NOTIONS FROM ROUGH SET THEORY IN A GENERALIZED DEPENDENCY RELATION CONTEXT

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Abstract. In this paper, we introduce a notion of generalized dependency relation between subsets of an arbitrary (not necessarily finite) set Ω starting with the classical Armstrong's rule. More specifically, we fix a given set system $\mathcal{F}$ on Ω and call any transitive binary relation $\leftarrow$ on the power set $\mathcal{P}(\Omega)$ such that

- $B \subseteq A \in \mathcal{F}$ implies $B \leftarrow A$;
- $B \leftarrow A$ if and only if $b \leftarrow A \ \ \forall b \in B$;

a $[\mathcal{F}]$-*dependency relation* on Ω. We use the generality of such a notion to investigate some common analogies between rough set theory on information tables, formal context analysis, Scott's information systems and possibility theory. More specifically, taking as inspirational models some classical notions derived by rough set theory on attribute set of an information table, we first generalize and study such notions for any $[\mathcal{F}]$-dependency relation. Next, we interpret such general results relatively to natural dependency relations derived by rough set theory on objects of an information table, formal context analysis, Scott's information systems and possibility theory. Finally, we study the generation of $[\mathcal{F}]$-dependency relations by starting from a fixed set $\mathcal{D}$ of subset ordered pairs of Ω.

1. Introduction

1.1. **General Premise.** Currently, information sciences, theoretical computer science and mathematics have reached very high levels of specialization, and each of their specific research field is going to develope very refined and sophisticated techniques and tools. Such a natural tendency is unavoidably irreversible. Nevertheless, we believe that an effort must be made to seek concepts and paradigms that are sufficiently general to establish bridges between the various specialized research sectors. To this regard, we consider some notions derived by rough set theory (briefly RST) suitable for our purposes. Although it has been arisen for practical reasons, RST had soon gained a notable theoretical and unifying power. Its fundamental starting point is the well-known *indiscernibility relation* for a given information table, where two objects are identified when they share the same properties provided by a fixed attribute subset. Based on such a notion, one can introduce the derived notions of *lower* and *upper approximation operators*, *discernibility matrix*, *core*, *reducts*, *positive region*, *attribute dependency measure* and so on.

In particular, the attribute dependency corresponds to the classical *functional dependency*, usually studied in database theory and related fields. Now, although the two definitions are mathematically equivalent, in literature they lead to different research fields and to different research perspectives. Furthermore, a notion of dependency has also been provided in formal context analysis (briefly FCA), in terms of *attribute subset implication*, though such notion is not mathematically equivalent to the previous ones. In addition, any *Scott's information system* is naturally endowed with a binary relation between subsets of a fixed set family, which, for specific aspects, has properties similar to the aforementioned subset relations. Finally, in possibility theory (briefly PT), it is also possible to define a type of binary relation between subsets of a given universe of a discourse, satisfying some properties analogous to those of the previous relations.

Nevertheless, let us report the lack of a standard and abstract notion of *dependency relation*. Indeed, as we deduce from the previous examples, the word *dependency* has used in several different contexts, but with non-equivalent meanings: functional dependency in database theory and data mining [49], attribute dependency in rough set theory and related fields [38, 45], implicational dependency in formal context analysis [31], dependency as a congruence relation [34], dependency as a property of particular subsets in matroid on rough sets [52, 53, 63].

The above considerations lead us to identify a unifying dependency framework in order to investigate the common properties of the previous specific dependency relations.

Let us make use of the following terminology. We call *attribute*-RST (abbreviated A-RST), the set of all RST notions which induce a set operator or a set system on the attribute set of information tables; for example, core, reducts, essential subsets, attribute dependency are typical examples of A-RST notions.

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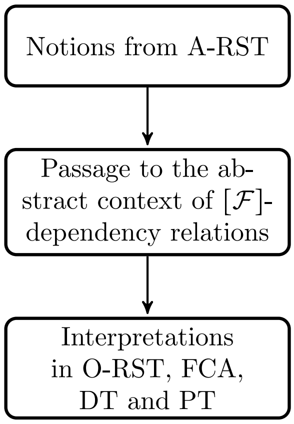
On the orther hand, we call *object*-RST (abbreviated O-RST), the set of all RST notions which induce a set operator or a set system on the object set of information tables; for example, indiscernibility relations, lower and upper approximation operators are typical examples of O-RST notions. Now, some typical A-RST notions are suitable to be generalized relatively to an arbitrary set endowed of an abstract dependency relation. Furthermore, these generalizations derived by A-RST can be interpreted and investigated for particular dependency relations derived in a natural way fromy O-RST, FCA, DT and PT.

Therefore, the basic motivations underlying the present work are:

(I) to establish a sufficiently general dependency notion (Sections 1 and 2);
(II) to interpret some classical A-RST notions in terms of such a generalized dependency (Section 3);
(III) to define and study the basic properties of such notions in a more generalized abstract context (Sections 4 and 5);
(IV) to interpret and investigate the general results established in (III) relatively to specific dependency relations derived by O-RST, FCA, DT and PT (Sections 6, 7, 8, 9);
(V) to outline the first elements for a study concerning the generation of abstract dependency relations (Section 10).

In the next figure we will outline the guidelines of what we have exposed so far.



The classical Armstrong's deduction rules represent the starting point of our analysis. We used them as specific types of axioms for a generic binary relation $\leftarrow$ between subsets of an arbitrary set, a priori not associated with an information table, relational database, formal context etc. Next, we provide specific interpretations when such a set is an attribute set of some of the aforementioned structures.

1.2. **A-RST and Armstrong's rules as Inspiration Models for Dependency Relations.** In order to better expose the content of this paper, we introduce a part of the mathematical formalism that we use in the sequel. Let Ω be a fixed non-empty set, $\mathcal{P}(\Omega)$ its power set, $SS(\Omega) := \mathcal{P}(\mathcal{P}(\Omega))$, $\mathcal{P}(\Omega)^2 := \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$ and $BREL(\Omega) := \mathcal{P}(\mathcal{P}(\Omega)^2)$. A family $\mathcal{F} \in SS(\Omega)$ is called a *set system* on Ω, whereas $\mathcal{D} \in BREL(\Omega)$ is called a *binary relation* on $\mathcal{P}(\Omega)$.

When Ω is a finite set of attributes relatively to a given relational scheme $R(\Omega)$ [28], Armstrong's axioms are the following three abstract inference rules between elements of $\mathcal{P}(\Omega)$ [4]:

$(A1)$ if $Y \subseteq X$ then $Y \leftarrow X$;
$(A2)$ if $Z \leftarrow Y$ and $Y \leftarrow X$ then $Z \leftarrow X$;
$(A3)$ if $Y \leftarrow X$ then $Y \cup Z \leftarrow X \cup Z$;

for all $X, Y, Z \in \mathcal{P}(\Omega)$.

The above axioms are used in $R(\Omega)$ in order to deduce all possible functional dependencies relatively to a fixed family $\mathcal{D} \subseteq \mathcal{P}(\Omega)^2$.

In this paper, we consider the aforementioned inference rules as three specific properties for a given binary relation $\leftarrow\, \in BREL(\Omega)$. Then, we will call any binary relation $\leftarrow$ on $\mathcal{P}(\Omega)$ having the three previous properties a *dependency relation* on Ω.

It is immediate to see that if a binary relation $\leftarrow\, \in BREL(\Omega)$ satisfies the properties $(A1)$ and $(A2)$, then $\leftarrow$ satisfies $(A3)$ if and only if it satisfies one of the following two properties:

$(A3')$ if $Y \leftarrow X$ and $Z \leftarrow X$ then $Y \cup Z \leftarrow X$;
$(A3'')$ if $y \leftarrow X$ for all $y \in Y$, then $Y \leftarrow X$;

for all $X, Y, Z \in \mathcal{P}(\Omega)$.
As we highlighted in the previous subsection, in mathematics and computer science, there are many situations in which one deals with formal structures which are characterized by binary relations on $\mathcal{P}(\Omega)$, where Ω is a finite or infinite set Ω of attributes or tokens. For example, just consider rough set theory (briefly RST), formal context analysis (briefly FCA), domain theory (briefly DT) and possibility theory (PT).
We take inspiration from the Armstrong's axioms used in database theory to propose an abstract definition of dependency relation relatively to an arbitrary fixed set Ω. However, in order to provide an unifying notion, we must recall that the classical *entailment relation* defined for Scott's information systems [1, 33, 48]) is a subset of $\mathcal{P}(\Omega) \times Con$, where Con is a subfamily of finite subsets of Ω. Therefore, the entailment relation is not defined on the whole $\mathcal{P}(\Omega)^2$. Now, in order to obtain a condition of generalized reflexivity similar to that stated in $(A1)$, it is necessary to restrict $(A1)$ to elements X belonging to a fixed set system on Ω. In Section 8, we will give the formal details concerning Scott's information systems. Let us see an example of Scott's information system (see Section 8 for further details).
Let $\Omega := \mathbb{N}$ and $\mathcal{F}$ be the family of all finite subsets of Ω. Let us consider the Scott's information system $\mathcal{S} := (\mathcal{F}, \leftarrow_{\mathcal{S}})$, where $\leftarrow_{\mathcal{S}}$ is the binary relation on $\Omega \times \mathcal{F}$ defined by $n \leftarrow_{\mathcal{S}} A$ if and only if there exists $a \in A$ such that $n \leq a$. Moreover, for any $B \in \mathcal{P}(\Omega)$, we set $B \leftarrow_{\mathcal{S}} A$ if and only if $b \leftarrow_{\mathcal{S}} A$ for all $b \in B$. Then, it is immediate to see that $(A2)$ and $(A3'')$ hold. Nevertheless, let us note that $(A1)$ holds only when the right member of the relation $\leftarrow_{\mathcal{S}}$ belongs to $\mathcal{F}$. Hence, in this case, we have that $B \subseteq A$ and $A \in \mathcal{F}$ imply $B \leftarrow_{\mathcal{S}} A$.
From what has been said before, our definition must depend on the choice of a fixed set system $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ which, in the most part of cases, coincides with $\mathcal{P}(\Omega)$ itself, but, in other important cases (namely for Scott's information systems), could be a suitable set system smaller than the whole $\mathcal{P}(\Omega)$. The generalization to a fixed set system $\mathcal{F} \in SS(\Omega)$ is thus necessary in order to obtain a sufficiently general dependency theory which also includes the Scott's information systems as model.
At this point, let us give the definition of $[\mathcal{F}]$-*dependency relation*.

Definition 1.1. *Let $\mathcal{F} \in SS(\Omega)$ and $\leftarrow \in BREL(\Omega)$. We say that $\leftarrow$ is a $[\mathcal{F}]$-*inclusive relation *on Ω if*

$(D1\mathcal{F})$ $B \subseteq A$ *and* $A \in \mathcal{F} \implies B \leftarrow A$.

When $\mathcal{F} = \mathcal{P}(\Omega)$, we simply say that $\leftarrow$ is an inclusive relation *on Ω, and we write simply $(D1)$ instead of $(D1\mathcal{F})$.*
*We say that $\leftarrow$ is a $[\mathcal{F}]$-*dependency relation *on Ω if $\leftarrow$ is $[\mathcal{F}]$-inclusive and it also satisfies the following two properties:*

- $(D2)$ $C \leftarrow B,\ B \leftarrow A \implies C \leftarrow A$ (transitivity)*;*
- $(D3)$ $B \leftarrow A \iff b \leftarrow A\ \forall b \in B$ (adjunctivity)*;*

for any $A, B, C \in \mathcal{P}(\Omega)$. Then, if $\leftarrow$ is a $[\mathcal{F}]$-dependency relation on Ω and $B \leftarrow A$, we say that B depends *on A. If $B \not\leftarrow A$, we say that B* does not depend *on A. In particular, when $\mathcal{F} = \mathcal{P}(\Omega)$ we simply say* dependency relation *on Ω instead of $[\mathcal{P}(\Omega)]$-dependency relation on Ω. W denote by $DREL(\mathcal{F}|\Omega)$ the set of all $[\mathcal{F}]$-dependency relations on Ω and simply by $DREL(\Omega)$ the set of all dependency relations on Ω.*

As a very simple and general example of a $[\mathcal{F}]$-inclusive relation on Ω, we can take any abstract simplicial complex $\mathcal{F}$ and consider the relation $B \leftarrow B' :\iff B' \setminus B \in \mathcal{F}$.
In this paper, we study in detail a $[\mathcal{F}]$-dependency relation $\leftarrow$ on Ω and determine two natural set operators and several set systems on Ω which are naturally induced by the relation $\leftarrow$. Furthermore, we establish general results concerning the $[\mathcal{F}]$-dependency relations, bearing in mind the standard set systems and set operators derived by RST in the context of an information table (see Section 3). Indeed, we study in our abstract framework the classical properties of well-known set systems (for example, the *reducts* [38]) and set operators derived by RST. On the other hand, we show that formal context analysis (FCA), domain theory (DT) and possibility theory (PT) find a natural collocation in this general and unifying context; in fact it is possible to fix three specific dependency relations in FCA, DT and PT and to interpret the aforementioned general results for these particular relations.
The attempt of a generalization of the well-known properties of such mathematical objects of RST is the basic motivation to delineate an axiomatic for an abstract dependency relation theory which includes in itself also the particular dependency relation induced by formal contexts, Scott's information systems and possibility measures. In general, let us notice that many notions derived by RST also have a theoretical mathematical relevance (for details see [43]); therefore a generalization of such notions relatively to an abstract dependency relation theory can be of interest also in purely mathematical contexts: topological

spaces, group actions, vector spaces endowed by bilinear forms, metric spaces, graphs and digraphs. A study towards this direction has been recently started in [25].

1.3. **Dependency in RST, FCA, DT and PT.** In RST and FCA the elements of Ω are *attributes* (equivalently, *properties*) and in DT they are *tokens*.
More in detail, in RST, these attributes of Ω are usually interrelated with a universe U of *objects* by means of a data table $\mathcal{I}$, called *information table*. An information table is a fundamental notion in classical RST [38, 39, 43], algebraic RST [10, 56], generalized RST [8, 12, 55], probabilistic RST [62] and in granular computing [42, 46, 57]. See also [27] for recent questions concerning the use of such a terminology in RST and related fields. Then, given an information table $\mathcal{I}$ with attribute set Ω and object set U, one can consider the Pawlak attribute dependency function $\gamma_{\mathcal{I}}$ [38] and to obtain the induced dependency relation $\leftarrow_{\mathcal{I}}$ given by $B \leftarrow_{\mathcal{I}} A$ if $\gamma_{\mathcal{I}}(A, B) = 1$.
The numerical function $\gamma_{\mathcal{I}}$ satisfies several interesting properties [60] and many of its variants, generalizations and refinements have been studied in RST [60], database theory [49], FCA [31] and more recently also in relation to some hypergraphs and graph models [13, 19]. Moreover, in Proposition 6.4 of [23] it has been proved that the usual Pawlak's exactness in the universe set U can be equivalently characterized in terms of dependency measure in an appropriate decision table.
In O-RST, relatively to a given information table $\mathcal{I}$, if we fix an attibute subset A, we can consider the dependency relation $\multimap_A$ on the power set $\mathcal{P}(U)$ induced by the Pawlak A-upper approximation operator $up_A : \mathcal{P}(U) \to \mathcal{P}(U)$. That is, for any object subsets X and Y, $\multimap_A$ is defined by $Y \multimap_A X$ if $Y \subseteq up_A(X)$.
In FCA, the attributes of Ω are related with the objects in U by means of a binary relation $\mathcal{R}$ induced by a formal context $\mathbb{K}$. Through the binary relation $\mathcal{R} \subseteq U \times \Omega$, another binary relation between subsets of Ω has been defined [31]. This relation is classically called *attribute subset implication* and, in our terminology, it is the standard model of dependency relation on Ω induced by a formal context having Ω as attribute set.
Particular types of dependency relations have been recently investigated in several contexts: discrete dynamical systems induced by graphs [2, 3] and sand piles [6], adjacency information tables induced by simple undirected graphs [16, 18, 47], digraphs [21].
In DT, the elements of Ω are usually called *tokens* and represent the basic units of information. The classical notion of Scott's information system $\mathcal{S}$ [48] requires a set system of finite subsets of tokens, denoted by Con and an *entailment* relation $\leftarrow_{\mathcal{S}} \subseteq \Omega \times Con$ that turns out to be a $[Con]$-dependency relation on Ω (Proposition 8.2).
In PT [29, 30, 61], Ω is usually interpreted as a *universe of discourse*, where is given a map $\Pi : \mathcal{P}(\Omega) \to [0, 1]$, called *possibility measure*, thanks to which we can introduce a binary relation $\leftarrow_{\Pi}$ that is a dependency relation on Ω (see Proposition 9.2).
In the specific case $\mathcal{F} = \mathcal{P}(\Omega)$ and Ω is finite, the three properties which we propose to characterize an abstract dependency relation are substantially equivalent to provide a closure operator σ on Ω. Therefore, in mathematics and information sciences, whenever we have a set Ω endowed with a closure operator σ, we also have a dependency relation and vice versa, in a bijective correspondence. Hence, to any dependency relation $\leftarrow$ on Ω there corresponds a closure operator $Dc_{\leftarrow}$ on Ω, which uniquely characterizes the indiscernibility relations on U when $\leftarrow$ coincides with $\leftarrow_{\mathcal{I}}$ (see [7, 26]) and the formal concepts of $\mathbb{K}$ when $\leftarrow$ coincides with $\leftarrow_{\mathbb{K}}$ (see [31]). We call any subset $A \subseteq \Omega$ such that $A = Dc_{\leftarrow}(A)$ $\leftarrow$*closed* and we denote by $CLOS(\leftarrow)$ the set of all $\leftarrow$closed subsets of Ω. The closed sets coincide with the maximum partitioners introduced in [17] when $\leftarrow$ is induced by an information table having Ω as attribute set. In a similar way, $CLOS(\leftarrow)$ coincides with the intent set system of a formal context when Ω is its attribute set.
The aforementioned correspondence between dependency relations and closure systems can also be established when the set Ω is not finite and one has some specific closure operator on Ω. For instance, this happens when one deals with abstract approximation spaces [9, 11], infinite information tables [43, 44], Scott topologies [1], metric spaces [49]. This consideration justifies the study of a dependency relation both in finite and non-finite case.
On the other hand, although there are very frequent situations in which one has a closure operator on Ω (topologies, matroids, posets etc), a very simple practical way to provide a closure system on Ω is that to use an information table having Ω as its attribute set.
Let us also note here that, when Ω is finite, the above general notion of dependency relation is related to the information tables on Ω also in an inverse way. Indeed, in [25], it has been proved that *any* closure system on a finite set Ω can be induced by means of an information table on Ω, namely:

Theorem 1.2. *For any closure system $\mathfrak{S}$ on Ω, there exists an information table $\mathcal{I}$ such that $CLOS(\leftarrow_{\mathcal{I}})$ coincides with $\mathfrak{S}$.*

The above discourse leads us to notice that the abstract notion of dependency relation provides a further unifying paradigm between those research fields of mathematics and information sciences which implicitly use specific relations falling within such a more general notion.

1.4. **Section Content.** We now briefly describe the content of the paper. In Section 2, we introduce the basic notions and provide some essential results concerning the $\mathcal{F}$-inclusive relations. In Section 3, we interpret the classical attribute subset dependency $\leftarrow_{\mathcal{I}}$ for Pawlak's information tables as a particular case of dependency relation. In Section 4, we study the main properties of $[\mathcal{F}]$-dependency relations, associating with them two set operators and the families of their fixed points. In Section 5, we introduce the notions of dependency reducts and essential subsets of a given $[\mathcal{F}]$-dependency relations and investigate their main properties. In Section 6 we apply the general notions treated in Section 4 and Section 5 to the dependency relation induced by the upper approximation operator on the object set associated with an information table. Thus, we investigate in an unusual way reducts, core, essentials and the related operators, which are instead classically studied by means of indiscernibility. In Section 7, we interpret the classical attribute subset implication $\leftarrow_{\mathbb{K}}$ as another case of dependency relation. Even in this case, we investigate the role of each structure and set operator associated with a dependency relation. In Section 8, we see that the entailment relation $\leftarrow_{\mathcal{S}}$ is a particular case of $[\mathcal{F}]$-dependency relation on a given set Ω. Therefore, we are enabled to provide some examples in order to understand the role of the aforementioned structures and set operators in the theory of Scott's information systems. In Section 9, we analyze the way in which dependency relations arise in possibility theory through the possibility measure. Therefore, even in this context, we provided a model for our abstract dependency theory. In Section 10, we define a closure operator on $\mathcal{P}(\Omega)^2$ associating with each binary relation $\mathcal{D}$ on Ω a $\mathcal{F}$-dependency relation, denoted by $\mathcal{D}^+_{\mathcal{F}}$ and called $\mathcal{F}$-*dependency envelope*, containing $\mathcal{D}$ itself. Under particular hypothesis, it is possible to generate $\mathcal{D}^+_{\mathcal{F}}$ inductively starting with $\mathcal{D}$. As an example, we find a subset family $\mathcal{E}_{ker}$ whose dependency envelope coincides with a given dependency relation $\mathcal{E}$ on a finite set.

