If $z, z' \in Z$ we have $z \equiv_A z'$ because $Z \preccurlyeq \pi_{\mathcal{I}}(A)$. Therefore, by (ii) of Proposition 6.2 it follows that $A \subseteq \Gamma(Z)$.
Let now $A \in MAXP(\mathcal{I})$ such that $A \subseteq \Gamma(Z)$. Let us prove that $Z \preccurlyeq \pi_{\mathcal{I}}(A)$. For this, let $z, z' \in Z$. Then $\forall a \in A$, since $a \in \Gamma(Z)$, $F(z, a) = F(z', a)$. Then $z \equiv_A z'$ and thus $Z \preccurlyeq \pi_{\mathcal{I}}(A)$. So $A \in MAXP(\mathcal{I}|Z)$. □

Corollary 6.6. *Let $Z \subseteq U$. Then the poset $\mathbb{M}(\mathcal{I}|Z)$ is a complete lattice and it coincides with the up-set of $\Gamma(Z)$ in $\mathbb{M}(\mathcal{I})$.*

Proof. It follows directly by Proposition 6.3 and Proposition 6.5. □

Example 6.7. Let us consider the information table $\mathcal{I}$ introduced in Example 3.13 and let $Z = \{u_1, u_4\}$. Then the generalized indiscernibility relation applied to Z is $\Gamma(Z) = \{2, 3, 4\}$ and the lattice $\mathbb{M}(\mathcal{I}|Z)$ is the below sub diagram having the edges dashed.

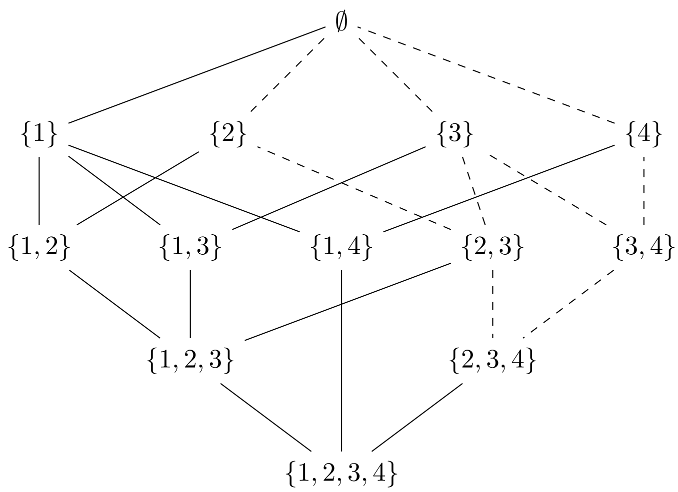


Figure 4:

In fact, it is easy to verify that $Z \preccurlyeq \pi_{\mathcal{I}}(A)$ if and only if A is some node in the above dashed diagram.

Proposition 6.8. *The posets $\mathbb{P}_{ind}(\mathcal{I}|Z)$ and $\mathbb{G}(\mathcal{I}|Z)$ are complete lattices both isomorphic to the lattice $\mathbb{M}(\mathcal{I}|Z)$.*

Proof. The map

$$\phi : MAXP(\mathfrak{I}) \to \Pi_{ind}(U) \text{ such that } \phi(A) := \pi_{\mathfrak{I}}(A)$$

is an order isomorphism between $\mathbb{M}(\mathfrak{I})$ and $\mathbb{P}_{ind}(\mathfrak{I})$ such that $\phi(MAXP(\mathfrak{I}|Z)) = \Pi_{ind}(\mathfrak{I}|Z)$. This implies that the order isomorphism ϕ carries the properties of the sub lattice $\mathbb{M}(\mathfrak{I}|Z)$ towards the sub poset $\mathbb{P}_{ind}(\mathfrak{I}|Z)$. Therefore, by Proposition 6.6 we obtain the thesis for the poset $\mathbb{P}_{ind}(\mathfrak{I}|Z)$. On the other hand, the map

$$\psi : MAXP(\mathfrak{I}) \to Gran(\mathfrak{I}) \text{ such that } \psi(A) := (A, \pi_{\mathfrak{I}}(A))$$

is an order isomorphism between $\mathbb{M}(\mathfrak{I})$ and $\mathbb{G}(\mathfrak{I})$ such that $\psi(MAXP(\mathfrak{I}|Z)) = \mathbb{G}(\mathfrak{I}|Z)$. Then the thesis for $\mathbb{G}(\mathfrak{I}|Z)$ follows as in the previous case. □

Definition 6.9. *If $Z \subseteq U$, we call indiscernibility partition lattice of $\mathfrak{I}$* conditioned *to Z the lattice $\mathbb{P}_{ind}(\mathfrak{I}|Z)$.*

The relevance of the set operator Γ is due to the following result proved in [14].

Theorem 6.10. *[14] Let $\pi = B_1|\dots|B_M \in \Pi(U)$ and let $A = \bigcap_{i=1}^{M} \Gamma(B_i)$. Then*

$$\pi \in \Pi_{ind}(\mathfrak{I}) \iff \pi = \pi_{\mathfrak{I}}(A), \tag{48}$$

and in this case we have that $A = Max(\pi) = M_{\mathfrak{I}}(A)$.

Therefore, in order to better investigate the connections between all the subsets $\Gamma(Z)$, for $Z \subseteq U$, it is convenient to introduce the following order structure.

Definition 6.11. *We set $ICLO(\mathfrak{I}) := \{\Gamma(Z) : Z \subseteq U\}$ and $\mathbb{I}(\mathfrak{I}) := (ICLO(\mathfrak{I}), \subseteq^*)$.*

.

The partial order $\subseteq^*$ is the dual order of the inclusion relation, and it is the same partial order of the lattice $\mathbb{M}(\mathfrak{I})$. Therefore $\mathbb{I}(\mathfrak{I})$ is a sub-poset of $\mathbb{M}(\mathfrak{I})$, but in general $\mathbb{I}(\mathfrak{I})$ is not a lattice, as we show in the next example.

Example 6.12. Let us consider the following information table $\mathfrak{I}$:

	1	2	3	4
u_1	0	0	1	0
u_2	0	0	1	1
u_3	1	1	0	2
u_4	2	1	0	2
u_5	3	1	1	3

The Hasse diagram of the lattice $\mathbb{M}(\mathfrak{I})$ is drawn in the next Figure 5.

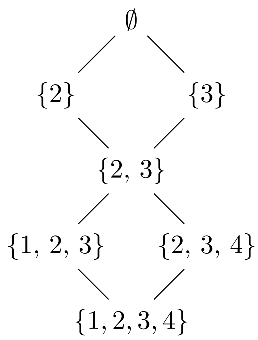


Figure 5: The lattice $\mathbb{M}(\mathfrak{I})$

It is easy to verify that the attribute set $\{2, 3\}$ is the unique maximum partitioner of $\mathfrak{I}$ which is not a generalized indiscernibility relation. The Hasse diagram of $\mathbb{I}(\mathfrak{I})$ is drawn in Figure 6.

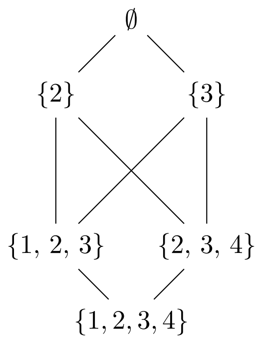


Figure 6: The poset $\mathbb{I}(\mathfrak{I})$

Thus, by Example 6.12 it follows that the poset $\mathbb{I}(\mathfrak{I})$ is not a lattice in general. On the other hand, in several cases it results that the poset $\mathbb{I}(\mathfrak{I})$ has also a lattice structure. Therefore a natural question, that we leave as an open issue, is to understand for which types of knowledge representation systems $\mathfrak{I}$ the poset $\mathbb{I}(\mathfrak{I})$ is also a lattice.

7. The Maximum Partitioner Lattice as a Measurer of Entropy and Dependency

In this section we show that the maximum partitioner lattice can be used as a *measurer* of entropy for any information table and of dependency in a decision system. Of course, due to the isomorphisms outlined until now, these results apply also to the indiscernibility partition lattice and to the granular partition lattice.

7.1. Entropy Growth in the Maximum Partitioner Lattice

In a situation of incomplete information, there is the need to measure the amount of uncertainty and this is typically done using entropy. Also in rough sets, we have several approaches to compute entropy, both in the classical rough sets based on an equivalence relation and in generalized contexts [67, 32, 3, 17, 16]. Often, a requirement of the uncertainty measure is its monotonicity with respect to knowledge changes [4, 5, 3]. In the following, we address this issue in the case of the maximum partitioner lattice.
If X is a finite set and $\pi = B_1|\cdots|B_N$ is any set partition of X, we can consider the probability distribution $(p_1, \ldots, p_N) := (|B_1|/|X|, \ldots, |B_N|/|X|)$, therefore we can compute the usual entropy of this probability distribution. This is called *entropy of the set partition* π (see [4, 5, 3] for details)

$$H(\pi) := \sum_{i=1}^{N} p_i \log \frac{1}{p_i} = \sum_{i=1}^{N} \frac{|B_i|}{|X|} \log \frac{|X|}{|B_i|}. \tag{49}$$

In [5], it has been proved that H is strictly anti-monotone with respect to the partial order $\preceq$, that is, if π and π' are two set partitions on X, then

$$\pi \prec \pi' \Longrightarrow H(\pi') < H(\pi). \tag{50}$$

Therefore, by (50), if $C = \{\pi_1 \prec \cdots \prec \pi_k\}$ is a k-chain of set partitions on X, then $H(\pi_1) > \cdots > H(\pi_k)$.

The entropy function H can be used to evaluate the uniformity of the distribution of the elements of X in the blocks of π. In fact, the value $H(\pi)$ increases with the uniformity of the distribution of the elements of X in the blocks of the set partition π (see [52]).

If $\mathfrak{I}$ is a knowledge representation system and A is an attribute subset of $\mathfrak{I}$, the *entropy generated by* A in $\mathfrak{I}$ is defined as

$$H_{\mathfrak{I}}(A) := H(\pi_{\mathfrak{I}}(A)). \tag{51}$$

Let us notice that if A and B are two attribute subsets of $\mathfrak{I}$, by (7), (51) and (50), it follows that

$$A \subseteq B \Longrightarrow H_{\mathfrak{I}}(A) \leq H_{\mathfrak{I}}(B). \tag{52}$$

Proposition 7.1. *Let B and B' be two maximum partitioners of $\mathfrak{I}$. Then:*

$$B' \subsetneqq B \Longrightarrow H_{\mathfrak{I}}(B') < H_{\mathfrak{I}}(B). \tag{53}$$

Proof. From the hypothesis and by Proposition 3.4 we have $M(B') = B' \subsetneqq B = M(B)$, therefore $\pi_{\mathfrak{I}}(B) \prec \pi_{\mathfrak{I}}(B')$ by (21). The thesis follows then by (51) and (50). □

Remark 7.2. *By Proposition 7.1, the entropy map $H_{\mathfrak{I}} : B \in MAXP(\mathfrak{I}) \mapsto H_{\mathfrak{I}}(B) \in \mathbb{R}$ induces a strictly anti-monotone real map on the lattice $\mathbb{M}(\mathfrak{I})$, since on this lattice we consider the dual order $\subseteq^*$ instead of $\subseteq$.*

We can represent the entropy values in a weighted lattice $\mathbb{M}_{ent}(\mathfrak{I})$ where each node of the Hasse diagram has the form $(B, H_{\mathfrak{I}}(B))$, with $B \in MAXP(\mathfrak{I})$.