2. Recalls, Notations and first Basic Results

We will denote by Ω a fixed non-empty set (not necessarily finite) and by $\mathcal{P}(\Omega)$ its power set. If $x \in \Omega$, we write simply x instead of $\{x\}$. We denote by $\mathcal{P}_f(\Omega)$ the family of all finite subsets of Ω and set $\mathcal{P}(\Omega)^2 := \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$.

In several cases we must distinguish ordered pairs $(B, A) \in \mathcal{P}(\Omega)^2$ such that $B \subseteq A$, which we call *trivial pairs*, or such that $B \nsubseteq A$, which we call *non-trivial pairs*. To describe these two different situations, we set

$$\hat{\Omega}_{tr} := \{(B, A) \in \mathcal{P}(\Omega)^2 : B \subseteq A\} \text{ and } \hat{\Omega}_{ntr} := \{(B, A) \in \mathcal{P}(\Omega)^2 : B \nsubseteq A\}$$

If $\mathcal{F} \in SS(\Omega)$, we set $\mathcal{F}^c := \mathcal{P}(\Omega) \setminus \mathcal{F}$.

In what follows, if a binary relation on $\mathcal{P}(\Omega)$ is denoted by $\leftarrow$ (or with a similar symbol) we write $B \leftarrow A$ instead of $(B, A) \in \leftarrow$. On the other hand, if the binary relation on $\mathcal{P}(\Omega)$ is denoted by $\mathcal{D}$ (or with a similar capital letter), we write $(B, A) \in \mathcal{D}$ instead of $B\,\mathcal{D}\,A$.

Let $\leftarrow \in BREL(\Omega)$. Starting with it, we can define the following induced relations, that we will use several times in this work.

- If $\mathcal{B}, \mathcal{A} \in SS(\Omega)$, we set

$$\mathcal{B} \leftarrow^{ext} \mathcal{A} :\Longleftrightarrow B \leftarrow A,\ \forall B \in \mathcal{B}, A \in \mathcal{A} \tag{1}$$

- If $A, B \in \mathcal{P}(\Omega)$, we set

$$B \nleftarrow_{\forall} A :\Longleftrightarrow b \nleftarrow A,\ \forall b \in B \tag{2}$$

A *set operator* on Ω is a map $\sigma : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, and we denote by $OP(\Omega)$ the set of all the set operators on Ω.

In this paper, a theoretical relevant role is played by the interrelations between the sets $SS(\Omega)$, $BREL(\Omega)$ and $OP(\Omega)$. Formally, these interrelations can be described by using some specific maps, which are defined as follows:

- $Fix : OP(\Omega) \to SS(\Omega)$, where

$$Fix(\sigma) := \{A \in \mathcal{P}(\Omega) : \sigma(A) = A\};$$

- $Int : SS(\Omega) \to OP(\Omega)$, where
$$Int_{\mathcal{F}}(C) := \bigcap\{A \in \mathcal{F} : C \subseteq A\};$$
- $Dc : SS(\Omega) \times BREL(\Omega) \to OP(\Omega)$, where
$$Dc_{\mathcal{F},\leftarrow}(C) := \bigcup\{B \in \mathcal{F} : B \leftarrow C\},$$
and, in particular, we set $Dc_{\leftarrow} := Dc_{\mathcal{P}(\Omega),\leftarrow}$;
- $Inc : OP(\Omega) \to BREL(\Omega)$, where
$$Inc(\sigma) := \{(B,A) \in \mathcal{P}(\Omega)^2 : B \subseteq \sigma(A)\};$$
- $DP_{\mathcal{F}} : BREL(\Omega) \to SS(\Omega)$, where
$$DP_{\mathcal{F}}(\mathcal{D}) := \{A \in \mathcal{F} :\ (B,C) \in \mathcal{D} \text{ and } C \subseteq A \implies B \subseteq A\},$$
and in particular, we set $DP := DP_{\mathcal{P}(\Omega)}$;

for any $\sigma \in OP(\Omega)$, $\mathcal{F} \in SS(\Omega)$, $C \in \mathcal{P}(\Omega)$ and $\mathcal{D} \in BREL(\Omega)$.
Since we mainly propose to investigate the general properties of the dependency relations relatively to a fixed set system $\mathcal{F} \in SS(\Omega)$, in this paper we will establish several types of results concerning the links between specific properties of set systems, set operators and binary relations relatively to the above introduced maps. To help the reader become familiar with these maps, we provide an example based on a specific kind of set systems and set operators that we also use when we will deal with the case of the Scott's information systems in Section 8.

Example 2.1. Let us take $\Omega := \mathbb{N}$, $\mathcal{F} := \mathcal{P}_f(\Omega)$ and $\mathcal{D} := \{(B,A) \in \mathcal{P}_f(\Omega)^2 :\ \forall b \in B\ \exists a \in A : b \le a\}$. Then it is easy to verify that $Dc_{\mathcal{F},\mathcal{D}} = \sigma$, where $\sigma : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is the set operator given by:
$$\sigma(A) := \begin{cases} \{0,1,\dots,\max(A)\} & \text{if } A \in \mathcal{P}_f(\Omega) \\ \varnothing & \text{otherwise} \end{cases}$$
Moreover, we also obtain the following identities:
$$Fix(\sigma) = \{A \in \mathcal{P}(\Omega) :\ A = \{0,1,\dots,n\} \text{ for some } n \in \mathbb{N}\},$$
$$Int_{\mathcal{F}}(A) = \begin{cases} A & \text{if } A \in \mathcal{P}_f(\Omega) \\ \varnothing & \text{otherwise,} \end{cases}$$
$$Inc(\sigma) = \{(\varnothing,A) :\ A \in \mathcal{P}(\Omega)\} \cup \{(B,A) \in \mathcal{P}_f(\Omega) \setminus \{(\varnothing,\varnothing)\} :\ B \subseteq \{0,1,\dots,\max(A)\}\},$$
$$DP_{\mathcal{F}}(\mathcal{D}) = Fix(\sigma) \cup \mathbb{N}.$$

An important role in the present work is played by the set systems that are closed with respect to arbitrary intersections. More specifically, a set system $\mathcal{F} \in SS(\Omega)$ is called a *closure system* on Ω [5] if $\Omega \in \mathcal{F}$ and whenever $\{A_i : i \in I\} \subseteq \mathcal{F}$ we have that $\bigcap_{i \in I} A_i \in \mathcal{F}$. Let $CLSY(\Omega)$ the set of all closure systems on Ω.
In the next result we establish two basic properties of the map $DP_{\mathcal{F}}$.

Proposition 2.2. *Let $\mathcal{D}, \mathcal{D}' \in BREL(\Omega)$, then*
$$\mathcal{D} \subseteq \mathcal{D}' \implies DP_{\mathcal{F}}(\mathcal{D}) \supseteq DP_{\mathcal{F}}(\mathcal{D}'). \tag{3}$$
Moreover, if $\mathcal{F} \in CLSY(\Omega)$, then $DP_{\mathcal{F}}(\mathcal{D}) \in CLSY(\Omega)$.

Proof. Since $\Omega \in \mathcal{F}$, clearly it also belongs to $DP_{\mathcal{F}}(\mathcal{D})$. Let $\{A_i : i \in I\} \subseteq DP_{\mathcal{F}}(\mathcal{D})$. By the fact that $\mathcal{F} \in CLSY(\Omega)$, we deduce that $A := \bigcap_{i \in I} A_i \in \mathcal{F}$. Let $(B,C) \in \mathcal{D}$ and assume that $C \subseteq A$. Then $C \subseteq A_i$ for any $i \in I$ and, thus, $B \subseteq A_i$ for any $i \in I$. This entails that $B \subseteq A$ and the proof concludes here. ☐

In several results of this paper, we use the first two properties characterizing a closure operator relatively to a given set system $\mathcal{F}$. Therefore we give the following definition.

Definition 2.3. *Let $\mathcal{F} \in SS(\Omega)$. We say that a set operator $\sigma \in OP(\Omega)$ is a $\mathcal{F}$-*preclosure operator *on Ω if for any $A, B \in \mathcal{F}$, the following hold:*

$(CL1)$ $A \subseteq \sigma(A)$ *(extensivity relatively to $\mathcal{F}$);*
$(CL2)$ $A \subseteq B$ *implies* $\sigma(A) \subseteq \sigma(B)$ *(monotonicity relatively to $\mathcal{F}$);*

*Moreover, we also say that σ is an $\mathcal{F}$-*closure operator *on Ω if it is a $\mathcal{F}$-preclosure operator and for any $A \in \mathcal{F}$, it results that:*

$(CL3)$ $\sigma(\sigma(A)) = \sigma(A)$ *(idempotence relatively to $\mathcal{F}$).*

Based on Definition 2.3, it follows that a classical closure operator on Ω (as given in [5]) coincides with a $\mathcal{P}(\Omega)$-closure operator. We denote by $CLOP(\mathcal{F}|\Omega)$ the collection of all $\mathcal{F}$-closure operators on Ω and (in particular) by $CLOP(\Omega)$ that of all closure operators on Ω. Let us recall a standard result of the theory of closure systems.

Theorem 2.4. *Let $\mathcal{F} \in CLSY(\Omega)$ and $\sigma \in CLOP(\Omega)$. Then $Int_{\mathcal{F}} \in CLOP(\Omega)$, $Fix(\sigma) \in CLSY(\Omega)$ and*

$$Fix(Int_{\mathcal{F}}) = \mathcal{F},\ Int_{Fix(\sigma)} = \sigma \tag{4}$$

Moreover, the poset $(\mathcal{F}, \subseteq)$ is a complete lattice in which meet means intersection.

The identities given in (4) assert that the notions of closure system and closure operator on the same set Ω can be identified by means of the maps Int and Fix.
In the sequel, we will deal with given set systems satisfying other properties as well as being intersection-closed. Let $\mathcal{F} \in SS(\Omega)$. Then:

- $\mathcal{F}$ is an *abstract simplicial complex* (or simply an *abstract complex*) on Ω if $\emptyset \in \mathcal{F}$ and whenever $Y \in \mathcal{F}$ and $Z \subseteq Y$, then $Z \in \mathcal{F}$.
  We denote by $AC(\Omega)$ the set of all abstract simplicial complexes on Ω.
- $\mathcal{F}$ is *union closed* if whenever $\{A_i : i \in I\} \subseteq \mathcal{F}$ then $\bigcap_{i\in I} A_i \in \mathcal{F}$.
  We denote by $UCL(\Omega)$ the set of all union closed systems on Ω.

Finally, we say that a subset $Y \subseteq \Omega$ is a *transversal* of $\mathcal{F}$ if $Y \cap A \neq \emptyset$ for each non-empty $A \in \mathcal{F}$. Moreover, a transversal A of $\mathcal{F}$ is *minimal* if no proper subset of A is again a transversal of $\mathcal{F}$. We denote by $Tr(\mathcal{F})$ the family of all minimal transversals of $\mathcal{F}$.

Remark 2.5. *When $\mathcal{F} = \mathcal{P}(\Omega)$, and $\leftarrow$ satisfies $(D1)$ and $(D2)$, then it is immediate to see that the condition $B \leftarrow A$ implies $b \leftarrow A$ $\forall b \in B$. Therefore, in order to show that a binary relation $\leftarrow$ is a dependency relation, it is sufficient to verify that $\leftarrow$ satisfies $(D1)$, $(D2)$ and the only part $\Longrightarrow$ in $(D3)$.*

In the next result, we give an alternative characterization for a dependency relation.

Proposition 2.6. *Let $\leftarrow\, \in BREL(\Omega)$ be inclusive and transitive. Then, $\leftarrow$ satisfies $(D3)$ if and only if the following condition holds:*

$(D3')$ $B_i \leftarrow A, \forall i \in I \Longrightarrow \bigcup_{i\in I} B_i \leftarrow A$ (union additivity)

Proof. Assume that $(D3)$ holds and that $B_i \leftarrow A$, $\forall i \in I$. Let $b \in \cup_{i\in I} B_i$. Then there exists $i \in I$ such that $b \in B_i$. Since $\leftarrow$ is inclusive, we have that $b \leftarrow B_i$. Then, since $B_i \leftarrow A$ and $\leftarrow$ is also transitive, we have $b \in A$. This shows that $b \in A$ for all $b \in \cup_{i\in I} B_i$. By $(D3)$ it follows then that $\bigcup_{i\in I} B_i \leftarrow A$.
Conversely, assume that $(D3')$ holds and that $b \leftarrow A$ for any $b \in B$. Then, by $(D3')$ we have that $B = \bigcup_{b\in B} b \leftarrow A$, that is $(D3)$ holds. □

As already discussed in the introductory section, the fixed points of the set operator $Dc_{\leftarrow}$ induced by an $[\mathcal{F}]$-dependency relation play a basic role in our work. Now, since the set operator $Dc_{\leftarrow}$ is a closure operator when $\leftarrow$ is a dependency relation, we are led to interpret the fixed points of $Dc_{\leftarrow}$ as *closed subsets* relatively to the relation $\leftarrow$, even in more general situations. For this reason, for basic motivations originating from RST and that will be discussed in detail in Section 3 and for the results established in Section 4, we will adopt the following terminology.

Definition 2.7. *Let $A \in \mathcal{P}(\Omega)$.*

- *We call* $(\leftarrow, \mathcal{F})$–Pawlak preclosure operator *the set operator $Dc_{\mathcal{F},\leftarrow}$ and* $(\leftarrow, \mathcal{F})$–Pawlak preclosure *of A the subset $Dc_{\mathcal{F},\leftarrow}(A)$.*
- *We say that A is a* $(\leftarrow, \mathcal{F})$–Pawlak preclosed subset *if $A = Dc_{\mathcal{F},\leftarrow}(A)$.*
- *We denote by $CLOS(\leftarrow)$ the set system of all $(\leftarrow, \mathcal{F})$–Pawlak preclosed subsets, that is*

  $$CLOS(\leftarrow) = Fix(Dc_{\mathcal{F},\leftarrow})$$

  Moreover, we also set $CLOS_{\leftarrow}(A) := \{A \cap B : B \in CLOS(\leftarrow)\}$, therefore, in particular, $CLOS(\leftarrow)$ coincides with $CLOS_{\leftarrow}(\Omega)$.
- *In Section 4 we will see that the set operator $Dc_{\mathcal{F},\leftarrow}$ is a closure operator on Ω when $\mathcal{F} = \mathcal{P}(\Omega)$, i.e. when $\leftarrow\, \in DREL(\Omega)$. Therefore, in such a case, we call simply* $\leftarrow$Pawlak closure operator *the set opeartor $Dc_{\leftarrow}$, $\leftarrow$Pawlak closure of A the subset $Dc_{\leftarrow}(A)$ and* $\leftarrow$Pawlak's closed subset *any fixed point of $Dc_{\leftarrow}$.*

Remark 2.8. *In this paper, we use dependency relations induced by information tables $\mathcal{I}$, formal contexts $\mathbb{K}$, Scott's information systems $\mathcal{S}$ and possibility measures Π. In all such cases, when we denote set operators, set systems and maps that depend on such specific dependency relations we will use simply the symbols $\mathcal{I}$, $\mathbb{K}$, $\mathcal{S}$ and Π instead of $\leftarrow_{\mathcal{I}}$, $\leftarrow_{\mathbb{K}}$, $\leftarrow_{\mathcal{S}}$ and $\leftarrow_{\Pi}$ respectively. For instance, if we deal with an information table $\mathcal{I}$, we write $Dc_{\mathcal{I}}$ instead of $Dc_{\leftarrow_{\mathcal{I}}}$, $CLOS_{\mathcal{I}}(A)$ instead of $CLOS_{\leftarrow_{\mathcal{I}}}(A)$, $CLOS(\mathcal{I})$ instead of $CLOS(\leftarrow_{\mathcal{I}})$ and, similarly, for the other set operators and set systems depending on $\leftarrow_{\mathcal{I}}$. Analogously, in reference to terminologies that we will introduce in the next sections, we will say that $Dc_{\mathcal{I}}$ is the $\mathcal{I}$-*dependency closure *instead of* $\leftarrow_{\mathcal{I}}$-dependency closure*, and so on.*

3. Dependency by A-RST

In this paper we establish general results concerning the $[\mathcal{F}]$-dependency relations, bearing in mind four basic models: information tables, formal contexts, Scott's information systems and universes of discourse relatively to the possibility theory. However, the basic source of our inspiration models is RST. The results of RST established in this section have been expressed in a form which is preliminary to understand the results and the various interpretations obtained in firstly more general perspective and, next, in FCA, DT and PT.

3.1. Models by Attribute Dependency.

Let us recall the basic notion of *information table* (for details concerning RST applied on information tables we refer the reader to [38, 39, 40, 41, 43]).
An *information table* [38] on Ω is a structure $\mathcal{I} = (U, F, \Lambda)$, where U and Λ are non-empty sets, and $F : U \times \Omega \to \Lambda$ is a given map relating the attributes of Ω and the elements of U through specific values of the set Λ. Traditionally, the elements of the set U are called *objects*, those of Λ *values* and F is called the *information map* of $\mathcal{I}$. Let $INFT(\Omega)$ be the collection of all information tables on Ω and $\mathcal{I} = (U, F, \Lambda) \in INFT(\Omega)$.
In [38], Pawlak introduced a numerical measure to evaluate a specific type of *dependency* between attribute subsets of $\mathcal{I}$, that is to measure a degree of correlation between attribute subsets. Such an attribute subset dependency measure is based on the following *A-indiscernibility relation* with respect to a given attribute subset $A \subseteq \Omega$: if $u, u' \in U$, then

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a),$$

for all $a \in A$. Let $[u]_A$ be the equivalence class of u with respect to $\equiv_A$, that we call *A-indiscernibility class* of u. Then, for any $A, B \in \mathcal{P}(\Omega)$, the Pawlak's $\mathcal{I}$-*positive region* relative to A and B is

$$\Gamma_{\mathcal{I}}(A, B) := \{u \in U : [u]_A \subseteq [u]_B\},$$

and the $\mathcal{I}$-*attribute dependency measure* between A and B is defined by

$$\gamma_{\mathcal{I}}(A, B) := \frac{|\Gamma_{\mathcal{I}}(A, B)|}{|U|},$$

when the object set U is finite.
Then, in the Pawlak's perspective, one says that B *depends* from A relatively to the information table $\mathcal{I}$, denoted by $B \leftarrow_{\mathcal{I}} A$, when $\Gamma_{\mathcal{I}}(A, B) = U$. In particular, when U is finite:

$$B \leftarrow_{\mathcal{I}} A :\Longleftrightarrow \Gamma_{\mathcal{I}}(A, B) = U \iff \gamma_{\mathcal{I}}(A, B) = 1 \tag{5}$$

It is immediate to verify that $\leftarrow_{\mathcal{I}}$ is a dependency relation on Ω in the sense of Definition 1.1.

Proposition 3.1. $\leftarrow_{\mathcal{I}} \in DREL(\Omega)$.

In the present work, relatively to the binary relation $\leftarrow_{\mathcal{I}}$ we use the following terminology.