Example 7.3. Let us consider the information table $\mathfrak{I}$ given in Example 3.13. Below we draw the maximum partitioner lattice $\mathbb{M}(\mathfrak{I})$ with entropy values attached to each node.

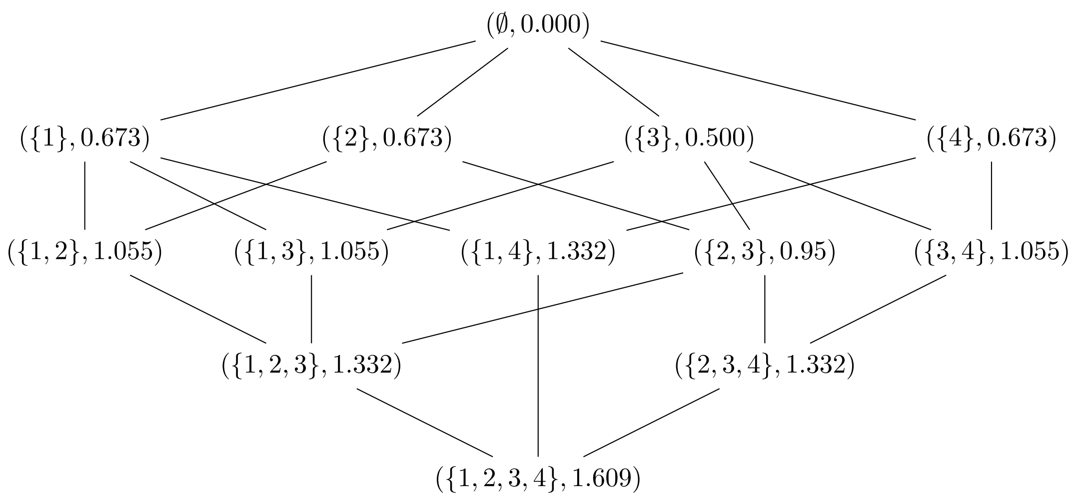


Figure 7: Lattice $\mathbb{M}(\mathfrak{I})$ weighted with entropy values

Then, if we take for example the two maximal chains

$$C_1 = \{\emptyset, \{1\}, \{1,4\}, \{1,2,3,4\}\} \text{ and } C_2 = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$$

the entropy growth of $\mathfrak{I}$ along C_1 is $(0.000, 0.673, 1.332, 1.609)$, whereas along C_2 is $(0.000, 0.673, 1.055, 1.332, 1.609)$.

Let us stress that the isomorphism between the lattices $\mathbb{M}(\mathfrak{I})$ and $\mathbb{P}_{co}(\mathfrak{I}_{ps})$, automatically introduces an entropy notion (and its monotonicity with respect to the lattice structure) also in the pattern concept lattice induced by a partition of an information table. Of course, the general issue of defining an entropy measure on a general pattern concept lattice is an interesting open problem.

7.2. The Role of the Maximum Partitioner Lattice for Decision Tables

Now, we add to an information table a new *decision attribute* d and a set of corresponding values $\xi_{u,d}$, for any $u \in U$. Then the structure $\hat{\mathfrak{I}} := \langle U, Att \cup \{d\}, \hat{F}, \hat{Val}\rangle$, where $\hat{Val} := Val \cup \{\xi_{u,d} : u \in U\}$ and $\hat{F}(u,a) := F(u,a)$, $\hat{F}(u,d) := \xi_{u,d}$ for any $u \in U$ and $a \in Att$, is a so called *decision system* or equivalently *decision table*.

Definition 7.4. *We call* evaluation dependency map *of the decision system $\hat{\mathfrak{I}}$ the application $\hat{\psi} : MAXP(\mathfrak{I}) \to [0,1]$ defined by*

$$\hat{\psi}(A) := \gamma_A(d),$$

for any $A \in MAXP(\mathfrak{I})$.

Proposition 7.5. *$\hat{\psi}$ is a real-valued order-reversing map on the maximum partitioner lattice $\mathbb{M}(\mathcal{I})$.*

Proof. It follows immediately by (13). □

Also in this case we can represent the dependency values $\gamma_A(d)$ in the weighted lattice $(\mathbb{M}_d(\mathcal{I}), \hat{\psi})$ where each node of the Hasse diagram has the form $(B, \hat{\psi}(B))$, with $B \in MAXP(\mathcal{I})$.

Example 7.6. We consider again the information table $\mathcal{I}$ given in Example 3.13 and we add a new decision attribute d in such a way to obtain the following decision table $\hat{\mathcal{I}}$:

	1	2	3	4	d
u_1	0	1	1	0	2
u_2	1	0	0	0	1
u_3	1	0	1	1	0
u_4	1	1	1	0	1
u_5	0	1	1	1	0

Then we have that $\pi_{\hat{\mathcal{I}}}(d) = u_1|u_2u_4|u_3u_5$. Below we draw the weighted maximum partitioner lattice $(\mathbb{M}(\mathcal{I}), \hat{\psi})$:

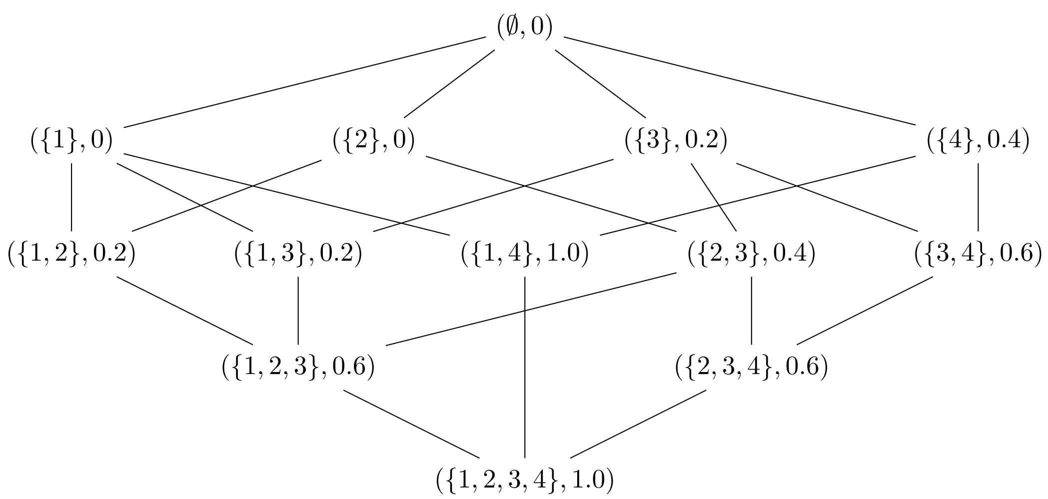


Figure 8: Lattice $\mathbb{M}(\mathcal{I})$ weighted with dependency values

Let us note now that by (12) it follows that

$$POS_{A\cap B}(d) \subseteq POS_A(d) \cap POS_B(d) \tag{54}$$

for any two attribute subsets $A, B \subseteq Att$, but in general the reverse inclusion in (54) does not hold. However, when also the reverse inclusion holds, we obtain an interesting result on the function $\hat{\psi}$.

Proposition 7.7. *Let $A, B \in MAXP(\mathcal{I})$ such that*

$$POS_{A\cap B}(d) = POS_A(d) \cap POS_B(d). \tag{55}$$

Then

$$\hat{\psi}(A \wedge B) + \hat{\psi}(A \vee B) \geq \hat{\psi}(A) + \hat{\psi}(B). \tag{56}$$

Proof. Since

$$|POS_A(d) \cup POS_B(d)| = |POS_A(d)| + |POS_B(d)| - |POS_A(d) \cap POS_B(d)|$$

and $A \vee B = A \cap B$, by (55) it follows that

$$|POS_A(d) \cup POS_B(d)| + |POS_{A\vee B}(d)| = |POS_A(d)| + |POS_B(d)|. \tag{57}$$

On the other hand, by Corollary 3.7 we have that $A \wedge B = M(A \cup B)$, therefore, since $A, B \subseteq M(A \cup B)$ it follows that

$$POS_A(d) \cup POS_B(d) \subseteq POS_{M(A\cup B)}(d) = POS_{A\wedge B}(d). \tag{58}$$

Then, by (57) and (58) we have that

$$|POS_{A\wedge B}(d)| + |POS_{A\vee B}(d)| \geq |POS_A(d)| + |POS_B(d)|,$$

that is equivalent to (56). Hence the thesis follows. □

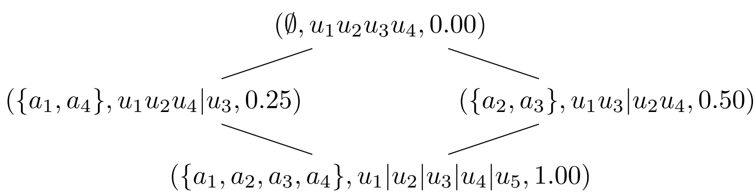
The result established in Proposition 7.7 leads us to formulate the following classification of decision tables.

Definition 7.8. *We say that the decision system* $\hat{\mathfrak{I}}$ *is:*
(i) POS-strongly intersecting *if for any* $A, B \in MAXP(\mathfrak{I})$ *the condition* (55) *is satisfied.*
(ii) POS-weakly intersecting *if for any* $A, B \in MAXP(\mathfrak{I})$ *the condition* (56) *is satisfied.*

Example 7.9. Let $\hat{\mathfrak{I}}$ be the decision table given below:

$\hat{\mathfrak{I}}$	a_1	a_2	a_3	a_4	d
u_1	0	1	0	0	1
u_2	0	0	1	0	0
u_3	1	1	0	1	2
u_4	0	0	1	0	1

We represent below the Hasse diagram of the lattice $\mathbb{G}(\hat{\mathfrak{I}})$.



Since we have $\pi_{\hat{\mathfrak{I}}}(d) = u_1|u_2u_4|u_3$, it is now immediate to note that (55) holds.

By Proposition 7.7 it follows that a POS-strongly intersection decision system is also a POS-weakly intersection decision system, but the reverse implication does not hold in general, as we show in the next example.

Example 7.10. Let $\hat{\mathfrak{I}}$ be the decision table given below:

$\hat{\mathfrak{I}}$	a_1	a_2	a_3	a_4	a_5	d
u_1	0	1	0	0	1	1
u_2	1	0	1	0	0	1
u_3	0	1	0	1	0	1
u_4	0	0	1	0	1	2
u_5	1	0	0	1	0	0

It is easy to see that $MAXP(\mathfrak{I}) = \{\emptyset, \{a_i\}, \{a_i, a_{i+1}\}, Att\}$, where $i = 1, \dots, 5$ and the index sum $i+1$ is taken mod(5). We observe that (56) holds. Moreover, if we take $A = \{a_2, a_3\}$ and $B = \{a_3, a_4\}$, we have $POS_A(d) \cap POS_B(d) = \{a_1\}$ while $POS_{A\cap B}(d) = POS_{\{a_3\}}(\hat{d}) = \emptyset$. Hence, (55) does not hold. Thus $\hat{\mathfrak{I}}$ is POS-weakly intersecting but it is not POS-strongly intersecting.