Definition 3.2. *We call $\leftarrow_{\mathcal{I}}$ the $\mathcal{I}$-*dependency relation *on Ω, or simply the $\mathcal{I}$-*dependency *on Ω and, if $B \leftarrow_{\mathcal{I}} A$, we say that B is $\mathcal{I}$-*dependent *on A*

For any $A \in \mathcal{P}(\Omega)$, we set

$$\pi_{\mathcal{I}}(A) := \{[u]_A : u \in U\},$$

i.e. the set partition on U induced by $\equiv_A$, that we call *A-indiscernibility partition* of U.
In terms of indiscernibility relations and indiscernibility partitions, the Pawlak's dependency $\leftarrow_{\mathcal{I}}$ can be expressed in the following equivalent ways:

$$B \leftarrow_{\mathcal{I}} A :\Longleftrightarrow (\forall u, u' \in U,\ u \equiv_A u' \implies u \equiv_B u') \iff \pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(B), \tag{6}$$

where the symbol $\preceq$ denotes the usual refining partial order between set partitions of U (see [5]). Moreover, in some cases we will write $\pi_{\mathcal{I}}(A) \prec \pi_{\mathcal{I}}(B)$, instead of $\pi_{\mathcal{I}}(A) \preceq \pi_{\mathcal{I}}(B)$ and $\pi_{\mathcal{I}}(A) \neq \pi_{\mathcal{I}}(B)$.
Furthermore, in terms of indiscernibility relation, the set operator $Dc_{\mathcal{I}}$ assumes the following form:

Proposition 3.3. *For any $A \in \mathcal{P}(\Omega)$ we have that*

$$Dc_{\mathcal{I}}(A) = \{b \in \Omega : \forall u, u' \in U,\ u \equiv_A u' \implies u \equiv_b u'\} \tag{7}$$

Proof. Let $b \in \Omega$ be such that $\forall u, u' \in U$, $u \equiv_A u' \implies u \equiv_b u'$. Then, $b \leftarrow_{\mathcal{I}} A$. This implies that $b \in Dc_{\mathcal{I}}(A)$. Conversely, let $b \in Dc_{\mathcal{I}}(A)$. Then, there exists $B \in \mathcal{P}(\Omega)$ such that $b \in B$ and $B \leftarrow_{\mathcal{I}} A$. In other terms, $\forall u, u' \in U$, $u \equiv_A u' \implies u \equiv_B u'$. This means that $F(u, b') = F(u', b')$ for any $b \in B$ and, in particular, also for the element b. This shows that $u \equiv_B u'$ and the claim follows. □

The set operator expressed in the form (7) has been widely studied relatively to the lattice of all indiscernibility partitions [17], in connection with specific micro and macro models in granular computing [7, 26], simple undirected graphs [14, 15, 20], knowledge pairing systems [22] and decision tables [23]. More specifically, in [17] it has been shown that $Dc_{\mathcal{I}}$ is a closure operator on Ω and its associated closure system

$$CLOS(\mathcal{I}) := Fix(Dc_{\mathcal{I}}) \tag{8}$$

becomes a complete lattice which is order isomorphic to the lattice

$$\mathbb{P}_{ind}(\mathcal{I}) := (\{\pi_{\mathcal{I}}(A) : A \in \mathcal{P}(\Omega)\}, \leq) \tag{9}$$

of all indiscernibility partitions (see also [58] for a discussion of such order structure in a granular computing context).
Since in the present paper we are particularly interested to understand how the closure property of $Dc_{\mathcal{I}}$ are interrelated to the $\mathcal{I}$-dependency, we use the following terminology.

Definition 3.4. *We call $Dc_{\mathcal{I}}$ the $\mathcal{I}$*-Pawlak's dependency closure operator.

Regarding the set system $DP(Dc_{\mathcal{I}})$ and the relation $Inc(Dc_{\mathcal{I}})$, we have the following identities:

$$DP(Dc_{\mathcal{I}}) = CLOS(\mathcal{I}), \tag{10}$$

and

$$Inc(Dc_{\mathcal{I}}) = \leftarrow_{\mathcal{I}}, \tag{11}$$

which will be proved in Section 4 for any abstract dependency relation. Actually, the identity given in (10) is a direct consequence of part (iii) in Proposition 4.12, whereas (11) follows by Theorem 4.8.
As a direct consequence of (6) and (7), we also deduce the following equivalent expression for the set operator $Dc_{\mathcal{I}}$:

$$Dc_{\mathcal{I}}(A) = \{b \in \Omega : b \leftarrow_{\mathcal{I}} A\} \tag{12}$$

The identity established in (12) is particularly relevant to relate the most part of this section to the generalizations concerning the abstract dependency relations.
To this regard, one of our main purposes will be that to generalize the classical notion of *Pawlak's core* to any abstract dependency relation. Then, it will be convenient to express the Pawlak's core as a specific set operator $I_{\mathcal{I}} \in OP(\Omega)$ and connect it explicitly with the $\mathcal{I}$-dependency.
We recall that, given $A \in \mathcal{P}(\Omega)$, an attribute $a \in A$ is called *indispensable* (relatively to the subset A) if $\pi_{\mathcal{I}}(A) \neq \pi_{\mathcal{I}}(A \setminus a)$ [38]. The subset of all indispensable attributes $a \in A$ is called the *$\mathcal{I}$-Pawlak's core* of A, and it is a basic tool in RST applied on information tables. Clearly, based on such a definition, we can interpret the $\mathcal{I}$-Pawlak's core as a set operator $Ic_{\mathcal{I}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined by

$$Ic_{\mathcal{I}}(A) = \{a \in A : \exists\, u, u' \in U :\ u \equiv_A u' \text{ and } u \not\equiv_{A \setminus a} u'\}, \tag{13}$$

for any $A \in \mathcal{P}(\Omega)$.
In [7, 24] it has been showed that the family of all fixed points of the set operator $Ic_{\mathcal{I}}$ is an abstract simplicial complex on Ω whose structure is strictly connected with the closure system $CLOS(\mathcal{I})$ as well as the independent set family of a matroid are connected with its flat subset family. Moreover, it has also been provided a characterization of the members of $CLOS(\mathcal{I})$ and $Fix(Ic_{\mathcal{I}})$, respectively, in terms of maximal and minimal elements in the equivalence classes of attribute subsets having the same indiscerinibility partition. Therefore, the elements of $CLOS(\mathcal{I})$ and $Fix(Ic_{\mathcal{I}})$ have been called respectively *maximal partitioners* and *minimal partitioners* relatively to the information table $\mathcal{I}$ (see [7, 26] for details).
By (6) and (13) we deduce that

$$Ic_{\mathcal{I}}(A) = \{a \in A : a \not\leftarrow_{\mathcal{I}} A \setminus a\} \tag{14}$$

Now, the set operators $Dc_{\mathcal{I}}$ and $Ic_{\mathcal{I}}$, respectively given in the forms (12) and (14), are the basic theoretical tools of our next investigation concerning the generalized dependency relations.

For example, it is natural to ask how we can interpret in terms of dependency relations the condition given in (14) that an element $a \in A$ satisfies the condition $a \not\leftarrow_{\mathcal{I}} A \smallsetminus a$, i.e. that the element $a \in A$ does not depend on the remaining elements of A (if we exclude it). Then, it is natural to think of such a property as a sort of *independency* of the element a relatively to the subset A. In other terms, we can image $Ic_{\mathcal{I}}(A)$ as the subset of all *independent* elements of A with respect to the relation $\leftarrow_{\mathcal{I}}$. Hence, the notion of Pawlak's indispensability naturally becomes a notion of independence with respect to a given relation. Subsequently, if $Ic_{\mathcal{I}}(A) = A$, we infer that all elements of A are independent. Therefore it is natural to call *independent subsets* of $\mathcal{I}$ (that is of $\leftarrow_{\mathcal{I}}$) the members of $Fix(Ic_{\mathcal{I}})$, and to set

$$INDP(\mathcal{I}) \coloneqq Fix(Ic_{\mathcal{I}})$$

In some cases, we localize to a fixed attribute subset A the previous set system $INDP(\mathcal{I})$; namely we set:

$$INDP_{\mathcal{I}}(A) \coloneqq \{B \in INDP(\mathcal{I}) : B \subseteq A\}, \tag{15}$$

for any $A \in \mathcal{P}(\Omega)$.
In view of the link between Pawlak's core and independency notion, we use the following terminology.

Definition 3.5. *We call $Ic_{\mathcal{I}}$ the $\mathcal{I}$-*Pawlak's independency core operator*.*

The closed and independent subsets, and their corresponding set operators, can be investigated for any $[\mathcal{F}]$-*dependency relation* on Ω. In such an investigation, that we will undertake in Section 4, we find properties and interrelations between closed and independent subsets when the system $\mathcal{F}$ satisfies specific conditions, for example when $\mathcal{F}$ is an abstract simplicial complex or when it is union closed. The condition of being an abstract simplicial complex or union closed families will be encountered when we study an information table. In fact, the set system $INDP(\mathcal{I})$ is an abstract simplicial complex (see [7]); whereas any equivalence class of attribute subsets having the same indiscernibility partition is union closed (see [17]).
In the finite case (i.e. when Ω is a finite set), the dependency relations induced by information tables characterize completely any dependency relation. In fact, in the next result we show that in the finite case any dependency relation on Ω is a $\mathcal{I}$-dependency relation, for some information table $\mathcal{I} \in INFT(\Omega)$.

Theorem 3.6. *Let Ω be a finite set and let $\leftarrow \in DREL(\Omega)$. Then there exists an information table $\mathcal{I} \in INFT(\Omega)$ such that $\leftarrow_{\mathcal{I}}$ coincides with $\leftarrow$.*

Proof. Let $\mathcal{H} \coloneqq CLOS(\leftarrow)$. Since $\mathcal{H}$ is a closure system, it is also a lattice with respect the set theoretical inclusion whose top element is Ω (see [5]). On the other hand, by Theorem 1.2, for any finite closure system there exists a information table on the same ground set whose closed subset family coincides with the given closure system. Therefore, in our case there exists a information table $\mathcal{I} \in INFT(\Omega)$ such that $\mathcal{H} = CLOS(\mathcal{I})$. This entails that $Dc_{\mathcal{I}} = Dc_{\leftarrow}$ and so, by (20), we deduce that the dependency relations $\leftarrow$ and $\leftarrow_{\mathcal{I}}$ coincide. □

Given an information table $\mathcal{I} \in INFT(\Omega)$, there is an equivalence relation $\approx_{\mathcal{I}}$ on $\mathcal{P}(\Omega)$, that is naturally associated with $\mathcal{I}$, of which we have already mentioned earlier. For any $A, B \in \mathcal{P}(\Omega)$, we set

$$A \approx_{\mathcal{I}} B :\Longleftrightarrow \pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(B)$$

This relation has been studied in [17] in connection with the Yao lattice model [58] and in [7] (where it was called *indistinguishability relation*) relatively to micro and macro models of lattices and posets of attribute subsets. Moreover, in [24] the relation $\approx_{\mathcal{I}}$ has been interpreted and studied as a type of generalized symmetry relation for several mathematical models.
In the present work, we interpret the equivalence relation $\approx_{\mathcal{I}}$ as the equivalence relation on $\mathcal{P}(\Omega)$ induced by the preorder $\leftarrow_{\mathcal{I}}$. In fact, by (6) we have that

$$A \approx_{\mathcal{I}} B \iff A \leftarrow_{\mathcal{I}} B \text{ and } B \leftarrow_{\mathcal{I}} A \tag{16}$$

Then, in Section 4 we use the equivalence established in (16) as inspiration to study the symmetrization of any $[\mathcal{F}]$-dependency relation.
Since the main purpose of the present paper consists of studying specific and abstract dependency relations, we will use the following terminology.

- We call $\approx_{\mathcal{I}}$ the $\mathcal{I}$-*dependency equivalence* on $\mathcal{P}(\Omega)$;
- if $A \approx_{\mathcal{I}} B$, we say that A and B are $\mathcal{I}$-*dependency equivalent* and we denote by $[A]_{\mathcal{I}}$ the equivalence class with respect to $\leftrightarrows_{\mathcal{I}}$, that we call $\mathcal{I}$-*equivalence dependency class* of A.

We close this subsection by expressing the $\mathcal{I}$-dependency equivalence in terms of $\mathcal{I}$-dependency closure operator.

Proposition 3.7. *For any $A, B \in \mathcal{P}(\Omega)$ we have that*

$$A \approx_{\mathcal{I}} B \iff Dc_{\mathcal{I}}(A) = Dc_{\mathcal{I}}(B) \tag{17}$$

Proof. Let us assume that $A \approx_{\mathcal{I}} B$. By (16) we get $A \leftarrow_{\mathcal{I}} B$ and $B \leftarrow_{\mathcal{I}}$. So, if $c \in Dc_{\mathcal{I}}(A)$, then $c \leftarrow_{\mathcal{I}} A \leftarrow_{\mathcal{I}} B$, i.e. $c \in Dc_{\mathcal{I}}(B)$ and, analogously, if $c \in Dc_{\mathcal{I}}(B)$, then $c \in Dc_{\mathcal{I}}(A)$. Therefore, $Dc_{\mathcal{I}}(A) = Dc_{\mathcal{I}}(B)$.
Conversely, let $Dc_{\mathcal{I}}(A) = Dc_{\mathcal{I}}(B)$. We claim that $A \approx_{\mathcal{I}} B$. For, let us note that $c \leftarrow_{\mathcal{I}} A$ if and only if $c \leftarrow_{\mathcal{I}} B$. Since $a \leftarrow_{\mathcal{I}} A$ for any $a \in A$ and $b \leftarrow_{\mathcal{I}} B$ for any $b \in B$, the claim holds trivially. □

3.2. **Reducts, Essentials and Discernibility.** A fundamental notion of RST applied on information tables is that of *reduct*. Let $\mathcal{I} \in INFT(\Omega)$ and $A \in \mathcal{P}(\Omega)$. We recall the notion of relative reduct.

Definition 3.8. [38] *A subset $B \subseteq A$ is called an $\mathcal{I}$-*reduct *of A if:*

(i) $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(B)$*;*
(ii) $\pi_{\mathcal{I}}(A) \neq \pi_{\mathcal{I}}(B')$ *for all* $B' \subsetneq B$*.*

Let $RED_{\mathcal{I}}(A)$ the set system of all $\mathcal{I}$-reducts of A and we set $RED(\mathcal{I}) := RED_{\mathcal{I}}(\Omega)$.

Now, in order to carry out the notion of reduct relatively to any generalized dependency relation, it is necessary to express any $\mathcal{I}$-reduct of A in terms of the $\mathcal{I}$-dependency relation $\leftarrow_{\mathcal{I}}$. To this regard, we have then the following characterization.

Proposition 3.9. *Let $B \subseteq A$. Then $B \in RED_{\mathcal{I}}(A)$ if and only if the two following conditions hold:*

(i) $C \leftarrow_{\mathcal{I}} A \Longrightarrow C \leftarrow_{\mathcal{I}} B$*;*
(ii) *for any $B' \subsetneq B$, there exists $C' \in \mathcal{P}(\Omega)$ such that $C' \leftarrow_{\mathcal{I}} A$ and $C' \not\leftarrow_{\mathcal{I}} B'$.*

Proof. Let $B \in RED_{\mathcal{I}}(A)$ and let $C \in \mathcal{P}(\Omega)$ be such that $C \leftarrow_{\mathcal{I}} A$. By (6), we deduce that $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(B) \leq \pi_{\mathcal{I}}(C)$, i.e. $C \leftarrow_{\mathcal{I}} B$. This proves (i). On the other hand, let $B' \subsetneq B$. Then, $\pi_{\mathcal{I}}(A) \neq \pi_{\mathcal{I}}(B')$. In particular, it follows that $\pi_{\mathcal{I}}(A) \prec \pi_{\mathcal{I}}(B')$, so there exist $u, u' \in U$ such that $u \equiv_{B'} u'$ but $u \not\equiv_A u'$. Set $C' := A$. By (6), it cannot happen that $C' \leftarrow_{\mathcal{I}} B'$, so (ii) has been verified.
Conversely, let $B \subseteq A$ satisfying both (i) and (ii). Clearly, we have that $\pi_{\mathcal{I}}(A) \leq \pi_{\mathcal{I}}(B)$. Moreover, since $A \leftarrow_{\mathcal{I}} A$, it follows that $A \leftarrow_{\mathcal{I}} B$, i.e. $\pi_{\mathcal{I}}(B) \leq \pi_{\mathcal{I}}(A)$. This shows that $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(B)$.
On the other hand, let $B' \subsetneq B$. We have to show that $\pi_{\mathcal{I}}(A) \neq \pi_{\mathcal{I}}(B')$. Clearly, it results that $\pi_{\mathcal{I}}(A) \leq \pi_{\mathcal{I}}(B')$. By our assumptions, there exists $C' \in \mathcal{P}(\Omega)$ such that $C' \leftarrow_{\mathcal{I}} A$ and $C' \not\leftarrow_{\mathcal{I}} B'$. Moreover, by (ii), there exist two elements $u, u' \in U$ such that $u \equiv_B u'$ but $u \not\equiv_{C'} u'$. Let us note that if $u \equiv_A u'$, then $u \equiv_{C'} u'$ by the fact that $C' \leftarrow_{\mathcal{I}} A$. Nevertheless, this leads to a contradiction. Therefore $u \not\equiv_A u'$, so $\pi_{\mathcal{I}}(A) \neq \pi_{\mathcal{I}}(B')$. □

In Section 5 we use the characterization of reduct given in Proposition 3.9 to define a general notion of reduct associated with any $[\mathcal{F}]$-dependency relation $\leftarrow$, regardless of the presence of one information table having Ω as attribute set. Moreover, next, we will characterize the specific form of these reducts when the relation $\leftarrow$ is obtained by formal contexts, Scott's information systems and possibility measures. Relatively to the link between reducts and independent subsets, we have the following results (for their proof see [7]):

Theorem 3.10. *For any $A \in \mathcal{P}(\Omega)$ we have that:*

(i) $RED_{\mathcal{I}}(A) \subseteq \max(INDP_{\mathcal{I}}(A))$*;*
(ii) $RED_{\mathcal{I}}(A) = \{B \in INDP_{\mathcal{I}}(A) : A \leftarrow_{\mathcal{I}} B\}$*.*

The notion of *relative essential subset* [16], which we can consider both a generalization of the usual Pawlak's core and a dual form of a reduct (see [7, 16] for details), is naturally associated with the notion of relative reduct for an information table.

Definition 3.11. *We say that $B \subseteq A$ is an $\mathcal{I}$-*essential subset *of A if:*

$(E1)$ $\pi_{\mathcal{I}}(A \setminus B) \neq \pi_{\mathcal{I}}(A)$*;*
$(E2)$ $\pi_{\mathcal{I}}(A \setminus B') = \pi_{\mathcal{I}}(A)$ *for all* $B' \subsetneq B$*.*

Let $ESS_{\mathcal{I}}(A)$ the family of all $\mathcal{I}$-essential subsets of A and we set $ESS(\mathcal{I}) := ESS_{\mathcal{I}}(\Omega)$.

Since we want to carry out the notion of essential subset in the more general context of the $[\mathcal{F}]$-dependency relations and, consequently, to characterize this notion for formal contexts, Scott's information systems and possibility measures, we must also express the essential subsets in terms of $\mathcal{I}$-dependency. To this regard, we have the following result.

Proposition 3.12. *Let $B \subseteq A$. Then $B \in ESS_{\mathcal{I}}(A)$ if and only if the two following conditions hold:*

(i) $B \not\leftarrow_{\mathcal{I}} A \setminus B$*;*

(ii) $B' \subsetneq B \implies B' \leftarrow_{\mathcal{I}} A \smallsetminus B'$.

Proof. Let $B \in ESS_{\mathcal{I}}(A)$ and assume by contradiction that $B \leftarrow_{\mathcal{I}} A \smallsetminus B$. Then for any $u, u' \in U$ such that $u \equiv_{A \smallsetminus B} u'$, it must result $u \equiv_B u'$, whence $u \equiv_A u'$. But this would mean that $\pi_{\mathcal{I}}(A \smallsetminus B) \leq \pi_{\mathcal{I}}(A)$. Since we always have $\pi_{\mathcal{I}}(A) \leq \pi_{\mathcal{I}}(A \smallsetminus B)$, we get $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(A \smallsetminus B)$, contradicting $(E1)$.
Let us now show (ii). To this regard, let $B' \subsetneq B$. By $(E2)$, it follows that $\pi_{\mathcal{I}}(A \smallsetminus B') = \pi_{\mathcal{I}}(A)$. So, by (16) we get $B' \leftarrow_{\mathcal{I}} A \smallsetminus B'$.
Conversely, let $B \subseteq A$ satisfying both (i) and (ii). Then, there exist $u, u' \in U$ such that $u \equiv_{A \smallsetminus B} u'$ and $u \not\equiv_B u'$. In particular, $u \not\equiv_A u'$, whence $\pi_{\mathcal{I}}(A \smallsetminus B) \neq \pi_{\mathcal{I}}(A)$. Moreover, let $B' \subsetneq B$. Then, for any $u, u' \in U$ such that $u \equiv_{A \smallsetminus B'} u'$, it must be $u \equiv_{B'} u'$. So, $u \equiv_A u'$, i.e. $\pi_{\mathcal{I}}(A \smallsetminus B') = \pi_{\mathcal{I}}(A)$. □

Analogously to the reduct case, in Section 5 we use the equivalence given in Proposition 3.9 to define a general notion of essential subsets associated with any $[\mathcal{F}]$-dependency relation $\leftarrow$, which naturally is connected by means of several results described in Section 5 to the corresponding notion of reduct associated with $\leftarrow$.
At this point, to conclude the subsection, we show as the duality of the notions of reduct and essential subset is also represented through the discernibility induced by the information map F.
When we consider a dependency relation induced by an information table, the dependency reducts and the essential subsets have a further interrelation, in view of the possibility to construct the discernibility matrix [50, 59].
More specifically, for any $u, v \in U$, let

$$\Delta_A(u,v) := \{a \in A : F(u,a) \neq F(v,a)\}, \tag{18}$$

In the finite case, the discernibility matrix relativized to A is the square matrix of order $|U| \times |U|$ having in the (u,v)-place the above subset $\Delta_A(u,v)$. On the other hand, in our case we can also have non finite sets, and, moreover, in order to formally describe the next two results, we introduce the following set system. We call A-*discernibility set system* the following set system on Ω:

$$DIS_{\mathcal{I}}(A) := \{\Delta_A(u,v) \neq \varnothing : u, v \in U,\, u \neq v\} \tag{19}$$

Let us note that $DIS_{\mathcal{I}}(A)$ is not a subset matrix, but it is a set system, therefore if it happens that $\Delta_A(u,v) = \Delta_A(u',v') = C$, for some two different ordered pairs $(u,v),(u',v') \in U \times U$, the subset $C \in \mathcal{P}(\Omega)$ appears one only time in $DIS_{\mathcal{I}}(A)$.
Relatively to the subsets B of A, the condition of being $\mathcal{I}$-dependency equivalent to A can be expressed in terms of transversals of the A-discernibility set system $\approx_{\mathcal{I}}$ in the following way.