If $X \in MAXP(\mathfrak{I})$ we set now

$$[X|\hat{d}] := \{Z \in MAXP(\mathfrak{I}) : X \subseteq^* Z \text{ and } \hat{\psi}(X) = \hat{\psi}(Z)\}. \tag{59}$$

Proposition 7.11. *Let* $\hat{\mathfrak{I}}$ *be a* POS*-weakly intersecting decision system. Then*

$$U, V \in [X|\hat{d}] \Longrightarrow U \vee V \in [X|\hat{d}] \tag{60}$$

Proof. Let $U, V \in [X|\hat{d}]$. Then

$$\hat{\psi}(U) = \hat{\psi}(V) = \hat{\psi}(X). \tag{61}$$

Since $X \subseteq^* U$ and $X \subseteq^* V$, we have $X \subseteq^* U \wedge V \subseteq^* U, V \subseteq^* U \vee V$. Therefore, from the anti-monotonicity of $\hat{\psi}$ we obtain

$$\hat{\psi}(X) \geq \hat{\psi}(U \wedge V) \geq \hat{\psi}(U), \hat{\psi}(V) \geq \hat{\psi}(U \vee V) \tag{62}$$

By (61) and by (62) we have

$$\hat{\psi}(U \wedge V) = \hat{\psi}(X). \tag{63}$$

Now, since $\hat{\mathfrak{I}}$ is POS-weakly intersecting we also have

$$\hat{\psi}(U \wedge V) + \hat{\psi}(U \vee V) \geq \hat{\psi}(U) + \hat{\psi}(V). \tag{64}$$

By (61), (62), (63) and (64) we obtain then

$$\hat{\psi}(U \vee V) = \hat{\psi}(X).$$

Hence the thesis follows. □

Theorem 7.12. *Let $\hat{\mathfrak{I}}$ be a POS-weakly intersecting decision table. Then:*
(i) $[X|\hat{d}]$ has a maximum element, here denoted by $\mu_{\hat{d}}(X)$;
(ii) the map $\mu_{\hat{d}} : MAXP(\mathfrak{I}) \to MAXP(\mathfrak{I})$ defines a closure operator on the maximum partitioner lattice $\mathbb{M}(\mathfrak{I})$, i.e.

$$X \subseteq^* \mu_{\hat{d}}(Y) \iff \mu_{\hat{d}}(X) \subseteq^* \mu_{\hat{d}}(Y), \tag{65}$$

for any $X, Y \in MAXP(\mathfrak{I})$.

Proof. (i) : This part is a direct consequence of Proposition 7.11 since for a decision table the set $[X|\hat{d}]$ is finite.
(ii) : Let $X, Y \in MAXP(\mathfrak{I})$. We assume firstly that $\mu_{\hat{d}}(X) \subseteq^* \mu_{\hat{d}}(Y)$. Since $\mu_{\hat{d}}(X)$ is the greatest element of $[X|\hat{d}]$ and $X \in [X|\hat{d}]$, we have $X \subseteq^* \mu_{\hat{d}}(X) \subseteq^* \mu_{\hat{d}}(Y)$. This proves the implication $\Longleftarrow$ in (65). We assume now that $X \subseteq^* \mu_{\hat{d}}(Y)$. Then, since $X \subseteq^* \mu_{\hat{d}}(X)$ we obtain $X \subseteq^* \mu_{\hat{d}}(X) \wedge \mu_{\hat{d}}(Y) \subseteq^* \mu_{\hat{d}}(X)$, and therefore

$$\hat{\psi}(X) \geq \hat{\psi}(\mu_{\hat{d}}(X) \wedge \mu_{\hat{d}}(Y)) \geq \hat{\psi}(\mu_{\hat{d}}(X)) = \hat{\psi}(X), \tag{66}$$

because $\hat{\psi}$ is anti-monotone and $\mu_{\hat{d}}(X)$ is an element of $[X|\hat{d}]$. Hence

$$\hat{\psi}(X) = \hat{\psi}(\mu_{\hat{d}}(X) \wedge \mu_{\hat{d}}(Y)) = \hat{\psi}(\mu_{\hat{d}}(X)). \tag{67}$$

On the other hand, since the decision table $\hat{\mathfrak{I}}$ is POS-weakly intersecting we have that

$$\hat{\psi}(\mu_{\hat{d}}(X) \wedge \mu_{\hat{d}}(Y)) + \hat{\psi}(\mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y)) \geq \hat{\psi}(\mu_{\hat{d}}(X)) + \hat{\psi}(\mu_{\hat{d}}(Y)). \tag{68}$$

By (67) and (68) we obtain then

$$\hat{\psi}(\mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y)) \geq \hat{\psi}(\mu_{\hat{d}}(Y)) = \hat{\psi}(Y). \tag{69}$$

Now, since $Y \subseteq^* \mu_{\hat{d}}(Y) \subseteq^* \mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y)$, from the anti-monotonicity of $\hat{\psi}$ we have

$$\hat{\psi}(Y) \geq \hat{\psi}(\mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y)). \tag{70}$$

By (69) and (70) it follows that

$$\hat{\psi}(Y) = \hat{\psi}(\mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y)). \tag{71}$$

Then, since $Y \subseteq \mu_{\hat{d}}(Y) \subseteq \mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y)$, by (71) we deduce that $\mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y) \in [Y|\hat{d}]$, and since $\mu_{\hat{d}}(Y)$ is the maximum of $[Y|\hat{d}]$ we obtain that $\mu_{\hat{d}}(X) \vee \mu_{\hat{d}}(Y) \subseteq^* \mu_{\hat{d}}(Y)$, therefore also $\mu_{\hat{d}}(X) \subseteq^* \mu_{\hat{d}}(Y)$. This proves the implication $\Longrightarrow$ in (65). □