Proposition 3.13. *Let $B \subseteq A$. Then $A \approx_{\mathcal{I}} B$ if and only if B is a transversal of $DIS_{\mathcal{I}}(A)$.*

Proof. Let us assume that $\pi_{\mathcal{I}}(B) = \pi_{\mathcal{I}}(A)$. Let moreover $D \in DIS_{\mathcal{I}}(A)$. We will prove that $B \cap D \neq \varnothing$. To this regard, note that the condition $D \in DIS_{\mathcal{I}}(A)$ implies the existence of $u, u' \in U$ such that $D = \Delta_A(u,u')$. In particular, this means that $u \not\equiv_A u'$ and, by our assumption, that $u \not\equiv_B u'$. Therefore, there must be some $b \in B$ such that $F(u,b) \neq F(u',b)$, i.e. $b \in B \cap D$. Thus, B is a transversal of $DIS_{\mathcal{I}}(A)$.
Conversely, assume that $B \cap D \neq \varnothing$ for each $D \in DIS_{\mathcal{I}}(A)$. Since $B \subseteq A$, we clearly have $\pi_{\mathcal{I}}(A) \leq \pi_{\mathcal{I}}(B)$. On the other hand, take two elements $u.u' \in U$ such that $u \not\equiv_A u'$. Set $D := \Delta_A(u,u')$. Then $D \neq \varnothing$ and, moreover, $B \cap D \neq \varnothing$. So, let $b \in B \cap D$. Thus, $F(u,b) \neq F(u',b)$ i.e. $u \not\equiv_B u'$. This proves that $\pi_{\mathcal{I}}(B) = \pi_{\mathcal{I}}(A)$. □

In the next two results we establish the basic link between $\mathcal{I}$-essential subsets and $\mathcal{I}$-reducts with respect to the A-discernibility set system.

Theorem 3.14. *Let $B \subseteq A$. Then the following conditions are equivalent:*

(i) $B \in ESS_{\mathcal{I}}(A)$.
(ii) $B \in \min(DIS_{\mathcal{I}}(A))$.

Proof. $(i) \implies (ii)$: Let $B \in ESS_{\mathcal{I}}(A)$. In order to prove the claim, it suffices to show that whenever two elements $u, v \in U$ satisfy the relation $\Delta_A(u,v) \subseteq B$, then $\Delta_A(vu,v) = B$. For, note that since $\pi_{\mathcal{I}}(A) \prec \pi_{\mathcal{I}}(A \smallsetminus B)$, it is possible to find two distinct elements $u, v \in U$ such that $u \equiv_{A \smallsetminus B} v$ and $u \not\equiv_B v$. This means that $A \smallsetminus B \subseteq A \smallsetminus \Delta_A(u,v)$ and, hence, $\Delta_A(u,v) \subseteq B$. Now, let us prove that $\Delta_A(u,v) = B$. To this regard, fix $b \in B$ and set $B' := B \smallsetminus \{b\} \subsetneq B$. By our assumptions on u and v, it results that $u \not\equiv_A v$. Moreover, since $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(A \smallsetminus B')$, we get $u \not\equiv_{A \smallsetminus B'} v$. In particular, it must necessarily be $u \not\equiv_b v$. So, $b \in \Delta_A(u,v)$ and, by the arbitrariness of $b \in B$, we have that $\Delta_A(v,w) = B$. This proves that $B \in \min(DIS_{\mathcal{I}}(A))$.

$(ii) \implies (i)$: Let $\varnothing \neq B = \Delta_A(u,v) \in \min(DIS_{\mathcal{I}}(A))$, for some $u, v \in U$. Then, $F(u,a) \neq F(v,a)$ for some $a \in A$, i.e. $u \not\equiv_A v$. Moreover, we also have $u \equiv_{A \smallsetminus B} v$. Thus, we infer that $\pi_{\mathcal{I}}(A \smallsetminus B) \neq \pi_{\mathcal{I}}(A)$.
On the other hand, let $B' \subsetneq B$. Minimality of B in $DIS_{\mathcal{I}}(A)$ implies the following condition:

$$\Delta_A(w,w') \neq \varnothing \implies \Delta_A(w,w') \nsubseteq B'$$

We claim that $\pi_{\mathcal{I}}(A \smallsetminus B') = \pi_{\mathcal{I}}(A)$. Clearly, $\pi_{\mathcal{I}}(A) \leq \pi_{\mathcal{I}}(A \smallsetminus B')$. Conversely, let $w \equiv_{A \smallsetminus B'} w'$ and assume by contradiction that $w \not\equiv_A w'$. Thus, we have that $\Delta_A(w,w') \subseteq B'$, that leads us to a contradiction. Therefore, we showed that $w \equiv_A w' \iff w \equiv_{A \smallsetminus B'} w'$ or, equivalently, let $\pi_{\mathcal{I}}(A \smallsetminus B') = \pi_{\mathcal{I}}(A)$. This ensures that $B \in ESS_{\mathcal{I}}(A)$. ☐

Theorem 3.15. *Let $B \subseteq A$. Then the following conditions are equivalent:*
- (i) $B \in RED_{\mathcal{I}}(A)$.
- (ii) $B \in Tr(\min(DIS_{\mathcal{I}}(A)))$.

Proof. (i): Let $B \in RED_{\mathcal{I}}(A)$. By Proposition 3.13, B is a transversal of $DIS_{\mathcal{I}}(A)$ since $\pi_{\mathcal{I}}(B) = \pi_{\mathcal{I}}(A)$. In particular, B is a transversal of $\min(DIS_{\mathcal{I}}(A)))$. Let us prove that $B \in Tr(\min(DIS_{\mathcal{I}}(A)))$. For, choose an arbitrary $b \in B$. Since $\pi_{\mathcal{I}}(B \smallsetminus \{b\}) \neq \pi_{\mathcal{I}}(A)$, it follows that $B \smallsetminus \{b\}$ is not a transversal of $DIS_{\mathcal{I}}(A)$. This ensures that B is a minimal transversal of $DIS_{\mathcal{I}}(A)$.
(ii): Let $B \in Tr(\min(DIS_{\mathcal{I}}(A)))$. Clearly, it follows that $B \in Tr(DIS_{\mathcal{I}}(A))$. By Proposition 3.13, we deduce that $\pi_{\mathcal{I}}(B) = \pi_{\mathcal{I}}(A)$. Let us now take $b \in B$. Then, the subset $B \smallsetminus \{b\}$ is not a transversal of $DIS_{\mathcal{I}}(A)$ because of the minimality of B. Therefore, again by Proposition 3.13, we get $\pi_{\mathcal{I}}(B \smallsetminus \{b\}) \neq \pi_{\mathcal{I}}(A)$. This proves that $B \in RED_{\mathcal{I}}(A)$. ☐

As a direct consequence of the above Theorems 3.14 and 3.15 we obtain the following basic link between reducts and essentials.

Corollary 3.16. *If $B \subseteq A$ we have that $RED_{\mathcal{I}}(A) = Tr(ESS_{\mathcal{I}}(A))$.*

The results established in Theorem 3.14 and Theorem 3.15 provide the starting ideas to find also in more general cases links between transversality, essentials and reducts. To this regard, in this paper we establish a basic result concerning such interrelations in Theorem 5.10.

4. Set Systems and Set Operators induced by $[\mathcal{F}]$-Dependency Relations

In this section we take inspiration from the notions and results established in Subsection 3.1 to understand how they can be transferred in a general context where we have an $[\mathcal{F}]$-dependency relation.
Therefore, we assume that $\mathcal{F} \in SS(\Omega)$ and $\leftarrow \in DREL(\mathcal{F}|\Omega)$. In the next two subsections we find the conditions on $\mathcal{F}$ so that $Dc_{\mathcal{F},\leftarrow}$ is a closure operator and, then, introduce a map $Ic_{\leftarrow}$ and investigate its main properties according to those satisfied by $\mathcal{F}$. The set operators $Dc_{\mathcal{F},\leftarrow}$ and $Ic_{\leftarrow}$ are obviously generalizations respectively of the $\mathcal{I}$-dependency closure operator and of the $\mathcal{I}$-independency core operator.

4.1. $\leftarrow$Dependency Closure.

The purpose of this subsection consists of investigating the main properties of the set operator $Dc_{\mathcal{F},\leftarrow}$.
Let us express the dependency of members $B \in \mathcal{F}$ in terms of inclusion with respect to the set operator $Dc_{\mathcal{F},\leftarrow}$ in the following way.

Proposition 4.1. *If $A \in \mathcal{P}(\Omega)$ and $B \in \mathcal{F}$ we have that*

$$B \leftarrow A \iff B \subseteq Dc_{\mathcal{F},\leftarrow}(A), \tag{20}$$

Proof. Let $B \leftarrow A$. Then $B \subseteq \bigcup\{C \in \mathcal{F} : C \leftarrow A\} := Dc_{\mathcal{F},\leftarrow}(A)$. Conversely, let $B \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. Then, for any $b \in B$ there exists $C_b \in \mathcal{F}$ such that $b \in C_b \leftarrow A$. Therefore, since $\leftarrow$ is $\mathcal{F}$-inclusive and transitive, we deduce that $b \leftarrow A$. Hence we have that $b \leftarrow A$ for any $b \in B$, so $B \leftarrow A$ by $(D3)$. ☐

In particular, when $\mathcal{F} = \mathcal{P}(\Omega)$ and $\leftarrow$ is a dependency relation, we obtain the following expected form for the set operator $Dc_{\leftarrow}$.

Corollary 4.2. *Let $\leftarrow \in DREL(\Omega)$ and $A \in \mathcal{P}(\Omega)$. Then*

$$Dc_{\leftarrow}(A) = \{b \in \Omega : b \leftarrow A\}. \tag{21}$$

Proof. In our hypothesis on $\leftarrow$ we assume that $\mathcal{F} = \mathcal{P}(\Omega)$. Therefore, if $b \in \Omega$, by (20) we have that $b \leftarrow A$ if and only if $b \in Dc_{\leftarrow}(A)$, and the thesis follows. ☐

Hence in (21) we obtained the natural generalization of (12), which was established for an $\mathcal{I}$-dependency closure operator.
In the next result, we characterize the condition $Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(B)$ in terms of dependency relation between A and B. In this way, we generalize to any $[\mathcal{F}]$-dependency relation the identities established in (16) and (17) for an $\mathcal{I}$-dependency closure operator.

Proposition 4.3. *Let $A, B \in \mathcal{F}$. Then*

$$Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(B) \iff A \leftarrow B \text{ and } B \leftarrow A.$$

Proof. Assume that $Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(B)$ and let $a \in A$. By $(D1)$, we have that $a \leftarrow A$ hence, by our assumption, $a \leftarrow B$. Thus, by $(D3)$, we have that $A \leftarrow B$. In the same way, we prove that $B \leftarrow A$.
Conversely, let $A \leftarrow B$ and $B \leftarrow A$. Then, we respectively have $A \subseteq Dc_{\mathcal{F},\leftarrow}(B)$ and $B \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. Let now $x \in Dc_{\mathcal{F},\leftarrow}(A)$; there necessarily exists $A_x \in \mathcal{F}$ such that $x \in A_x$ and $A_x \leftarrow A$. Now, by the fact that $\leftarrow$ is $[\mathcal{F}]$-inclusive and transitive, we have that $x \leftarrow A_x \leftarrow A \leftarrow B$, i.e. $x \leftarrow B$. This proves that $Dc_{\mathcal{F},\leftarrow}(A) \subseteq Dc_{\mathcal{F},\leftarrow}(B)$. Similarly, we prove that $Dc_{\mathcal{F},\leftarrow}(B) \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. □

In view of Proposition 4.3, we consider now the equivalence relation $\leftrightarrows$ on $\mathcal{F}$ defined by

$$A \leftrightarrows B :\iff A \leftarrow B \text{ and } B \leftarrow A \tag{22}$$

The above equivalence relation $\leftrightarrows$ is the natural generalization of the $\mathcal{I}$-dependency equivalence relatively to any $[\mathcal{F}]$-dependency relation.

Definition 4.4. *We call $\leftrightarrows$ the* $\leftarrow$dependency equivalence *on $\mathcal{F}$. If $A \leftrightarrows B$, we say that A and B are* $\leftarrow$dependency equivalent*, and we denote by $[A]_{\leftrightarrows}$ the equivalence class of A with respect to $\leftrightarrows$, which we call* $\leftarrow$equivalence dependency class *of A.*

In [17], it has been proved that for any information table $\mathcal{I} \in INFT(\Omega)$ the $\mathcal{I}$-dependency class $[A]_{\mathcal{I}}$ is union closed and $Dc_{\mathcal{I}}(A)$ is the only maximal element of $[A]_{\mathcal{I}}$. Then it is natural to ask under which conditions on the set system $\mathcal{F}$ we have a similar result. In the following result we show that this happens when the set system $\mathcal{F}$ is union closed and A is a member of $\mathcal{F}$.

Theorem 4.5. *Let $\mathcal{F} \in UCL(\Omega)$ and $A \in \mathcal{F}$. Then the set system $[A]_{\leftrightarrows}$ is also union closed and the maximum element of $[A]_{\leftrightarrows}$ is $Dc_{\mathcal{F},\leftarrow}(A)$.*

Proof. We first show that $Dc_{\mathcal{F},\leftarrow}(A)$ is the maximum element of the set system $[A]_{\leftrightarrows}$. Clearly, $Dc_{\mathcal{F},\leftarrow}(A) \in \mathcal{F}$. Hence, by the fact that $\leftarrow$ is $[\mathcal{F}]$-inclusive, we have that $A \leftarrow Dc_{\mathcal{F},\leftarrow}(A)$. Conversely, if $b \in Dc_{\mathcal{F},\leftarrow}(A)$, then $b \leftarrow A$ by (20). By $(D3)$ we deduce then that $Dc_{\mathcal{F},\leftarrow}(A) \leftarrow A$. Hence $Dc_{\mathcal{F},\leftarrow}(A) \in [A]_{\leftrightarrows}$.
Let now $B \in [A]_{\leftrightarrows}$. For any $b \in B$ we have $b \leftarrow B \leftarrow A$, therefore, by $(D2)$, $b \leftarrow A$, that is $b \in Dc_{\mathcal{F},\leftarrow}(A)$. Hence $B \subseteq Dc_{\mathcal{F},\leftarrow}(A)$ for any $B \in [A]_{\leftrightarrows}$.
We now have to show that the set system $[A]_{\leftrightarrows}$ is union closed. For, let $\{B_i : i \in I\} \subseteq [A]_{\leftrightarrows}$. Clearly, $\bigcup_{i\in I} B_i \in \mathcal{F}$. Since $B_i \leftarrow A$ for all $i \in I$, by $(D1)$ and $(D2)$, it follows that $b \leftarrow A$ for any B_i and any $i \in I$. Therefore, by $(D3)$ we have that $\bigcup_{i\in I}\{b \in B_i\} = \bigcup_{i\in I} B_i \leftarrow A$. On the contrary, for any index $i \in I$ we have $A \leftarrow B_i \leftarrow \bigcup_{i\in I} B_i$, therefore, by $(D2)$, it follows that $A \leftarrow \bigcup_{i\in I} B_i$. Hence $\bigcup_{i\in I} B_i \in [A]_{\leftrightarrows}$. □

We denote by

$$EDC(\leftarrow) \coloneqq \{[A]_{\leftrightarrows} : A \in \mathcal{F}\},$$

the quotient set of $\mathcal{F}$ with respect to the $\leftarrow$dependency equivalence. $EDC(\leftarrow)$ is a family of set systems on Ω on which we consider the extended relation $\leftarrow^{ext}$ (see (1)). Then $\leftarrow^{ext}$ is a partial order on $EDC(\leftarrow)$:

Proposition 4.6. $(EDC(\leftarrow), \leftarrow^{ext})$ *is a poset.*

Proof. Clearly, $A \leftarrow^{ext} A$ for any $A \in \mathcal{F}$ since $A \leftarrow A$. Moreover assume that $[B]_{\leftrightarrows} \leftarrow^{ext} [A]_{\leftrightarrows}$ and $[A]_{\leftrightarrows} \leftarrow^{ext} [B]_{\leftrightarrows}$. Hence $B' \leftarrow A'$ and $A' \leftarrow B'$ for any $B' \in [B]_{\leftrightarrows}$ and any $A' \in [A]_{\leftrightarrows}$. So, we get $B' \leftrightarrows A'$ for any $B' \in [B]_{\leftrightarrows}$ and any $A' \in [A]_{\leftrightarrows}$, i.e. $[B]_{\leftrightarrows} = [A]_{\leftrightarrows}$. Thus, $\leftarrow^{ext}$ is antisymmetric.
Finally, let us assume that $[C]_{\leftrightarrows} \leftarrow^{ext} [B]_{\leftrightarrows}$ and that $[B]_{\leftrightarrows} \leftarrow^{ext} [A]_{\leftrightarrows}$. Hence, we have that $C' \leftarrow B'$ for any $C' \in [C]_{\leftrightarrows}$ and any $B' \in [B]_{\leftrightarrows}$ and $B' \leftarrow A'$ for any $B' \in [B]_{\leftrightarrows}$ and any $A' \in [A]_{\leftrightarrows}$. By $(D2)$, we get $C' \leftarrow A'$ for any $C' \in [C]_{\leftrightarrows}$ and any $A' \in [A]_{\leftrightarrows}$, i.e. $[C]_{\leftrightarrows} \leftarrow^{ext} [A]_{\leftrightarrows}$. □

The poset $(EDC(\leftarrow), \leftarrow^{ext})$ is the natural candidate to extend the role of the closure system $CLOS(\mathcal{I})$ (see (8)) and of the corresponding isomorphic indiscernibility partition lattice $\mathbb{P}_{ind}(\mathcal{I})$ to the general case of any dependency relation. In fact, in Proposition 4.12 we will show that the above poset has a complete lattice structure and it is order isomorphic to the closure lattice $(CLOS(\leftarrow), \subseteq)$ when $\mathcal{F} = \mathcal{P}(\Omega)$.
In the next result we prove that the set operator $Dc_{\mathcal{F},\leftarrow}$ is a $\mathcal{F}$-preclosure operator and moreover, we also show that $Dc_{\mathcal{F},\leftarrow} \in CLOP(\mathcal{F}|\Omega)$ when the set system $\mathcal{F}$ is union closed.

Proposition 4.7. *The following hold:*

(*i*) $Dc_{\mathcal{F},\leftarrow}$ *is a* $\mathcal{F}$*-preclosure operator on* Ω*;*
(*ii*) *if* $\mathcal{F} \in UCL(\Omega)$*, then* $Dc_{\mathcal{F},\leftarrow} \in CLOP(\mathcal{F}|\Omega)$*;*
(*iii*) *if* $\sigma \in CLOP(\mathcal{F}|\Omega)$*, then* $Inc(\sigma)$ *is* $[\mathcal{F}]$*-inclusive;*

Proof. (*i*): Let $A \in \mathcal{F}$. First of all, since $A \leftarrow A$, we have that $A \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. This shows that $Dc_{\mathcal{F},\leftarrow}$ satisfies $(CL1)$. Let now $B \in \mathcal{F}$ such that $B \subseteq A$. For any $c \in Dc_{\mathcal{F},\leftarrow}(B)$ we have $c \leftarrow B$ by (20). Moreover, since $\leftarrow$ is $[\mathcal{F}]$-inclusive, we have that $B \leftarrow A$, therefore $c \leftarrow A$ by $(D2)$, that is $c \in Dc_{\mathcal{F},\leftarrow}(A)$. Hence $Dc_{\mathcal{F},\leftarrow}(B) \subseteq Dc_{\mathcal{F},\leftarrow}(A)$, i.e. $Dc_{\mathcal{F},\leftarrow}$ satisfies $(CL2)$.
(*ii*): As a direct consequence of part (*i*) and by the fact that $\mathcal{F} \in UCL(\Omega)$, we have that $Dc_{\mathcal{F},\leftarrow}(A) \subseteq Dc^2_{\mathcal{F},\leftarrow}(A)$. On the contrary, let $b \in Dc^2_{\mathcal{F},\leftarrow}(A)$. By (20), we have that $b \leftarrow Dc_{\mathcal{F},\leftarrow}(A)$. Moreover, if $c \in Dc_{\mathcal{F},\leftarrow}(A)$, then $c \leftarrow A$ by (20) so, by $(D3)$ we deduce then that $Dc_{\mathcal{F},\leftarrow}(A) \leftarrow A$. Therefore, by $(D2)$ we conclude that $b \leftarrow A$ for any $b \in Dc^2_{\mathcal{F},\leftarrow}(A)$, i.e. by $(D3)$, $Dc^2_{\mathcal{F},\leftarrow}(A) \leftarrow A$ or, equivalently, $Dc^2_{\mathcal{F},\leftarrow}(A) \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. This shows that $Dc^2_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(A)$, i.e. $Dc_{\mathcal{F},\leftarrow}$ satisfies $(CL3)$. Hence $Dc_{\mathcal{F},\leftarrow} \in CLOP(\mathcal{F}|\Omega)$.
(*iii*): Let $B \subseteq B'$, where $B' \in \mathcal{F}$. Since $B' \subseteq \sigma(B')$, it follows that $B \subseteq \sigma(B')$, i.e. $(B, B') \in Inc(\sigma)$. □

Now, we can establish the equivalence between the notions of dependency relation and closure operator on Ω.

Theorem 4.8. *If* $\leftarrow \in DREL(\Omega)$ *and* $\sigma \in CLOP(\Omega)$*, then* $Dc_{\leftarrow}$ *is a closure operator on* Ω*,* $Inc(\sigma)$ *is a dependency relation on* Ω*, and*

$$Inc(Dc_{\leftarrow}) = \leftarrow \quad \text{and} \quad Dc_{Inc(\sigma)} = \sigma \tag{23}$$

Hence the map Dc *induces a bijection* $DREL(\Omega) \to CLOP(\Omega)$*, whose inverse* $CLOP(\Omega) \to DREL(\Omega)$ *is the map induced by* Inc*.*

Proof. By part (*ii*) of Proposition 4.7, we have that $Dc_{\leftarrow}$ is a closure operator on Ω. We set now $\leftarrow_\sigma := Inc(\sigma)$. Then, by definition of $Inc(\sigma)$ we have that $B \leftarrow_\sigma A$ if and only if $B \subseteq \sigma(A)$. Then, if $B \subseteq A$, we have $B \subseteq \sigma(B) \subseteq \sigma(A)$, therefore $B \leftarrow_\sigma A$. Moreover, if $C \leftarrow_\sigma B$ and $B \leftarrow_\sigma A$, then $C \subseteq \sigma(B)$ and $B \subseteq \sigma(A)$. Therefore $C \subseteq \sigma(B) \subseteq \sigma^2(A) = \sigma(A)$, that is $C \leftarrow_\sigma A$. Finally, $B \leftarrow_\sigma A$ if and only if $b \in B \subseteq \sigma(A)$, that is equivalent to say that $b \leftarrow_\sigma A$ for all $b \in B$. Hence $\leftarrow_\sigma$ also satisfies $(D3)$, and therefore it is a dependency relation on Ω.
Let now $\tau := Dc_{\leftarrow}$ and $\leftarrow_\tau := Inc(\tau)$. Then, by (20), we have that

$$B \leftarrow_\tau A \iff B \subseteq \tau(A) = Dc_{\leftarrow}(A) \iff B \leftarrow A,$$

and this shows that $Inc(Dc_{\leftarrow}) = \leftarrow$.
Finally, we set again $\leftarrow_\sigma := Inc(\sigma)$. Then

$$Dc_{\leftarrow_\sigma}(A) := \bigcup_{B \leftarrow_\sigma A} B = \bigcup_{B \subseteq \sigma(A)} B = \sigma(A),$$

and this shows that $Dc_{Inc(\sigma)} = \sigma$. □

The equivalence between the notions of dependency relations and closure operators established in Theorem 4.8 tells us that substantial parts of RST and database theory (all those parts which are connected with the Pawlak's dependency or to the functional dependency) have an equivalent reformulation in terms of closure operators. Conversely, if we start with a closure operator σ on Ω, we can consider its corresponding associate dependency relation $\leftarrow$ described in Theorem 4.8. Then, if Ω is finite, we can build an information table $\mathcal{I} \in INFT(\Omega)$ such that $\leftarrow_{\mathcal{I}} = \leftarrow$. In this way, we study the closure operator σ in terms of a dependency closure operator associated with an information table. Let us note that if one wants to build explicitly this information table, starting with the closure operator σ, one must to use Theorem 3.6 in connection with the constructive proof of Theorem 1.2 provided in [25] (Theorem 6.1).
When a binary relation $\leftarrow$ satisfies the conditions $(D1\mathcal{F})$ and $(D2)$, we obtain the following characterization for the elements $A \in \mathcal{F} \cap Fix(Dc_{\leftarrow})$.

Proposition 4.9. *Let* $A \in \mathcal{F}$ *and let* $\leftarrow$ *be a* $[\mathcal{F}]$*-inclusive transitive relation. Then the following are equivalent:*

(*i*) $A \in DP_{\mathcal{F}}(\leftarrow)$*;*
(*ii*) $A = Dc_{\mathcal{F},\leftarrow}(A)$.

Proof. $(i) \implies (ii)$: Since $A \in \mathcal{F}$ and $\leftarrow$ is $[\mathcal{F}]$-inclusive, by $(D1)$ we have that $A \leftarrow A$, so $A \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. Conversely, let $x \in Dc_{\mathcal{F},\leftarrow}(A)$. Then, there exists $B_x \in \mathcal{P}(\Omega)$ such that $x \in B_x$ and $B_x \leftarrow A$. By hypothesis, (i) holds, therefore $B_x \subseteq A$. Hence $x \in A$. This shows that $A = Dc_{\mathcal{F},\leftarrow}(A)$.
$(ii) \implies (i)$: Let $B \leftarrow C$ and $C \subseteq A$. We must prove that $B \subseteq A$. Since $A \in \mathcal{F}$ and $\leftarrow$ is $[\mathcal{F}]$-inclusive, by $(D1)$ we have that $C \leftarrow A$. Then, since $\leftarrow$ is also transitive, by $(D2)$ it follows that $B \leftarrow A$. Then, by the definition of $Dc_{\mathcal{F},\leftarrow}$, we have that $B \subseteq Dc_{\mathcal{F},\leftarrow}(A) = A$ and this concludes the proof. □

In the next proposition, we establish the main properties for the set operator $Dc_{\mathcal{F},\leftarrow}$. More in detail, the results provided in (i)-(v) are relative to members of the set system $\mathcal{F}$. In (i) we prove that the condition of dependency of B from A is expressible in terms of inclusion with respect $Dc_{\mathcal{F},\leftarrow}$. The result given in (ii) establishes that when the set system is union closed, the dependency of B from A can be transfer to the dependency of any member of $[B]_{\leftrightarrows}$ by any member of $[A]_{\leftrightarrows}$. Part (iii) tells us that the dependency condition between $\leftarrow$closed subsets is equivalent to their inclusion. In (iv) an expected minimality of condition with respect to the $\leftarrow$closure has been established. In (v) it is proved that the $\leftarrow$closed subsets of A are a closure system on A when $\mathcal{F}$ is a closure system on Ω.

Proposition 4.10. *Let $A, B \in \mathcal{F}$. The following conditions hold:*

(i) $B \leftarrow A \iff Dc_{\mathcal{F},\leftarrow}(B) \subseteq Dc_{\mathcal{F},\leftarrow}(A)$;
(ii) *if* $\mathcal{F} \in UCL(\Omega)$, *then*
$$B \leftarrow A \iff Dc_{\mathcal{F},\leftarrow}(B) \leftarrow Dc_{\mathcal{F},\leftarrow}(A) \iff [B]_{\leftrightarrows} \leftarrow^{ext} [A]_{\leftrightarrows}$$
(iii) *if* $A, B \in CLOS(\leftarrow)$, *then* $B \leftarrow A$ *if and only if* $B \subseteq A$;
(iv) $Dc_{\mathcal{F},\leftarrow}(A)$ *is the smallest* $\leftarrow$*closed subset contaning* A;
(v) *if* $\mathcal{F} \in CLSY(\Omega)$, *then* $CLOS_{\leftarrow}(A) \in CLSY(A)$.

Proof. (i): It follows immediately by (20) and by the fact that $Dc_{\mathcal{F},\leftarrow}$ satisfies $(CL1)$ and $(CL2)$.
(ii): Assume that $B \leftarrow A$ and let $B' \in [B]_{\leftrightarrows}$ and $A' \in [A]_{\leftrightarrows}$. Clearly, we have $B' \leftarrow B \leftarrow A \leftarrow A'$, thus by $(D2)$, we have $B' \leftarrow A'$, i.e. $[B]_{\leftrightarrows} \leftarrow^{ext} [A]_{\leftrightarrows}$.
Let now $[B]_{\leftrightarrows} \leftarrow^{ext} [A]_{\leftrightarrows}$. Then, obviously, $Dc_{\mathcal{F},\leftarrow}(B) \leftarrow Dc_{\mathcal{F},\leftarrow}(A)$ and, by the fact that $Dc_{\mathcal{F},\leftarrow}(A) \in [A]_{\leftrightarrows}$ and $Dc_{\mathcal{F},\leftarrow}(B) \in [B]_{\leftrightarrows}$, we get $B \leftarrow A$. This proves the claim.
(iii): It is an immediate consequence of part (i).
(iv): Let B a $\leftarrow$closed subset containing A. Then $Dc_{\mathcal{F},\leftarrow}(A) \subseteq Dc_{\mathcal{F},\leftarrow}(B) = B$ and the claim has been proved.
(v): Let $\{B_i' : i \in I\} \subseteq CLOS_{\leftarrow}(A)$, then $B_i' = A \cap B_i$, where $B_i \in CLOS(\leftarrow)$. Clearly, $\bigcap_{i\in I} B_i' = A \cap \bigcap_{i\in I} B_i$, hence, by the fact that $\mathcal{F} \in CLSY(\Omega)$, it follows that $B := \bigcap_{i\in I} B_i \in \mathcal{F}$. Now, we will show that $B \in CLOS(\leftarrow)$. For, let us note that $B \subseteq B_i$ for any $i \in I$, therefore $Dc_{\mathcal{F},\leftarrow}(B) \subseteq Dc_{\mathcal{F},\leftarrow}(B_i) = B_i$ for any $i \in I$ and, hence, $Dc_{\mathcal{F},\leftarrow}(B) \subseteq B$. On the other hand, clearly, $B \subseteq Dc_{\mathcal{F},\leftarrow}(B)$. This proves that $B \in CLOS(\leftarrow)$. □

As an immediate consequence of (ii) of Proposition 4.10, we deduce that the $\leftarrow$equivalence between members of a union closed family can be evaluated in terms of identity with respect to the set operator $Dc_{\mathcal{F},\leftarrow}$.

Corollary 4.11. *Let $\mathcal{F} \in UCL(\Omega)$ and $A, B \in \mathcal{F}$. Then:*
$$A \leftrightarrows B \iff Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(B)$$

When the set system $\mathcal{F}$ coincides with the whole $\mathcal{P}(\Omega)$, we can prove the following further properties of the relation $\leftarrow$. In particular, in part (ii) it is proved that the poset $(EDC(\leftarrow), \leftarrow^{ext})$ is an order isomorphic version of the complete lattice induced by the closure system $CLOS(\leftarrow)$. Moreover, in part (iii) it is showed that the closure system $CLOS(\leftarrow)$ can be equivalently expressed in terms of the map DP. Finally, the result established in part (iv) asserts that any dependency relation is completely determined by its closed sets.

Proposition 4.12. *If $\leftarrow \in DREL(\Omega)$ we have that:*

(i) *the restriction of* $Dc_{\leftarrow}$ *on* $\mathcal{P}(A)$ *is the closure operator associated with the closure system* $CLOS_{\leftarrow}(A)$;
(ii) $(EDC(\leftarrow), \leftarrow^{ext})$ *is a complete lattice order isomorphic to the closure lattice* $(CLOS(\leftarrow), \subseteq)$;
(iii) $DP(\leftarrow) = CLOS(\leftarrow)$;
(iv) *if* $\leftarrow' \in DREL(\Omega)$ *and* $CLOS(\leftarrow) = CLOS(\leftarrow')$, *then* $\leftarrow = \leftarrow'$.

Proof. (i): It follows by the fact that $Dc_{\leftarrow}$ is a closure operator on Ω and by part (iii) of Proposition 4.10.

(ii): Just take the map associating $[A]_{\leftrightarrows}$ to $Dc_{\leftarrow}(A)$ and use part (i) of Proposition 4.10.
(iii): By Proposition 4.9, it results that $A \in DP(\leftarrow)$ if and only if $Dc_{\leftarrow}(A) = A$, i.e. if and only if $A \in CLOS(\leftarrow)$.
(iv): Let $A \notin CLOS(\leftarrow)$. Let us prove that $Dc_{\leftarrow}(A) = Dc_{\leftarrow'}(A)$. Indeed, we have that $A \subseteq Dc_{\leftarrow}(A)$, hence $Dc_{\leftarrow'}(A) \subseteq Dc_{\leftarrow'}(Dc_{\leftarrow}(A)) = Dc_{\leftarrow}(A)$ by the fact that $CLOS(\leftarrow) = CLOS(\leftarrow')$. The reverse inclusion is symmetric. This shows that $Dc_{\leftarrow} = Dc_{\leftarrow'}$ so, by Theorem 4.8, we conclude that $\leftarrow = \leftarrow'$. □

4.2. **The $\leftarrow$Pawlak's Independency Core Operator.** In this subsection, we study the version of the Pawlak's core operator $I_{\mathcal{I}}$ generalized to a fixed set system $\mathcal{F} \in SS(\Omega)$ and to a relation $\leftarrow \in DREL(\mathcal{F}|\Omega)$. Therefore, by taking inspiration from the identity given in (14) for the Pawlak's core operator, we introduce and study the map $Ic_{\leftarrow} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined as follows

$$Ic_{\leftarrow}(A) := \{a \in A : a \not\leftarrow A \smallsetminus a\}, \tag{24}$$

for any $A \in \mathcal{P}(\Omega)$.
In particular, we set $Ic_{\leftarrow} := Ic_{\mathcal{P}(\Omega),\leftarrow}$. Clearly, $Ic_{\leftarrow}(A) \subseteq A$.
In the next proposition, we establish two first basic properties of the map $Ic_{\leftarrow}$ when the set system $\mathcal{F}$ is an abstract simplicial complex on Ω.

Proposition 4.13. *Let $\mathcal{F} \in AC(\Omega)$ and $A, B \in \mathcal{F}$. Then:*
(i) if $A \subseteq B$, then $Ic_{\leftarrow}(A) \supseteq A \cap Ic_{\leftarrow}(B)$;
(ii) $Ic^2_{\leftarrow}(A) = Ic_{\leftarrow}(A)$.

Proof. (i): Let $a \in A \cap Ic_{\leftarrow}(B)$, then $a \not\leftarrow B \smallsetminus a$. Assume by contradiction that $a \leftarrow A \smallsetminus a$. Since $\mathcal{F} \in AC(\Omega)$, we have that $B \smallsetminus a \in \mathcal{F}$, so by $(D1\mathcal{F})$ we have that $A \smallsetminus a \leftarrow B \smallsetminus a$ and, by $(D2)$, it follows that $a \leftarrow B \smallsetminus a$, which is a contradiction. This proves that $a \not\leftarrow A \smallsetminus a$, i.e. $a \in Ic_{\leftarrow}(A)$.
(ii): By part (i), we have that $Ic^2_{\leftarrow}(A) \subseteq Ic_{\leftarrow}(A)$. So, it suffices to show the reverse inclusion. Set $D := Ic_{\leftarrow}(A)$. Let us assume by contradiction that $a \in D$ and that $a \notin Ic_{\leftarrow}(D)$, i.e. $a \leftarrow D \smallsetminus a$. Now, since $D \smallsetminus a \subseteq A \smallsetminus a$ and $\mathcal{F} \in AC(\Omega)$, by $(D1\mathcal{F})$, we have that $D \smallsetminus a \leftarrow A \smallsetminus a$ and so, by $(D2)$, that $a \leftarrow A \smallsetminus a$, contradicting the fact that $a \in D = Ic_{\leftarrow}(A)$. This concludes the proof. □

Analogously to the terminology used in Definition 2.7 and based on the above discussed motivation, it is also natural to adopt the following corresponding terminology relatively to the set operator $Ic_{\leftarrow}$.

Definition 4.14. *Let $A \in \mathcal{P}(\Omega)$.*

- *We call* $\leftarrow$Pawlak's independency core operator *the set operator $Ic_{\leftarrow}$ and* $\leftarrow$Pawlak's independency core *of A the subset $Ic_{\leftarrow}(A)$.*
- *We say that A is a* $\leftarrow$Pawlak's independent subset *if $a \not\leftarrow A \smallsetminus a$ for all $a \in A$, in other terms, if $A = Ic_{\leftarrow}(A)$.*
- *We denote by $INDP(\leftarrow)$ the set of all $\leftarrow$Pawlak's independent subsets, that is*
$$INDP(\leftarrow) = Fix(Ic_{\leftarrow})$$
Moreover, we set $INDP_{\leftarrow}(A) := \{B \in INDP(\leftarrow) : B \subseteq A\}$. In particular, when $A = \Omega$, we have $INDP(\leftarrow) := INDP_{\leftarrow}(\Omega)$.

As an immediate consequence of (ii) of Proposition 4.13, we have the following result.

Corollary 4.15. *Let $A \in \mathcal{F}$. Then $Ic_{\leftarrow}(A)$ is a $\leftarrow$Pawlak's independent subset.*

As we had already mentioned in Section 3, for any information table $\mathcal{I} \in INFT(\Omega)$ the set system $INDP(\mathcal{I})$ of all $\mathcal{I}$-Pawlak's independent subsets is an abstract simplicial complex on Ω. Moreover, in [7] it has been also proved that

$$INDP(\mathcal{I}) = \bigcup\{\min([A]_{\mathcal{I}}) : A \in \mathcal{P}(\Omega)\} \tag{25}$$

In other terms, the $\mathcal{I}$-Pawlak's independent subsets coincide with the minimal members of all I-dependency classes. Then, in the next result we show that similar properties also hold for a general $[\mathcal{F}]$-dependency relation $\leftarrow$ when the set system is an abstract simplicial complex on Ω.

Theorem 4.16. *Let $\mathcal{F} \in AC(\Omega)$. Then:*
(i) for any $A \in \mathcal{P}(\Omega)$, the set system $INDP_{\leftarrow}(A)$ is an abstract simplicial complex on A;
(ii) we have that

$$INDP(\leftarrow) = \bigcup\{\min([A]_{\leftrightarrows}) : A \in \mathcal{F}\} \tag{26}$$

Proof. (i): Let us prove that $INDP(\leftarrow)$ is an abstract simplicial complex on Ω. Clearly, $\varnothing \in INDP(\leftarrow)$. Let now $A \in INDP(\leftarrow)$ and $B \subseteq A$. Obviously, $Ic_{\leftarrow}(B) \subseteq B$. Let $b \in B$. Assume by contradiction that $b \leftarrow B \smallsetminus b$. Moreover, since $\mathcal{F} \in AC(\Omega)$ and $A \in \mathcal{F}$, by $(D1\mathcal{F})$ we deduce that $B \smallsetminus b \leftarrow A \smallsetminus b$ and, by $(D2)$, it follows that $b \leftarrow A \smallsetminus b$, i.e. $b \notin Ic_{\leftarrow}(A)$. This is a contradiction. Hence $b \not\leftarrow B \smallsetminus b$, i.e. $Ic_{\leftarrow}(B) = B$. This shows that $INDP(\leftarrow)$ is an abstract simplicial complex on Ω.
The claim concerning $INDP_{\leftarrow}(A)$ is now straightforward.
(ii): Let $B \in \min([A]_{\leftrightarrows})$ for some $A \in \mathcal{F}$. Let us show that $B \in INDP(\leftarrow)$. Clearly, $Ic_{\leftarrow}(B) \subseteq B$. On the other hand, let $b \in B$. Then $B \smallsetminus b \leftarrow B$ by $(D1\mathcal{F})$. By the minimality of B, it follows that $B \not\leftarrow B \smallsetminus b$, i.e. by $(D3)$, there exists $b' \in B$ such that $b' \not\leftarrow B \smallsetminus b$. By the fact that $B \in \mathcal{F}$ and that $\mathcal{F} \in AC(\Omega)$, we have that $B \smallsetminus b \leftarrow B \smallsetminus b$ or, equivalently by $(D3)$, $b'' \leftarrow B \smallsetminus b$ for any $b'' \in B \smallsetminus b$. This forces that $b' = b$. Thus, $Ic_{\leftarrow}(B) = B$.
Conversely, let $B \in INDP(\leftarrow)$. Let us prove that $B \in \min([A]_{\leftrightarrows})$. For, assume by contradiction that there exists $B' \subsetneq B$ such that $B' \in \min([A]_{\leftrightarrows})$. Hence, $A \leftarrow B'$ and $B' \leftarrow A$. Furthermore, there exists $b \in B$ such that $B' \subseteq B \smallsetminus b$. It is also straightforward to see that $A \leftarrow B' \leftarrow B \smallsetminus b \leftarrow B \leftarrow A$ so, $B \smallsetminus b \in [A]_{\leftrightarrows}$. In particular, we have that $B \leftarrow B \smallsetminus b$, hence by $(D3)$, for any $b' \in B$, it results that $b' \leftarrow B \smallsetminus b$. So, $b \leftarrow B \smallsetminus b$, i.e. $Ic_{\leftarrow}(B) \neq B$, contradicting our choice of B. This shows that $B \in \min([A]_{\leftrightarrows})$. □

In particular, when $\leftarrow \in DREL(\Omega)$, the set operator $Ic_{\leftarrow}$ has further properties, as we show in the next result.