8. Conclusion

The lattice arising from all the indiscernibility partitions generated by attributes of a knowledge representation system has been studied. In order to prove that it is closed under join and meet also in the infinite case, we first proved its isomorphism with the maximum partitioner lattice. A further isomorphic structure is the granular partition lattice which is also linked to pattern structures. Finally, by taking into account the generalized discernibility matrix, other sub-posets, although not necessarily sub-lattices, of the previous ones have been defined and studied. In Figure 9, the relationship among all these posets is put in evidence.

$$\begin{array}{c} \mathbb{P}(\mathfrak{I}) \\ | \\ \mathbb{P}_{ind}(\mathfrak{I}) \cong \mathbb{M}(\mathfrak{I}) \cong \mathbb{G}(\mathfrak{I}) = \mathbb{P}_{co}(\mathfrak{I}_{ps})* \cong \mathbb{S}_{ind}(\mathfrak{I}) \\ | \\ \mathbb{P}_{ind}(\mathfrak{I}|Z) \cong \mathbb{M}(\mathfrak{I}|Z) \cong \mathbb{G}(\mathfrak{I}|Z) \\ | \\ \mathbb{I}(\mathfrak{I}) \end{array}$$

Figure 9: Summary of the relation among the introduced posets.

From top to bottom, we move to a lattice with a smaller set of elements, obtaining a poset, but not necessarily a sub-lattice.

Thus, in this picture we can find several typical tools of rough set theory at work. We have the indiscernibility relation generating the lattice $\mathbb{P}_{ind}(\mathfrak{I})$, the (generalized) discernibility matrix at the basis of the lattice $\mathbb{P}_{ind}(\mathfrak{I}|Z)$ and the recently introduced notion of maximum partitioner generating the lattice $\mathbb{M}(\mathfrak{I})$. Links can also be find with pattern structures and hence formal contexts through the lattice $\mathbb{G}(\mathfrak{I})$. For the future, we plan to compare our approach to multi-granulation rough sets [47, 73, 35], which are based on several equivalence relations, and hence they naturally give rise to a poset of partitions, and to the already mentioned Scott information systems and to Polkowski's chains of indiscernible relations. Moreover, we envisage a comprehensive investigation of the micro and macro granular properties of the partition lattice as well as the condition in order that the poset $\mathbb{I}(\mathfrak{I})$ is also a lattice.

[1] Baixeries, J., Kaytoue, M., Napoli, A., 2014. Characterizing functional dependencies in formal concept analysis with pattern structures. Ann. Math. Artif. Intell. 72 (1-2), 129–149.

[2] Balla, I., Bollobás, B., Eccles, T., 2013. Union-closed families of sets. Journal of Combinatorial Theory, Series A 120 (3), 531 – 544.

[3] Bianucci, D., Cattaneo, G., 2009. Information entropy and granulation co-entropy of partitions and coverings: A summary. Transactions on Rough Sets X 5656, 15–66.

[4] Bianucci, D., Cattaneo, G., Ciucci, D., 2007. Entropies and co-entropies for incomplete information systems. In: Yao, J., Lingras, P., Wu, W.-Z., Szczuka, M. S., Cercone, N., Slezak, D. (Eds.), RSKT Proceedings. Vol. 4481 of LNCS. pp. 84–92.

[5] Bianucci, D., Cattaneo, G., Ciucci, D., 2007. Entropies and co-entropies of coverings with application to incomplete information systems. Fundamenta Informaticae 75 (1-4), 77–105.

[6] Cattaneo, G., Chiaselotti, G., Ciucci, D., Gentile, T., 2016. On the connection of hypergraph theory with formal concept analysis and rough set theory. Information Sciences 330, 342–357.

[7] Chen, C., 2014. Ranked highly-arc-transitive digraphs as colimits. Order 31 (2), 143–158.

[8] Chen, G., Zhong, N., Yao, Y., 2008. A hypergraph model of granular computing. In: The 2008 IEEE International Conference on Granular Computing, GrC 2008, Proceedings. IEEE, pp. 130–135.

[9] Chiaselotti, G., Ciucci, D., Gentile, T., 2015. Simple undirected graphs as formal contexts. In: Baixeries, J., Sacarea, C., Ojeda-Aciego, M. (Eds.), Proc. ICFCA 2015. Vol. 9113 of Lecture Notes in Computer Science. Springer, pp. 287–302.

[10] Chiaselotti, G., Ciucci, D., Gentile, T., 2016. Simple graphs in granular computing. Information Sciences 340–341, 279 –304.

[11] Chiaselotti, G., Ciucci, D., Gentile, T., Infusino, F., 2015. Preclusivity and simple graphs. In: [77], pp. 127–137.

[12] Chiaselotti, G., Ciucci, D., Gentile, T., Infusino, F., 2015. Preclusivity and simple graphs: The n-cycle and n-path cases. In: [77], pp. 138–148.

[13] Chiaselotti, G., Ciucci, D., Gentile, T., Infusino, F., 2015. Rough set theory applied to simple undirected graphs. In: Ciucci, D., Wang, G., Mitra, S., Wu, W. (Eds.), Proc. RSKT 2015. Vol. 9436 of Lecture Notes in Computer Science. Springer, pp. 423–434.

[14] Chiaselotti, G., Ciucci, D., Gentile, T., Infusino, F., 2016. Generalizations of rough set tools inspired by graph theory. Fundamenta InformaticaeSubmitted.

[15] Chiaselotti, G., Gentile, T., Infusino, F., Oliverio, P., 2016. Rough sets for n-cycles and n-paths. Applied Mathematics and Information Sciences 10 (1), 117–124.

[16] Dai, J., Huang, D., Su, H., Tian, H., Yang, T., 2014. Uncertainty measurement for covering rough sets. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 22 (2), 217–234.
[17] Dai, J., Tian, H., 2013. Entropy measures and granularity measures for set-valued information systems. Inf. Sci. 240, 72–82.
[18] Date, C., 2004. An Introduction to Database Systems. Pearson.
[19] Davey, B. A., Priestley, H. A., 2002. Introduction to lattices and order. Cambridge University Press, New York.
[20] G. Chiaselotti, D. C., Gentile, T., 2015. Simple Graphs in Granular Computing. Information SciencesSubmitted.
[21] Ganter, B., Wille, R., 1999. Formal Concept Analysis: Mathematical Foundations. Springer–Verlag, Berlin, Heidelberg.
[22] Gaume, B., Navarro, E., Prade, H., 2013. Clustering bipartite graphs in terms of approximate formal concepts and sub-contexts. Int. J. Computational Intelligence Systems 6 (6), 1125–1142.
[23] Honko, P., 2008. Description and classification of complex structured objects by applying similarity measures. Int. J. Approx. Reasoning 49 (3), 539–554.
[24] Honko, P., 2012. Relational pattern updating. Inf. Sci. 189, 208–218.
[25] Honko, P., 2013. Association discovery from relational data via granular computing. Inf. Sci. 234, 136–149.
[26] Hu, X., Lin, T. Y., Han, J., 2003. A new rough sets model based on database systems. In: Verlag, S. (Ed.), Lecture Notes on Artificial Intelligence.
[27] Huang, A., Lin, Z., Zhu, W., 2014. Matrix approaches to rough sets through vector matroids over fields. IJGCRSIS 3 (3), 179–194.
[28] Huang, A., Zhao, H., Zhu, W., 2014. Nullity-based matroid of rough sets and its application to attribute reduction. Inf. Sci. 263, 153–165.
[29] Kang, X., Li, D., Wang, S., Qu, K., 2012. Formal concept analysis based on fuzzy granularity base for different granulations. Fuzzy Sets and Systems 203, 33 – 48.
[30] Kreinovich, V., 2008. Interval Computation as an Important Part of Granular Computing: An Introduction. Wiley, pp. 3–32.
[31] Lee, T., 1987. An information-theoretic analysis of relational databases - part i: Data dependencies and metric. IEEE Transactions on Software Engineering SE-13, 1049–1061.
[32] Liang, J., Shi, Z., 2004. The information entropy, rough entropy and knowledge granulation in rough set theory. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 12, 37–46.
[33] Liau, C. J., Lin, T. Y., 2005. Granular computing and rough sets. The Data Mining and Knowledge Discovery Handbook, 535–561.
[34] Liau, C. J., Lin, T. Y., 2010. Granular computing and rough sets - an incremental development. Data Mining and Knowledge Discovery Handbook, 445–468.
[35] Lin, G., Liang, J., Qian, Y., 2014. Topological approach to multigranulation rough sets. Int. J. Machine Learning & Cybernetics 5 (2), 233–243.
[36] Lin, T. Y., 1999. Data mining: Granular computing approach, in methodologies for knowledge discovery and data mining. Lecture Notes in Computer Science 1574, 24–33.
[37] Lin, T. Y., 2000. Data mining and machine oriented modeling: A granular approach. Applied Intelligence 13, 113–124.
[38] Lin, T. Y., 2002. Database mining on derived attributes: Granular and rough computing approach. In: Alpigini, Peters Skowron, Z. e. (Ed.), Lecture Notes on Artificial Intelligence. pp. 14–32.
[39] Lin, T. Y., 2002. Fuzzy sets, rough set and probability. In: Keller, J., Nasraoui, O. (Eds.), Annual Meeting of the North American Fuzzy Information Processing Society Proceedings. pp. 302–305.
[40] Lin, T. Y., Liu, Y., Huang, W., 2013. Unifying rough set theories via large scaled granular computing. Fundamenta Informaticae 127 (1-4), 413–428.
[41] Pawlak, Z., 1991. Rough Sets. Theoretical Aspects of Reasoning about Data. Kluwer, Dordrecht.
[42] Pedrycz, A., Hirota, K., Pedrycz, W., Dong, F., 2012. Granular representation and granular computing with fuzzy sets. Fuzzy Sets and Systems 203, 17–32.
[43] Polkowski, L., 1993. Mathematical morphology of rough sets. Bull. Polish Acad. Sci. Math. 41, 241–273.
[44] Polkowski, L., 1999. Approximation mathematical morphology. rough set approach. In: Skowron, A., Pal, S. (Eds.), Rough Fuzzy Hybridization. A New Trend in Decision-Making. Springer–Verlag Singapore, pp. 151–162.
[45] Polkowski, L., 2002. On fractal dimension in information systems. toward exact sets in infinite information systems. Fundamenta Informaticae 50 (3-4), 305–314.
[46] Polkowski, L., Tsumoto, S., Lin, T. Y., 2000. Rough Set Methods and Applications. New Developments in Knowledge Discovery in Information Systems. Springer–Verlag.
[47] Qian, Y., Liang, J., Yao, Y., Dang, C., 2010. MGRS: A multi-granulation rough set. Information Sciences 180 (6), 949–970.
[48] Qiu, T., Chen, X., Liu, Q., Huang, H., 2010. Granular computing approach to finding association rules in relational database. Int. J. Intell. Syst. 25 (2), 165–179.
[49] Sanahuja, S. M., 2016. New rough approximations for n-cycles and n-paths. Applied Mathematics and Computation 276, 96–108.
[50] Scott, D. (Ed.), 1982. Domains for denotational semantics. Vol. 140 of Lecture Notes in Computer Science. Springer, Berlin.
[51] Simmons, H., 2011. An Introduction to Category Theory. Cambridge University Press.
[52] Simovici, D., Djeraba, C., 2014. Mathematical Tools for Data Mining. Springer-Verlag, London.
[53] Skowron, A., Rauszer, C., 1992. The discernibility functions matrices and functions in information systems. In: Slowinski, R. (Ed.), Intelligent Decision Support. Handbook of Applications and Advances of the Rough Set Theory. Kluwer Academic Publishers, pp. 331–362.
[54] Skowron, A., Wasilewski, P., 2011. Information systems in modeling interactive computations on granules. Theor. Comput. Sci. 412 (42), 5939–5959.
[55] Stanley, R., 2012. Enumerative combinatorics. Vol.1. Second Edition. Vol. 49 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge.
[56] Stell, J. G., 1999. Granulation for graphs. In: Freksa, C., Mark, D. M. (Eds.), Spatial Information Theory: Cognitive and Computational Foundations of Geographic Information Science, International Conference COSIT '99, Proceedings. Vol. 1661 of Lecture Notes in Computer Science. Springer, pp. 417–432.