Theorem 4.17. *Let $\leftarrow \in DREL(\Omega)$. Then the set operator $Ic_{\leftarrow}$ satisfies the following properties:*
- $(I1)$ $Ic_{\leftarrow}(A) \subseteq A$;
- $(I2)$ *if* $A \subseteq B$ *then* $Ic_{\leftarrow}(A) \supseteq A \cap Ic_{\leftarrow}(B)$;
- $(I3)$ *if* $b \in \Omega \smallsetminus (A \cup Ic_{\leftarrow}(A \cup \{b\}))$ *and* $c \in Ic_{\leftarrow}(A \cup \{c\}) \smallsetminus A$, *then* $c \in Ic_{\leftarrow}(A \cup \{b, c\})$;
- $(I4)$ $Ic^2_{\leftarrow}(A) = Ic_{\leftarrow}(A)$;
- $(I5)$ $Ic_{\leftarrow}(Dc_{\leftarrow}(A)) \subseteq A$;

for any $A, B \in \mathcal{P}(\Omega)$.

Proof. $(I1)$, $(I2)$ and $(I4)$ have been already proved (more in general) in Proposition 4.13. We now prove $(I3)$. To this regard, let $b \in \Omega \smallsetminus (A \cup Ic_{\leftarrow}(A \cup \{b\}))$ and $c \in Ic_{\leftarrow}(A \cup \{c\}) \smallsetminus A$. It is immediate to see that $c \not\leftarrow A$ and $b \leftarrow A$ because of our assumptions. Moreover, by (20), we have that $c \notin Dc_{\leftarrow}(A)$ and $b \in Dc_{\leftarrow}(A)$. Since $Dc_{\leftarrow} \in CLOP(\Omega)$, we also have $Dc_{\leftarrow}(A) = Dc_{\leftarrow}(A \cup \{b\})$, so $c \notin Dc_{\leftarrow}(A \cup \{b\})$. Again by (20), the previous conditions is equivalent to say that $c \not\leftarrow A \cup \{b, c\}$, i.e. $c \in Ic_{\leftarrow}(A \cup \{b, c\})$. Then $Ic_{\leftarrow}$ satisfies $(I3)$.
We now prove $(I5)$. Let $a \in Ic_{\leftarrow}(Dc_{\leftarrow}(A))$ and suppose by contradiction that $a \in Dc_{\leftarrow}(A) \smallsetminus A$. This means that $A \subseteq Dc_{\leftarrow}(A) \smallsetminus a$, so we have $a \leftarrow A \leftarrow Dc_{\leftarrow}(A) \smallsetminus a$, i.e. $a \leftarrow Dc_{\leftarrow}(A) \smallsetminus a$ by $(D2)$, but this contradicts the fact that $a \in Ic_{\leftarrow}(Dc_{\leftarrow}(A))$. Hence $(I5)$ is proved. □

Let us recall here that a *choice function* on Ω is a set operator $\sigma \in OP(\Omega)$ which satisfies the intensivity property $(I1)$. Moreover, in choice theory [37] the property established in $(I2)$ is known as *Chernoff's Axiom*, and it is a well investigated property relatively to a wide class of choice functions having specific interpretations in economic theory and related fields [32, 35, 36]. Therefore, in this perspective, we can consider the $\leftarrow$Pawlak's independency core operator as an idempotent choice function satisfying the Chernoff's axiom.
Let us consider now property $(I3)$. This property is satisfied for any $\leftarrow$Pawlak's independency core operator, hence, in particular also for the Pawlak's core operator associated with an information table. Such a property, jointly with $(I1)$ and $(I2)$, characterizes somehow any set operator on Ω which satisfies these three properties. In fact, in the next representation result, we show that if a set operator on Ω satisfies the properties $(I1), (I2)$ and $(I3)$, then it coincides with the $\leftarrow$Pawlak's independency core operator, for some dependency relation $\leftarrow$ on Ω.

Theorem 4.18. *If $\sigma \in OP(\Omega)$ satisfies the properties $(I1)$, $(I2)$ and $(I3)$, then there exists a dependency relation $\leftarrow \in DREL(\Omega)$ such that $\sigma = Ic_{\leftarrow}$.*

Proof. Let now σ be a set operator satisfying $(I1), (I2)$ and $(I3)$. We will prove that there exists a dependency relation $\leftarrow$ on Ω such that $\sigma = Ic_{\leftarrow}$. Let us consider the set operator $\varphi_\sigma : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ defined as follows:

$$\Psi_\sigma(A) := A \cup A^\flat, \tag{27}$$

where

$$A^\flat := \{b \in \Omega : b \in \Omega \smallsetminus (A \cup \sigma(A \cup \{b\}))\}$$

We will show that $\Psi_\sigma \in CLOP(\Omega)$. By (27), $A \subseteq \Psi_\sigma(A)$ for any $A \in \mathcal{P}(\Omega)$. Let now $A \subseteq B$. It suffices to show that $A^\flat \subseteq \Psi_\sigma(B)$. Let $b \in A^\flat$. Assume that $b \in \Omega \smallsetminus B$. Then, $A \cup \{b\} \subseteq B \cup \{b\}$, therefore, by $(I2)$, we have that $\sigma(A \cup \{b\}) \supseteq \sigma(B \cup \{b\}) \cap (A \cup \{b\})$. But since $b \in \Omega \smallsetminus \sigma(A \cup \{b\})$, it follows that $b \in \Omega \smallsetminus ((A \cup \{b\}) \cap \sigma(B \cup \{b\}))$. Clearly, it implies that $b \in \Omega \smallsetminus \sigma(B \cup \{b\})$ and this proves that $b \in \Omega \smallsetminus (B \cup \sigma(B \cup \{b\})) = B^\flat$, i.e. $b \in \Psi_\sigma(B)$. Conversely, if $b \in B$, it is obvious that $b \in \Psi_\sigma(B)$. This proves $\Psi_\sigma(A) \subseteq \Psi_\sigma(B)$ whenever $A \subseteq B$.
We now show that Ψ_σ is idempotent. Clearly, $\Psi_\sigma(A) \subseteq \Psi_\sigma(\Psi_\sigma(A))$. Moreover, if $(\Psi_\sigma(A))^\flat = \varnothing$, then $\Psi_\sigma(\Psi_\sigma(A)) \subseteq \Psi_\sigma(A)$. Therefore, assume by contradiction that $(\Psi_\sigma(A))^\flat \neq \varnothing$ and let $b \in (\Psi_\sigma(A))^\flat$. Then, $b \in \Omega \smallsetminus (\Psi_\sigma(A) \cup \sigma(\Psi_\sigma(A) \cup \{b\})) = \Omega \smallsetminus ((A \cup A^\flat) \cup \sigma(A \cup A^\flat \cup \{b\}))$. Let us prove that the condition $b \in \Omega \smallsetminus A \cup A^\flat$ implies $b \in \sigma(A \cup A^\flat \cup \{b\})$.
Let us observe that since $b \in \Omega \smallsetminus \Psi_\sigma(A)$, then $b \in \sigma(A \cup \{b\})$. Therefore, let us fix an integer m and assume that $b \in \sigma(A \cup B \cup \{b\})$ for any $B \subseteq A^\flat$ such that $|B| = m-1$. Let now $B \subseteq A^\flat$ such that $|B| = m$ and fix $c \in B$. We will prove that $c \in \Omega \smallsetminus \sigma((A \cup (B \smallsetminus c)) \cup \{c\})$ and that $b \in \sigma((A \cup B \smallsetminus c) \cup \{b\})$.
Let us note that $c \in \Omega \smallsetminus \sigma((A \cup (B \smallsetminus c)) \cup \{c\})$. As a matter of fact, we have that $c \in A^\flat$, so $c \in \Omega \smallsetminus (A \cup \{c\})$. Furthermore, to say that $c \in B$ means that $A \cup \{c\} \subseteq A \cup B$ so, by $(I2)$, $\sigma(A \cup \{c\}) \supseteq \sigma(A \cup B) \cap (A \cup \{c\})$. In other terms, $c \in \Omega \smallsetminus \sigma(A \cup B)$, i.e. $c \in \Omega \smallsetminus \sigma((A \cup (B \smallsetminus c)) \cup \{c\})$. On the other hand, since $|B \smallsetminus c| = m-1$, by the inductive hypothesis on $B \smallsetminus c \subseteq A^\flat$, we have that $b \in \sigma((A \cup B \smallsetminus c) \cup \{b\})$. Hence, we can apply $(I3)$ that, in this case, yields $c \in \sigma((A \cup B \smallsetminus c) \cup \{b, c\}) = \sigma(A \cup B \cup \{b\})$, showing the claim for B. In particular, the claim holds for $B = A^\flat$. Then, $b \in \sigma(A \cup A^\flat \cup \{b\})$, contradicting our choice of b. Necessarily, it must be $(\Psi_\sigma(A))^\flat = \varnothing$. This proves that $\Psi_\sigma(\Psi_\sigma(A)) = \Psi_\sigma(A)$, so $\Psi_\sigma \in CLOP(\Omega)$.
By part (iii) of Proposition 4.7, there exists a dependency relation $\leftarrow$ on Ω such that $\Psi_\sigma = Dc_\leftarrow$. We must show that $\sigma = Ic_\leftarrow$. To this regard, let $a \in Ic_\leftarrow(A)$, then $a \not\leftarrow A \smallsetminus a$ that, by (20), is equivalent to require that $a \notin Dc_\leftarrow(A \smallsetminus a) = (A \smallsetminus a) \cup (A \smallsetminus a)^\flat$. Thus, $a \notin (A \smallsetminus a)^\flat$, i.e. $a \in (A \smallsetminus a) \cup \sigma((A \smallsetminus a) \cup \{a\})$. In particular, it follows that $a \in \sigma(A)$.
On the contrary, let $a \in \sigma(A)$ and assume by contradiction that $a \leftarrow A \smallsetminus a$. Then, $a \in Dc_\leftarrow(A \smallsetminus a) = \Psi_\sigma(A \smallsetminus a) = (A \smallsetminus a) \cup (A \smallsetminus a)^\flat$ by (20). It must necessarily be $a \in (A \smallsetminus a)^\flat$, i.e. $a \in \Omega \smallsetminus ((A \smallsetminus a) \cup \sigma((A \smallsetminus a) \cup \{a\})) = \Omega \smallsetminus ((A \smallsetminus a) \cup \sigma(A))$. This entails $a \notin \sigma(A)$, contradicting our assumption. This proves that $\sigma = Ic_\leftarrow$ and concludes the proof. $\square$

5. Essential Subsets and Dependency Reducts

In Subsection 3.2 we have treated the role of reducts, essentials and discernibility relatively to the dependency relation induced by an information table. In this section we generalize the notion of reducts and essentials relatively to any $[\mathcal{F}]$-dependency relation on Ω and establish the corresponding results concerning such generalizations. Therefore, the basic motivations behind the results and definitions given in this section derive from those described in the Subsection 3.2.
In this section we assume that $\mathcal{F} \in SS(\Omega)$ and $\leftarrow$ be a $[\mathcal{F}]$-dependency relation on Ω.
In the next definition we generalize the notion of Pawlak reduct to the relation $\leftarrow$.

Definition 5.1. *Let $A, B \in \mathcal{P}(\Omega)$. We say that B is a* $\leftarrow$dependency reduct *of A, denoted by $B \leftarrow_{dr} A$, if:*
$(B1)$ $B \subseteq A$;
$(B2)$ $C \leftarrow A \implies C \leftarrow B$;
$(B3)$ B is minimal with respect to $(B2)$. That is, for any $B' \subsetneq B$, there exists $C' \in \mathcal{P}(\Omega)$ such that $C' \leftarrow A$ and $C' \not\leftarrow B'$.
We set $RED_\leftarrow(A) := \{B \in \mathcal{P}(A) : B \leftarrow_{dr} A\}$.

The basic links between reducts and independent subsets relatively to an information table are described in Theorem 3.10. Such relations can be generalized to the $[\mathcal{F}]$-dependency relation $\leftarrow$ when the system $\mathcal{F}$ is an abstract simplicial complex on Ω. To this regard, part (i) of the next theorem generalizes the corresponding part (i) of Theorem 3.10.

Theorem 5.2. *Let $\mathcal{F} \in AC(\Omega)$ and $A \in \mathcal{F}$. We have that:*
(i) $RED_\leftarrow(A) \subseteq \max(INDP_\leftarrow(A))$;
(ii) if $B \subseteq A$ and $B \in [A]_\leftrightarrows$, then $RED_\leftarrow(B) \subseteq RED_\leftarrow(A)$.

Proof. (i) : Let $B \in RED_\leftarrow(A)$, then $B \subseteq A$ and, hence, $B\mathcal{F}$. Let us prove that $B \leftrightarrows A$. Since $B \subseteq A$, by $(D1\mathcal{F})$ it follows that $B \leftarrow A$; conversely, since $A \leftarrow A$, by $(B2)$, it must necessarily be $A \leftarrow B$. This shows that $B \leftrightarrows A$. Now, let us show that $B \in \min([A]_\leftrightarrows)$. For, let $B' \subsetneq B$ be such that $B' \min([A]_\leftrightarrows)$. Moreover, let $C \in \mathcal{P}(\Omega)$ be such that $C \leftarrow A$. Since $A \leftarrow B'$, by $(B2)$, we have that $C \leftarrow B'$. This means that B' satisfies $(B2)$, but this contradicts the minimality of B. This shows that $B \in \min([A]_\leftrightarrows)$. By

(ii) of Theorem 4.16, we deduce that $B \in INDP_{\leftarrow}(A)$.
Finally, we prove that $B \in \max(INDP_{\leftarrow}(A))$. To this regard, assume by contradiction it were false, that is there exists $B'' \supsetneq B$ such that $B'' \in INDP_{\leftarrow}(A)$. Thus, $B'' \in \min([A]_{\leftrightarrows})$, contradicting what we have shown for B.
(ii): Let $B \subseteq A$ be such that $B \in [A]_{\leftrightarrows}$ and let $C \in RED_{\leftarrow}(B)$. Then $C \subseteq B \subseteq A$, so $C \in \mathcal{F}$. We will prove that C satisfies $(B2)$. For, first of all let us notice that $C \in [B]_{\leftrightarrows}$ by the proof of part (i). Now, let $D \leftarrow A$. By $(D2)$, it also follows that $D \leftarrow B$, so, again by $(D2)$, $D \leftarrow C$. This shows that C satisfies $(B2)$. In particular, we have that $C \in [A]_{\leftrightarrows}$. We finally prove $(B3)$. To this regard, let $C' \subsetneq C$ be such that $C' \in RED_{\leftarrow}(A)$. By using $(D2)$, it is straightforward to see that $C' \in RED_{\leftarrow}(B)$, and this contradicts the fact that $C \in RED_{\leftarrow}(B)$. Thus, $C \leftarrow_{dr} A$. □

When the set system $\mathcal{F}$ is an abstract simplicial complex on Ω and A is a member of $\mathcal{F}$, the set system $RED_{\leftarrow}(A)$ can be characterized as the family of all $\leftarrow$Pawlak's independent subsets of A from which A depends. This claim is the generalization of part (ii) of Theorem 3.10 an it is proved in the following theorem.

Theorem 5.3. *Let $\mathcal{F} \in AC(\Omega)$ and $A \in \mathcal{F}$. Then*

$$RED_{\leftarrow}(A) = \{B \in INDP_{\leftarrow}(A) : A \leftarrow B\}$$

Proof. Let $B \in RED_{\leftarrow}(A)$ and set $\mathcal{H}(A) := \{B \in INDP_{\leftarrow}(A) : A \leftarrow B\}$. By (i) of Theorem 5.2, we have $B \in INDP_{\leftarrow}(A)$. Moreover, in order to show that $B \in \mathcal{H}(A)$, it suffices only to show that $A \leftarrow B$. To this regard, since $A \leftarrow A$, by $(B2)$ it must be necessarily $A \leftarrow B$.
Conversely, let $B \in \mathcal{H}(A)$. If $C \leftarrow A$, by $(D2)$ it must necessarily be $C \leftarrow B$, i.e. B satisfies $(B2)$. Let us prove that B satisfies $(B3)$. For, assume by contradiction that there exists $B' \subsetneq B$ satisfying both $(B2)$ and $(B3)$, i.e. $B' \leftarrow_{dr} A$. In particular, there exists $b \in B$ such that $B' \subseteq B \smallsetminus b$, so by the fact that $\mathcal{F} \in AC(\Omega)$ and by $(D1\mathcal{F})$, we deduce that $B' \leftarrow B \smallsetminus b$. Moreover, since $B \leftarrow A$, by $(B2)$ it must result $B \leftarrow B'$. This means that $B \leftarrow B \smallsetminus b$ by $(D2)$. At this point, by $(D3)$, we conclude that $b \leftarrow B \smallsetminus b$, i.e. $Ic_{\leftarrow}(B) \neq B$, so $B \notin INDP(\leftarrow)$, that is in contrast with our assumptions. Therefore B satisfies $(B3)$, i.e. $B \in RED_{\leftarrow}(A)$. □

A fundamental result of RST is that the Pawlak's core coincides with the intersection of all reducts, relatively to a given information table. For our general $[\mathcal{F}]$-dependency relation, we are able to show that $Ic_{\leftarrow}(A)$ is contained in $\bigcap RED \leftarrow (A)$ for any $A \in \mathcal{F}$ when the set system $\mathcal{F}$ is an abstract simplicial complex. Moreover, under the same hypothesis, we also prove that $Ic_{\leftarrow}(A)$ coincides with $\bigcap RED \leftarrow (A)$ when Ω is finite.

Theorem 5.4. *Let $\mathcal{F} \in AC(\Omega)$ and $A \in \mathcal{F}$. Hence, $Ic_{\leftarrow}(A) \subseteq \bigcap RED_{\leftarrow}(A)$. Equality holds when Ω is a finite set.*

Proof. Let $a \in Ic_{\leftarrow}(A)$ and assume, by contradiction, that there exists some $B \in RED_{\leftarrow}(A)$ such that $a \notin B$. By $(D1\mathcal{F})$, we have that $a \leftarrow A$, hence by $(B2)$ we deduce that $a \leftarrow B \subseteq A \smallsetminus a$, that is $a \leftarrow A \smallsetminus a$ again by the fact that $\leftarrow$ is $[\mathcal{F}]$-inclusive and by transitivity. This is in contrast with the hypothesis that $a \in Ic_{\leftarrow}(A)$. This shows that $Ic_{\leftarrow}(A) \subseteq \bigcap RED_{\leftarrow}(A)$.
Conversely, assume that Ω is finite and let $a \in \bigcap RED_{\leftarrow}(A)$. Let us also suppose by contradiction that $a \notin Ic_{\leftarrow}(A)$, so that $a \leftarrow A \smallsetminus a$. Then $A \smallsetminus a$ satisfies the property $(B2)$. In fact, let $C \in \mathcal{P}(\Omega)$ such that $C \leftarrow A$. By $(D3)$, we have that $A \leftarrow A \smallsetminus a$ so, by transitivity it must necessarily be $C \leftarrow A \smallsetminus a$. Let us consider now the set system $\mathcal{G}$ of all subsets of $A \smallsetminus a$ satisfying $(B2)$. Then $\mathcal{G}$ is non-empty because $A \smallsetminus a \in \mathcal{G}$. Thus, there exists a minimal element $B \in \mathcal{G}$. Therefore, $a \notin B$ and $B \in RED_{\leftarrow}(A)$, but this is in contrast with the hypothesis that $a \in \bigcap RED_{\leftarrow}(A)$. This proves that $Ic_{\leftarrow}(A) \supseteq \bigcap RED_{\leftarrow}(A)$ and the proof concludes here. □

When the set system $\mathcal{F}$ is both union closed and an abstract simplicial complex, for any member $A \in \mathcal{F}$ we can provide two further equivalent conditions in order that a subset of A is a $\leftarrow$dependency reduct of A.