[57] Stell, J. G., 2007. Relations in mathematical morphology with applications to graphs and rough sets. In: Winter, S., Duckham, M., Kulik, L., Kuipers, B. (Eds.), Spatial Information Theory, 8th International Conference, COSIT 2007, Proceedings. Vol. 4736 of Lecture Notes in Computer Science. Springer, pp. 438–454.
[58] Stell, J. G., 2010. Relational granularity for hypergraphs. In: Szczuka, M. S., Kryszkiewicz, M., Ramanna, S., Jensen, R., Hu, Q. (Eds.), Rough Sets and Current Trends in Computing - 7th International Conference, RSCTC 2010, Proceedings. Vol. 6086 of Lecture Notes in Computer Science. Springer, pp. 267–276.
[59] Stell, J. G., 2012. Relations on hypergraphs. In: Kahl, W., Griffin, T. G. (Eds.), Relational and Algebraic Methods in Computer Science - 13th International Conference, RAMiCS 2012, Proceedings. Vol. 7560 of Lecture Notes in Computer Science. Springer, pp. 326–341.
[60] Stell, J. G., 2014. Formal concept analysis over graphs and hypergraphs. In: Croitoru, M., Rudolph, S., Woltran, S., Gonzales, C. (Eds.), Graph Structures for Knowledge Representation and Reasoning - Third International Workshop, GKR 2013, Revised Selected Papers. Vol. 8323 of Lecture Notes in Computer Science. Springer, pp. 165–179.
[61] Stell, J. G., 2015. Symmetric heyting relation algebras with applications to hypergraphs. J. Log. Algebr. Meth. Program. 84 (3), 440–455.
[62] Syau, Y. R., Fan, J., Lin, T. Y., 2012. Generalized infinitive rough sets based on reflexive relation. In: IEEE International Conference on Granular Computing.
[63] Tang, J., She, K., Min, F., Zhu, W., 2013. A matroidal approach to rough set theory. Theor. Comput. Sci. 471, 1–11.
[64] Trotter, W. T., Moore, J. I., 1976. Characterization problems for graphs, partially ordered sets, lattices, and families of sets. Discrete Mathematics 16 (4), 361 – 381.
[65] Wang, S., Zhu, Q., Zhu, W., Min, F., 2014. Graph and matrix approaches to rough sets through matroids. Inf. Sci. 288, 1–11.
[66] Wang, S., Zhu, Q., Zhu, W., Min, F., 2014. Rough set characterization for 2-circuit matroid. Fundam. Inform. 129 (4), 377–393.
[67] Wierman, M., 1999. Measuring uncertainty in rough set theory. International Journal of General Systems 28, 283–297.
[68] Wong, S. K. M., Ziarko, W., Ye, R. L., 1986. Comparison of rough-set and statistical methods in inductive learning. International Journal of Man-Machine Studies 25 (1), 53–72.
[69] Wu, W., Leung, Y., Mi, J., 2009. Granular computing and knowledge reduction in formal contexts. IEEE Trans. Knowl. Data Eng. 21 (10), 1461–1474.
[70] Wu, W.-Z., 2011. On some mathematical structures of t-fuzzy rough set algebras in infinite universes of discourse. Fundamenta Informaticae 108, 337–369.
[71] Wu, W.-Z., Xu, Y.-H., 2009. On fuzzy rough set algebras in infinite universes. In: Wen, P., Li, Y., Polkowski, L., Yao, Y., Tsumoto, S., Wang, G. (Eds.), Proceedings of RSKT 2009. pp. 312–319.
[72] Xie, J., Lin, T. Y., Zhu, W., 2012. Granular and rough computing on covering. In: IEEE International Conference on Granular Computing.
[73] Yang, X., Qian, Y., Yang, J., 2012. Hierarchical structures on multigranulation spaces. J. Comput. Sci. Technol. 27 (6), 1169–1183.
[74] Yao, J. T., Yao, Y. Y., 2002. A granular computing approach to machine learning. In: Wang, L., Halgamuge, S. K., Yao, X. (Eds.), FSDK'02, Proceedings of the 1st International Conference on Fuzzy Systems and Knowledge Discovery: Computational Intelligence for the E-Age. pp. 732–736.
[75] Yao, Y., 2001. On modeling data mining with granular computing. In: 25th International Computer Software and Applications Conference (COMPSAC 2001). IEEE Computer Society, p. 638.
[76] Yao, Y., 2004. A partition model of granular computing. Transactions on Rough Sets I I, 232–253.
[77] Yao, Y., Hu, Q., Yu, H., Grzymala-Busse, J. W. (Eds.), 2015. Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing, RSFDGrC 2015, Proceedings. Vol. 9437 of Lecture Notes in Computer Science. Springer.
[78] Zadeh, L. A., 1979. Fuzzy sets and information granularity. North-Holland, Amsterdam, pp. 3–18.
[79] Zadeh, L. A., 1997. Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic. Fuzzy Sets and Systems 90 (2), 111 –127.
[80] Zhu, W., Wang, S., 2013. Rough matroids based on relations. Inf. Sci. 232, 241–252.