Theorem 5.5. *Let $\mathcal{F} \in AC(\Omega) \cap UCL(\Omega)$, $A \in \mathcal{F}$ and $B \subseteq A$. Then the following conditions are equivalent:*

(i) $B \in RED_{\leftarrow}(A)$.
(ii) $Dc_{\mathcal{F},\leftarrow}(B) = Dc_{\mathcal{F},\leftarrow}(A)$ *and* $Dc_{\mathcal{F},\leftarrow}(B') \subsetneq Dc_{\mathcal{F},\leftarrow}(A)$ *for all* $B' \subsetneq B$.
(iii) $Dc_{\mathcal{F},\leftarrow}(B) = Dc_{\mathcal{F},\leftarrow}(A)$ *and* $Ic_{\leftarrow}(B) = B$.

Proof. $(i) \implies (ii)$: The claim is immediate by (i) of Theorem 5.2 and by Corollary 4.11.
$(ii) \implies (iii)$: Let us assume that $Dc_{\mathcal{F},\leftarrow}(B) = Dc_{\mathcal{F},\leftarrow}(A)$ and $Dc_{\mathcal{F},\leftarrow}(B') \subsetneq Dc_{\mathcal{F},\leftarrow}(A)$ for all $B' \subsetneq B$. We now prove that $Ic_{\leftarrow}(B) = B$. For, take $B' = B \setminus \{b\}$, for some $b \in B$. Assume by contradiction that $b \leftarrow B \setminus b$. By (20), it follows that $b \in Dc_{\mathcal{F},\leftarrow}(B')$, so, by the fact that $Dc_{\mathcal{F},\leftarrow} \in CLOP(\mathcal{F}|\Omega)$, we have that $Dc_{\mathcal{F},\leftarrow}(B') = Dc_{\mathcal{F},\leftarrow}(B) = Dc_{\mathcal{F},\leftarrow}(A)$, contradicting our hypothesis. Then, $b \not\leftarrow B'$, that is $b \in Ic_{\leftarrow}(B)$. By the arbitrariness of b, we deduce that $Ic_{\leftarrow}(B) = B$.
$(iii) \implies (i)$: Let $B \subseteq A$ be such that $Dc_{\mathcal{F},\leftarrow}(B) = Dc_{\mathcal{F},\leftarrow}(A)$ and $Ic_{\leftarrow}(B) = B$. Clearly, $(B1)$ is satisfied. We must prove that B satisfies $(B2)$. To this regard, let $C \leftarrow A$. By (20), $C \subseteq Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(B)$, i.e. $C \leftarrow B$. Finally, let us show $(B3)$. Let $B' \subsetneq B$ satisfying $(B2)$. Then, there exists $b \in B$ such that $B' \subseteq B \setminus b$. Since $B' \subsetneq A$ and $A \in \mathcal{F}$, we have that $B' \leftarrow A$. On the other hand, we also have that $A \leftarrow B'$ and, in particular, by $(D3)$, that $b \leftarrow B'$. This is absurd, since $Ic_{\leftarrow}(B) = B$, i.e. $b \not\leftarrow B'$. This proves that B satisfies $(B3)$, i.e. $B \leftarrow_{dr} A$. □

In Section 3 (see Definition 3.11), after fixing an information table $\mathcal{I} \in INFT(\Omega)$ and an attribute subset $A \in \mathcal{P}(\Omega)$, we discussed the role of the $\mathcal{I}$-essential subsets of A in RST as a type of dual version of the $\mathcal{I}$-reducts of A. In the next definition we introduce the corresponding notion of essential subset relatively to our $[\mathcal{F}]$-dependency relation $\leftarrow$. This means that the condition for which an element $a \in A$ satisfies the condition $a \not\leftarrow A \setminus a$ can be considered also relatively to a subset B of A in the following way.

Definition 5.6. *Let $A, B \in \mathcal{P}(\Omega)$. We say that B is an* $\leftarrow$essential subset *of A, denoted by $B \leftarrow_{ess} A$, if:*
$(E1)$ *$B \subseteq A$;*
$(E2)$ *$B \not\leftarrow A \setminus B$;*
$(E3)$ *B is minimal with respect to the property $(E2)$, that is, $B' \subsetneq B \implies B' \leftarrow A \setminus B'$.*
We set $ESS_{\leftarrow}(A) := \{B \in \mathcal{P}(A) : B \leftarrow_{ess} A\}$.

As in the above reduct case, even for the essential subsets we will provide a characterizion in terms of the set operator $Dc_{\mathcal{F},\leftarrow}$, when the set system $\mathcal{F}$ is both union closed and an abstract simplicial complex, for any member $A \in \mathcal{F}$.

Proposition 5.7. *Let $\mathcal{F} \in AC(\Omega) \cap UCL(\Omega)$, $A \in \mathcal{F}$ and $B \subseteq A$. Then the following conditions are equivalent:*

- (i) *$B \leftarrow_{ess} A$.*
- (ii) *$Dc_{\mathcal{F},\leftarrow}(A \setminus B) \subsetneq Dc_{\mathcal{F},\leftarrow}(A)$ and $Dc_{\mathcal{F},\leftarrow}(A \setminus B') = Dc_{\mathcal{F},\leftarrow}(A)$ for all $B' \subsetneq B$.*

Proof. $(i) \implies (ii)$: Since $B \in \mathcal{F}$ and $A \setminus B \subseteq A$ for any $B \subseteq A$, by $(D1\mathcal{F})$ it follows that $Dc_{\mathcal{F},\leftarrow}(A \setminus B) \subseteq Dc_{\mathcal{F},\leftarrow}(A)$. Vice versa, let $B \in ESS_{\leftarrow}(A)$. By $(E2)$, $B \not\leftarrow A \setminus B$ and, by (20), we conclude that $B \notin Dc_{\mathcal{F},\leftarrow}(A \setminus B)$. But since $B \subseteq Dc_{\mathcal{F},\leftarrow}(A)$, it must necessarily be $Dc_{\mathcal{F},\leftarrow}(A \setminus B) \subsetneq Dc_{\mathcal{F},\leftarrow}(A)$.
Finally, by $(E3)$, we have that if $B' \subsetneq B$, then $B' \leftarrow A \setminus B'$. Therefore, by (20), it follows that $B' \subseteq Dc_{\mathcal{F},\leftarrow}(A \setminus B')$, so $A \setminus B' \cup B' = A \subseteq Dc_{\mathcal{F},\leftarrow}(A \setminus B')$. This means that $Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(A \setminus B')$ and this shows the claim.
$(ii) \implies (i)$: Let $B \in \mathcal{P}(A)$ be such that $Dc_{\mathcal{F},\leftarrow}(A \setminus B) \subsetneq Dc_{\mathcal{F},\leftarrow}(A)$ and $Dc_{\mathcal{F},\leftarrow}(A \setminus B') = Dc_{\mathcal{F},\leftarrow}(A)$ for all $B' \subsetneq B$. Clearly, $(E1)$ is verified. Now, let us prove $(E2)$. To this regard, assume by contradiction that $B \leftarrow A \setminus B$. Then, by (20), it follows that $B \subseteq Dc_{\mathcal{F},\leftarrow}(A \setminus B)$, that means that $Dc_{\mathcal{F},\leftarrow}(A) = Dc_{\mathcal{F},\leftarrow}(A \setminus B)$, contradicting our assumption. Hence, $B \not\leftarrow A \setminus B$. Finally, let $B' \subsetneq B$ be such that $B' \not\leftarrow A \setminus B'$. Then $B' \notin Dc_{\mathcal{F},\leftarrow}(A \setminus B')$, so $Dc_{\mathcal{F},\leftarrow}(A) \neq Dc_{\mathcal{F},\leftarrow}(A \setminus B')$, contradicting our assumption. This proves that $(E3)$ and, in particular, that $B \leftarrow_{ess} A$. □

Remark 5.8. *When $\mathcal{F} = \mathcal{P}(\Omega)$, that is $\leftarrow \in DREL(\Omega)$, let us notice that $Ic_{\leftarrow}(A)$ coincides with the subset of all $\leftarrow$essential elements of A.*

In the particular case when $\mathcal{F}$ coincides with the whole $\mathcal{P}(\Omega)$, in other terms if $\leftarrow \in DREL(\Omega)$, we can establish a direct link between the $\leftarrow$essential subsets and the covering relation in the closure lattice $(CLOS_{\leftarrow}(A), \subseteq)$.

Theorem 5.9. *Let $\leftarrow \in DREL(\Omega)$, $A \in \mathcal{P}(\Omega)$ and $B \subseteq A$. Then the following conditions are equivalent:*

- (i) *$B \in ESS_{\leftarrow}(A)$.*
- (ii) *$Dc_{\leftarrow}(A \setminus B) \subsetneq Dc_{\leftarrow}(A)$ and $Dc_{\leftarrow}(A \setminus B') = Dc_{\leftarrow}(A)$ for all $B' \subsetneq B$.*
- (iii) *$A \setminus B$ covers A in the lattice $(CLOS_{\leftarrow}(A), \subseteq)$.*

Proof. $(i) \iff (ii)$: It has been already proved in Proposition 5.7.
$(ii) \implies (iii)$: Let $B \subseteq A$ such that $Dc_{\leftarrow}(A \setminus B) \subsetneq Dc_{\leftarrow}(A)$ and $Dc_{\leftarrow}(A \setminus B') = Dc_{\leftarrow}(A)$ for all $B' \subsetneq B$. Let us show that $A \setminus B = Dc_{\leftarrow}(A \setminus B) \cap A$. Clearly, $A \setminus B \subseteq Dc_{\leftarrow}(A \setminus B) \cap A$. Vice versa, let $a \in Dc_{\leftarrow}(A \setminus B) \cap A$ and assume that $a \notin A \setminus B$. Hence $a \in B$.

Set now $B' := B \smallsetminus A$. It is straightforward to show that $Dc_{\leftarrow}(A \smallsetminus B) = Dc_{\leftarrow}(A \smallsetminus B')$. Nevertheless, we would have $Dc_{\leftarrow}(A \smallsetminus B) = Dc_{\leftarrow}(A \smallsetminus B') = Dc_{\leftarrow}(A)$, contradicting our assumption. This means that $A \smallsetminus B = Dc_{\leftarrow}(A \smallsetminus B) \cap A$, i.e. $A \smallsetminus B \in CLOS_{\leftarrow}(A)$. We must show that $A \smallsetminus B$ covers A in the lattice $CLOS_{\leftarrow}(A)$. To this regard, suppose by contradiction it were false. Then, there exists $A \smallsetminus B \subsetneq C \subsetneq A$ such that C covers A in the lattice $CLOS_{\leftarrow}(A)$. However, this ensures that $C = A \smallsetminus B'$ for some non-empty $B' \subsetneq B$. Let us prove that $Dc_{\leftarrow}(C) \subsetneq Dc_{\leftarrow}(A)$. For, suppose by absurd that equality holds, namely $Dc_{\leftarrow}(C) = Dc_{\leftarrow}(A)$. Since $C = A \cap M$ for some $M \in CLOS(\leftarrow)$, we would have $Dc_{\leftarrow}(C) = Dc_{\leftarrow}(A) \subseteq M$, i.e. $A \subseteq M$. But, this implies that $A \cap M = C \supseteq A$ and this is impossible. So, $Dc_{\leftarrow}(C) \subsetneq Dc_{\leftarrow}(A)$. However, the existence of such a subset C contradicts our hypothesis on B. This proves that $A \smallsetminus B$ covers A in the lattice $CLOS_{\leftarrow}(A)$.
$(iii) \implies (ii)$: Let B be a non-empty subset of A such that $A \smallsetminus B$ covers A in the lattice $CLOS_{\leftarrow}(A)$. Assume by contradiction that $Dc_{\leftarrow}(A \smallsetminus B) = Dc_{\leftarrow}(A)$. Since $A \smallsetminus B = A \cap C$ for some $C \in CLOS(\leftarrow)$, we have that

$$Dc_{\leftarrow}(A \cap C) = Dc_{\leftarrow}(A \smallsetminus B) = Dc_{\leftarrow}(A) \subseteq Dc_{\leftarrow}(A) \cap Dc_{\leftarrow}(C) = Dc_{\leftarrow}(A) \cap C,$$

i.e. $Dc_{\leftarrow}(A) \subseteq C$. In other terms, we showed that $A = A \cap Dc_{\leftarrow}(A) \subseteq A \cap C = A \smallsetminus B$ or, equivalently $A = A \smallsetminus B$, that is a contradiction. Hence $Dc_{\leftarrow}(A \smallsetminus B) \subsetneq Dc_{\leftarrow}(A)$. Furthermore, let $B' \subsetneq B$ and assume by contradiction that $Dc_{\leftarrow}(A \smallsetminus B') \subsetneq Dc_{\leftarrow}(A)$. Then, $A \smallsetminus B \subsetneq A \smallsetminus B' \subseteq Dc_{\leftarrow}(A \smallsetminus B') \cap A$. In particular, we also have that $Dc_{\leftarrow}(A \smallsetminus B') \cap A \subsetneq A$, otherwise $A \subseteq Dc_{\leftarrow}(A \smallsetminus B')$, i.e. $Dc_{\leftarrow}(A) = Dc_{\leftarrow}(A \smallsetminus B')$, contradicting our assumption. In this way, we showed that $A \smallsetminus B$ does not cover A in the lattice $CLOS_{\leftarrow}(A)$, that is absurd. This proves the claim. □

In the last result of this section, we investigate the interrelation between dependency reducts and essential subsets when $\leftarrow \in DREL(\Omega)$. It can be considered the natural generalization of the corresponding result established in Corollary 3.16 concerning reducts and essentials of information tables.

Theorem 5.10. *Let $\leftarrow \in DREL(\Omega)$ and $A \in \mathcal{P}(\Omega)$. If $B \subseteq A$ satisfies $(B2)$, then B is a transversal of $ESS_{\leftarrow}(A)$.*

Proof. Let $B \subseteq A$ satisfying $(B2)$. Let us prove that it is transversal of the set system $\mathcal{G}(A) := \{D \subseteq A : D \not\leftarrow A \smallsetminus D\}$. Indeed, assume by contradiction that there exists $D \in \mathcal{G}(A)$ such that $B \cap D = \emptyset$. This implies that $B \subseteq A \smallsetminus D$, i.e. $B \leftarrow A \smallsetminus D$. But, by the fact that $A \leftarrow A$, by $(B2)$, we have $A \leftarrow B$, so by $(D2)$, we would have $A \leftarrow A \smallsetminus D$ and, hence, again by $(D2)$, $D \leftarrow A \smallsetminus D$, contradicting the fact that $D \in \mathcal{G}(A)$. This shows that B is a transversal of $\mathcal{G}(A)$. In particular, it is transversal of $ESS_{\leftarrow}(A) = \min(\mathcal{G}(A))$. □

6. Dependency by O-RST

Let $\mathcal{I} \in INFT(\Omega)$ a fixed information table having Ω as attribute set. The Pawlak's dependency relation $\leftarrow_{\mathcal{I}}$ investigated in Section 3 is equivalent to the functional dependency classically used in relational database theory [49]. However, in RST it is possible to build also a different type of dependency relation, which is naturally induced by the rough approximation operators on the object set U, relatively to the information table $\mathcal{I}$. In order to describe such a dependency relation, we recall the definition of the two classical set operators of RST.
For any $A \in \mathcal{P}(\Omega)$, the *$A$-lower* and *$A$-upper approximation operators* $lw_A, up_A : \mathcal{P}(U) \to \mathcal{P}(U)$ are defined by (see [38]):

$$lw_A(Y) := \{x \in U : [x]_A \subseteq Y\} = \bigcup\{C \in \pi_{\mathcal{I}}(A) : C \subseteq Y\},$$

and

$$up_A(Y) := \{x \in U : [x]_A \cap Y \neq \emptyset\} = \bigcup\{C \in \pi_{\mathcal{I}}(A) : C \cap Y \neq \emptyset\},$$

for any $Y \in \mathcal{P}(U)$.
Then

$$lw_A(Y) \subseteq Y \subseteq up_A(Y),$$

and the subset Y is called *A-exact* (see [38]) if and only if $lw_A(Y) = up_A(Y)$ and *A-rough* otherwise.
Three basic properties of the above approximation operators are described in the following proposition.

Proposition 6.1. *The following properties hold:*
(i) X is A-exact if and only if $up_A(X) = X$;
(ii) $up_A = up_{Dc_{\mathcal{I}}(A)}$, for any $A \in \mathcal{P}(\Omega)$;
(iii) if $B \subseteq A$, then $up_A(X) \subseteq up_B(X)$ for any $X \in \mathcal{P}(\Omega)$

Proof. Straightforward. □

Let now A be a fixed attribute subset of Ω, then we consider the following binary relation $\multimap_A$ on $\mathcal{P}(\Omega)$:

$$Y \multimap_A X :\iff Y \subseteq up_A(X), \tag{28}$$

for any $X, Y \in \mathcal{P}(U)$.
Let us prove that $\multimap_A$ is a dependency relation on U.

Proposition 6.2. *For any $A \in \mathcal{P}(\Omega)$ we have that $\multimap_A \in DREL(U)$.*

Proof. Clearly, by the definition of $up_A(Y)$, we have that $Y \subseteq up_A(Y)$, so $(D1)$ easily follows.
Now, let us show property $(D2)$. To this regard, assume that $Z \multimap_A Y$ and $Y \multimap_A X$, i.e. $Z \subseteq up_A(Y)$ and $Y \subseteq up_A(X)$. By our assumptions, for any $z \in Z$, it results that $[z]_A \cap Y \neq \varnothing$ and, for any $y \in Y$, $[y]_A \cap X \neq \varnothing$. Fix $z \in Z$ and let $y \in Y$ such that $y \equiv_A z$. Then, $[z]_A \cap X = [y]_A \cap X \neq \varnothing$, whence $z \in up_A(X)$.
Finally, as an immediate consequence of both $(D1)$ and $(D2)$, we have that for $y \multimap_A X$ for each $y \in Y$. The converse is obvious by (28). □

Relatively to the set operator $Dc_{\multimap_A}$ we have the following result, asserting that it coincides with the set operator up_A.

Proposition 6.3. *For any $A \in \mathcal{P}(\Omega)$, we have that:*
- (i) *$Dc_{\multimap_A} = up_A$;*
- (ii) *$CLOS(\multimap_A)$ coincides with the family of all A-exact subsets.*

Proof. (i): Let $X \in \mathcal{P}(U)$. Let moreover $x \in Dc_{\multimap_A}(X)$. Then, there exists some $Y \in \mathcal{P}(\Omega)$ containing x and such that $Y \multimap_A X$, i.e. $x \in Y \subseteq up_A(X)$.
Conversely, let $x \in u_A(X)$. By (28), it follows that $x \multimap_A X$, i.e. $x \in Dc_{\multimap_A}(X)$.
(ii): It is a direct consequence of the previous part (i), since it is well-known that the set of all A-exact subsets of U coincides with $Fix(up_A) = Fix(lw_A)$. □

By part (ii) of Proposition 6.3 we have that the A-exact subsets are exactly the closed subsets relatively to the dependency relation $\multimap_A$.
We now investigate the specific properties of the Pawlak's independency core operator relatively to the dependency relation $\multimap_A$. More in detail, in the next result, we provide the expression of the image of the set operator $Ic_{\multimap_A}$ and we also relate A-exactness with the properties of the $\multimap_A$Pawlak's independency core operator.

Proposition 6.4. *We have that:*
(i) *$Ic_{\multimap_A}(X) = \{x \in X : [x]_A \cap X = \{x\}\}$;*
(ii) *if X is A-exact, then $Ic_{\multimap_A}(X) = \{x \in X : [x]_A = \{x\}\}$;*
(iii) *let $x \in X$ such that $[x]_A = \{x\}$, then $x \in lw_A(X) \cap Ic_{\multimap_A}(X)$.*

Proof. (i): By definition, $Ic_{\multimap_A}(X) = \{x \in X : x \not\multimap_A X \setminus x\}$, that is equivalent to say that $x \notin up_A(X \setminus x)$, i.e. $[x]_A \cap X \setminus x = \varnothing$. In other terms, $y \notin X \setminus x$ for each $y \equiv_A x$. But this means that $[x]_A \cap X = \{x\}$.
(ii): It is an immediate consequence of part (ii) and of (i) of Proposition 6.1.
(iii): By part (ii), we have that $x \in Ic_{\multimap_A}(X)$. Moreover, $[x]_A \subseteq X$, whence $x \in lw_A(X)$. □

In order to establish the basic links between the $\mathcal{I}$-dependency and the set of all dependency relations $\multimap_A$, it is also convenient to identify the dependency relation $\multimap_A$ with its corresponding version in terms of subset ordered pair family

$$\mathcal{U}_A = \{(Y, X) \in \mathcal{P}(U)^2 : Y \multimap_A X\}$$

We also consider the family of all dependency relations $\mathcal{U}_A$:

$$\Xi(\mathcal{I}) \coloneqq \{\mathcal{U}_A : A \in \mathcal{P}(\Omega)\} = \{\mathcal{U}_A : A \in CLOS(\mathcal{I})\}$$

Then in the next theorem we show that we can express the $\mathcal{I}$-dependency in terms of set theoretical inclusion between the dependency relations $\mathcal{U}_A$. Moreover, as a consequence, we also obtain an order isomorphism between the closure lattices $(CLOS(\mathcal{I}), \subseteq)$ and $(\Xi(\mathcal{I}), \subseteq^*)$.

Theorem 6.5. *Let $A, B \in \mathcal{P}(\Omega)$. Then:*
(i) *$B \leftarrow_{\mathcal{I}} A \iff \mathcal{U}_A \subseteq \mathcal{U}_B$;*
(ii) *the map $A \in \mathcal{P}(\Omega) \mapsto \mathcal{U}_A \in \Xi(\mathcal{I})$ induces an order isomorphism between the posets $(CLOS(\mathcal{I}), \subseteq)$ and $(\Xi(\mathcal{I}), \subseteq^*)$;*
(iii) *$(\Xi(\mathcal{I}), \subseteq)$ is a complete lattice having maximum element $\mathcal{U}_\varnothing$, minimum element $\mathcal{U}_\Omega$ and where join and meet have respectively the following form:*

$$\bigwedge\{\mathcal{U}_{A_i} : i \in I\} = \mathcal{U}_{\bigcup_{i\in I} Dc_{\mathcal{I}}(A_i)} \text{ and } \bigvee\{\mathcal{U}_{A_i} : i \in I\} = \mathcal{U}_{\bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)}$$

Proof. (i): Let $(Y,X) \in \mathcal{U}_A$, therefore $Y \multimap_A X$ or, equivalently, $Y \subseteq up_A(X)$. Now, by our hypothesis, we have that $B \subseteq Dc_{\mathcal{I}}(A)$, hence, by (ii) and (iii) of Proposition 6.1, we conclude that $up_A(X) \subseteq up_B(X)$. This implies that $Y \subseteq up_B(X)$, i.e. $Y \multimap_B X$ or, equivalently, $(Y,X) \in \mathcal{U}_B$. Thus, $\mathcal{U}_A \subseteq \mathcal{U}_B$.
Conversely, assume that $\mathcal{U}_A \subseteq \mathcal{U}_B$. We will $B \leftarrow_{\mathcal{I}} A$. To this regard, consider $x \in \Omega$ and let $y \equiv_A x$. Therefore, $[y]_A \cap \{x\} \neq \emptyset$ and, by our assumption, this entails $[y]_B \cap \{x\} \neq \emptyset$. Thus $y \equiv_B x$. This proves that $B \leftarrow_{\mathcal{I}} A$, as claimed.
(ii): Let us consider the restricted map $A \in CLOS(\mathcal{I}) \mapsto \mathcal{U}_A \in \Xi(\mathcal{I})$ and take $A,B \in CLOS(\mathcal{I})$. By definition, the map is surjective. Furthermore, it is easy to show that this map is injective, in fact if $\mathcal{U}_A = \mathcal{U}_B$, then $A \leftrightarrows_{\mathcal{I}} B$ by part (v), whence $A = B$. Thus, the map is a bijection. Finally, part (v) implies that this map is an order isomorphism.
(iii): A bijection between a complete lattice and a poset induces a complete lattice structure on the poset. In particular, it is clear that the maximum of $(\Xi(\mathcal{I}), \subseteq)$ is $\mathcal{U}_\emptyset$, while its minimum is $\mathcal{U}_\Omega$. Moreover, in view of the above isomorphism, it must necessarily be $\bigwedge\{\mathcal{U}_{A_i} : i \in I\} = \mathcal{U}_{\bigcup_{i\in I} Dc_{\mathcal{I}}(A_i)}$ and $\bigvee\{\mathcal{U}_{A_i} : i \in I\} = \mathcal{U}_{\bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)}$. In fact,

$$\bigvee_{i\in I} \mathcal{U}_{A_i} = \bigvee_{i\in I} \mathcal{U}_{Dc_{\mathcal{I}}(A_i)} = \bigvee \phi(Dc_{\mathcal{I}}(A_i)) = \phi(\bigwedge_{i\in I} Dc_{\mathcal{I}}(A_i)) = \phi(\bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)) = \mathcal{U}_A$$

where $A := \bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)$ and ϕ denotes the isomorphism found in part (v). The case of $\bigwedge\{\mathcal{U}_{A_i} : i \in I\}$ can be proved in a similar way. □

Relatively to the lattice $\Xi(\mathcal{I})$, the join and meet operations have the following behavior with respect to the set theoretical intersection and union.

Proposition 6.6. *Let $\{A_i :\in I\} \subseteq CLOS(\mathcal{I})$. Then*

$$\bigwedge\{\mathcal{U}_{A_i} : i \in I\} \subseteq \bigcap_{i\in I} \mathcal{U}_{A_i}$$

and

$$\bigcup\{\mathcal{U}_{A_i} : i \in I\} \subseteq \bigvee\{\mathcal{U}_{A_i} : i \in I\}$$

Proof. Let us prove that $\mathcal{U}_{\bigcup_{i\in I} Dc_{\mathcal{I}}(A_i)} = \bigcap\{\mathcal{U}_{A_i} : i \in I\}$. To this regard, let $(Y,X) \in \mathcal{U}_A$, where $A := \bigcup_{i\in I} Dc_{\mathcal{I}}(A_i)$. Hence, $Y \subseteq up_A(X)$, i.e. for any $y \in Y$ it must be $[y]_A \cap X \neq \emptyset$. Since $[y]_A \subseteq [y]_{A_i}$ for any $i \in I$, we deduce that $y \in up_{A_i}(X)$ for any $y \in Y$, i.e. $(Y,X) \in \bigcap_{i\in I} \mathcal{U}_{A_i}$.
We now show that $\bigcup\{\mathcal{U}_{A_i} : i \in I\} \subseteq \mathcal{U}_{\bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)}$. For, let $(Y,X) \in \bigcup\{\mathcal{U}_{A_i} : i \in I\}$, i.e. there exists some index j such that $Y \subseteq up_{A_j}(X)$. Hence, for any $y \in Y$, it results that $[y]_{A_j} \cap X \neq \emptyset$. But since $[y]_{A_j} \subseteq [y]_A$, where $A := \bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)$, we conclude that $[y]_A \cap X \neq \emptyset$ for any $y \in Y$, that is $(Y,X) \in \mathcal{U}_{\bigcap_{i\in I} Dc_{\mathcal{I}}(A_i)}$. □

The Pawlak's independent subsets have the following characterization relatively to the dependency relation $\multimap_A$.

Proposition 6.7. $INDP(\multimap_A) = \{X \in \mathcal{P}(U) : x \not\equiv_A x'\ \forall x, x' \in X\}$.

Proof. Let $X \in INDP(\multimap_A)$. Hence, $Ic_{\multimap_A}(X) = X$, that is, by (i) of Proposition 6.4, $[x]_A \cap X = \{x\}$ for any $x \in X$. In particular, this is equivalent to say that $x \not\equiv_A x'$ for each distinct elements $x, x' \in X$. □

In the next result we provide a characterization for the $\multimap_A$dependency reducts.

Proposition 6.8. *Let $X \in \mathcal{P}(U)$ and $Y \subseteq X$. Then the following conditions are equivalent:*

- (i) *$Y \in RED_{\multimap_A}(X)$;*
- (ii) *for any $z \in U$ we have that*

$$[z]_A \cap X \neq \emptyset \implies |[z]_A \cap Y| = 1$$

Proof. Let $Y \in RED_{\multimap_A}(X)$, that is, by (iii) of Proposition 5.5, $up_A(Y) = up_A(X)$ and $Ic_{\multimap_A}(Y) = Y$. Let now $[z]_A \cap X \neq \emptyset$. By our assumptions, it follows that $[z]_A \cap Y \neq \emptyset$ and, in particular, $[z]_A \cap Y = \{y\}$ for some $y \in Y$. This proves that $|[z]_A \cap Y| = 1$.
Conversely, let $Y \subseteq X$ such that $[z]_A \cap X \neq \emptyset \implies |[z]_A \cap Y| = 1$. Clearly, $up_A(Y) \subseteq up_A(X)$. On the other hand, observe that whenever $[z]_A \cap X \neq \emptyset$, it also holds $[z]_A \cap Y \neq \emptyset$, i.e. $up_A(Y) \supseteq up_A(X)$. On the other hand, let $y \in Y$. Then $[y]_A \cap Y \neq \emptyset$ and let $y' \in [y]_A \cap Y$. By the fact that $[y]_A \cap Y \subseteq [y]_A \cap X \neq \emptyset$, it follows that $|[y]_A \cap Y| = 1$, that forces $y' = y$. In this way, we showed that $Ic_{\multimap_A}(Y) = Y$, so $Y \in RED_{\multimap_A}(X)$. □

Finally, for the $\multimap_A$essential subsets we have the following result.

Proposition 6.9. *Let $X \in \mathcal{P}(U)$ and $Y \subseteq X$. Then $Y \in ESS_{\multimap_A}(X)$ if and only if the two following conditions hold:*

	Temperature	Headache	Cold	Muscle-Pain	Cough	Stomach Ache
u_1	Very High	No	Yes	Yes	No	No
u_2	Normal	Yes	No	No	No	Yes
u_3	Very High	No	Yes	Yes	Yes	No
u_4	High	Yes	Yes	No	Yes	No
u_5	High	Yes	Yes	No	No	No
u_6	Very High	No	No	Yes	Yes	Yes
u_7	High	No	Yes	No	No	Yes
u_8	Very High	No	Yes	Yes	Yes	No
u_9	Normal	No	Yes	No	Yes	No
u_{10}	High	No	Yes	No	Yes	No
u_{11}	Normal	Yes	No	Yes	No	Yes
u_{12}	High	No	Yes	No	No	No

TABLE 1. Information Table of Example 6.10

(*i*) *for any* $z \in U$ *we have that*

$$[z]_A \cap X \neq \varnothing \iff [z]_A \cap ((X \smallsetminus Y) \cup \{y\}) \neq \varnothing \ \forall y \in Y; \tag{29}$$

(*ii*) *there exists* $w \in U$ *such that* $[w]_A \cap X = Y$.

Proof. Let $Y \in ESS_{\multimap_A}(X)$. By (ii) of Theorem 5.9, it follows that:

(1) $up_A(X \smallsetminus Y) \subsetneq up_A(X)$;
(2) $up_A(X \smallsetminus Y') = up_A(X)$ for any $Y' \subsetneq Y$.

Let us note that condition (2) is equivalent to the condition $up_A((X \smallsetminus Y) \cup \{y\}) = up_A(X)$ for any $y \in Y$. But this is clearly equivalent to (29). On the other hand, condition (1) ensures the existence of some $z \in U$ such that $[z]_A \cap X \neq \varnothing$ but $[z]_A \cap (X \smallsetminus Y) = \varnothing$, whence $[z]_A \cap Y \neq \varnothing$. For such an element z, by (29), it is possible to find an element w such that $w \in [z]_A \cap X$ if and only if $w \in Y$, i.e. $Y = [z]_A \cap X$. On the contrary, let Y be a subset of X satisfying the properties (i) and (ii). Clearly, $up_A(X \smallsetminus Y) \subseteq up_A(X)$. Just take the element w of our assumption (ii). Then it is immediate to verify that $[w]_A \cap X \neq \varnothing$ but $[w]_A \cap (X \smallsetminus Y) = \varnothing$, i.e. $up_A(X \smallsetminus Y) \subsetneq up_A(X)$. Finally, the condition described in part (I) is clearly equivalent to say that $up_A(X \smallsetminus Y') = up_A(X)$ for any $Y' \subsetneq Y$. This proves that $Y \in ESS_{\multimap_A}(X)$. □

Example 6.10. Twelve people made a check-up at the hospital to to determine if they have contracted the flu or not. In each patient's medical record, the collected data concerns the measurement of the temperature, that has been classified in three levels, namely Normal, High and Very High; the presence of other symptoms such as headache, cold, muscle-pain, cough and stomach ache.
The situation can be formalized by means of an information table $\mathcal{I}$ whose objects are the patients $u_1, \dots, u_{12}$ and whose attributes are Temperature, Headache, Cold, Muscle-Pain, Cough, Stomach Ache. Table 1 represents the given information table.
However, a doctor receives their answers concerning the attributes of the subset $A := \{$Temperature, Headache, Cold, Muscle-Pain$\}$. Moreover, there are some patients that prefer to have the diagnosis in the shortest possible time. They form the set $X = \{u_1, u_2, u_3, u_5, u_7, u_8\}$. Being forced to make a diagnosis based on previous data, the doctor considers the A-indiscernibility set partition induced by A on U, namely:

$$\pi_{\mathcal{I}}(A) = u_1u_3u_8|u_2|u_9|u_{11}|u_4u_5|u_7u_{10}u_{12}$$

Then, he computes both the A-lower and A-upper approximations of X in order to understand which patients can have a sure diagnosis and which of them are probably sick. In this case, one has respectively:

$$lw_A(X) = \{u_1, u_2, u_3, u_8\}$$

and

$$up_A(X) = \{u_1, u_2, u_3, u_4, u_5, u_7, u_8, u_{10}, u_{12}\}$$

Let us note that u_1, u_3, u_8 display symptoms thanks to which the doctor can classify them with certainty as having flu, whereas u_2 certainly has no flu. On the other hand, the symptoms displayed by u_5 and u_7 cannot induce the doctor to make a correct diagnosis. For instance, regarding the patient u_7, there may not be a correlation between the cold and his high temperature. Furthermore, notice that

$$Ic_{\multimap_A}(X) = \{u_2, u_5, u_7\}$$

In terms of $\multimap_A$Pawlak's independency core, we are saying that the deletion of one of the patients u_2, u_5 or u_7 (that we denote in general by x) causes a changement in the upper approximation of the remaining subset $X \smallsetminus x$. In fact, the previous elements are the only objects of the respective A-indiscernibility classes belonging to X.
For the sake of completeness, below we express the family of all $\multimap_A$Pawlak's independent subsets, relatively to the attribute subset A and to the set of all patients U:

(30)
$$INDP(\multimap_A) = \{\{u_1,u_2,u_4,u_6,u_7,u_9,u_{11}\},\{u_1,u_2,u_4,u_6,u_9,u_{10},u_{11}\},\{u_1,u_2,u_4,u_6,u_9,u_{11},u_{12}\},$$
$$\{u_1,u_2,u_5,u_6,u_7,u_9,u_{11}\},\{u_1,u_2,u_5,u_6,u_9,u_{10},u_{11}\},\{u_1,u_2,u_5,u_6,u_9,u_{11},u_{12}\},\{u_2,u_3,u_4,u_6,u_7,u_9,u_{11}\},$$
$$\{u_2,u_3,u_4,u_6,u_9,u_{10},u_{11}\},\{u_2,u_3,u_4,u_6,u_9,u_{11},u_{12}\},\{u_2,u_3,u_5,u_6,u_7,u_9,u_{11}\},\{u_2,u_3,u_5,u_6,u_9,u_{10},u_{11}\},$$
$$\{u_2,u_3,u_5,u_6,u_9,u_{10},u_{12}\},\{u_2,u_4,u_6,u_7,u_8,u_9,u_{11}\},\{u_2,u_4,u_6,u_8,u_9,u_{10},u_{11}\},\{u_2,u_4,u_6,u_8,u_9,u_{11},u_{12}\},$$
$$\{u_2,u_5,u_6,u_7,u_8,u_9,u_{11}\},\{u_2,u_5,u_6,u_8,u_9,u_{10},u_{11}\},\{u_2,u_5,u_6,u_8,u_9,u_{11},u_{12}\}\}$$

By intersecting with X the previous subsets, we will obtain the family of all $\multimap_A$Pawlak's independent subsets, relatively to the attribute subset A, that belong to X itself.
Finally, let us compute the smallest subsets Y of X inducing the same set partition of X. They are exactly
$$RED_{\multimap_A}(X) = \{\{u_1,u_2,u_5,u_7\},\{u_2,u_3,u_5,u_7\},\{u_2,u_5,u_7,u_8\}\}$$
Therefore, by taking each of the previous subsets and compute their upper approximations, we observe that they all coincide with $up_A(X)$. Moreover, the deletion of one single element from them induces a modification of the upper approximation. This means that the doctor could study one of the members of $RED_{\multimap_A}(X)$ in order to achieve the same diagnosis he made with the subset X. Finally, we get
$$ESS_{\multimap_A}(X) = \{\{u_2\},\{u_5\},\{u_7\},\{u_1,u_3,u_8\}\}$$
In fact, the deletion of one of the previous sets involves a changement in the upper approximation of the remaining set, that is $up_A(X \smallsetminus Y) \subsetneqq up_A(X)$. In other terms, the deletion of one of the previous sets yields a changement in the diagnosis of the doctor.

7. Dependency Relations Induced by Formal Contexts

In this section we study the natural dependency relation associated with a formal context having Ω as its attribute set.
A *formal context* on Ω is a pair $\mathbb{K} = (U,\mathcal{R})$, where U is a finite non-empty set whose elements are called the *objects* of $\mathbb{K}$ and $\mathcal{R}$ is a binary relation between U and Ω, that is $\mathcal{R} \subseteq U \times \Omega$. In this case, the elements of Ω are called *attributes* of $\mathbb{K}$.
If $V \subseteq U$ and $A \subseteq \Omega$, one usually defines
$$V^{\uparrow} := \{b \in \Omega : (\forall v \in V)\, v\mathcal{R}b\},\ \ A^{\downarrow} := \{u \in U : (\forall a \in A)\, u\mathcal{R}a\}$$
Consequently, the classical attribute subset implication $\leftarrow_{\mathbb{K}}$ induced by the formal context $\mathbb{K}$ (see [31]) is defined as follows:

$$B \leftarrow_{\mathbb{K}} A :\Longleftrightarrow B^{\downarrow} \supseteq A^{\downarrow}, \tag{31}$$

for any $A,B \in \mathcal{P}(\Omega)$. Also in this case, the binary relation $\leftarrow_{\mathbb{K}}$ is a well studied binary relation in FCA (see [31]). We call $\mathbb{K}$-*dependency* on Ω the binary relation $\leftarrow_{\mathbb{K}}$.

Let $\mathbb{K} = (U,\mathcal{R})$ be a formal context on Ω.
We set
$$CONC(\mathbb{K}) := \{(A^{\downarrow}, A^{\downarrow\uparrow}) : A \in \mathcal{P}(\Omega)\}$$
Then, in FCA an element $(A^{\downarrow}, A^{\downarrow\uparrow}) \in CONC(\mathbb{K})$ is called a *formal concept* of $\mathbb{K}$, with *extent* $A^{\downarrow}$ and *intent* $A^{\downarrow\uparrow}$.
An equivalent alternative characterization of $CONC(\mathbb{K})$ is the following:
$$CONC(\mathbb{K}) := \{(V,A) \in \mathcal{P}(U) \times \mathcal{P}(\Omega) : V^{\uparrow} = A,\ A^{\downarrow} = V\}$$
In the next result we establish the first basic properties of the $\mathbb{K}$-dependency.

Proposition 7.1. *Let $A \in \mathcal{P}(\Omega)$. Then:*
(i) $Dc_{\mathbb{K}}(A) = A^{\downarrow\uparrow}$ *for all* $A \in \mathcal{P}(\Omega)$.
(ii) $CLOS(\mathbb{K}) = \{A^{\downarrow\uparrow} :\ A \in \mathcal{P}(\Omega)\}$.
(iii) $Ic_{\mathbb{K}}(A) = \{a \in A :\ (A \smallsetminus a)^{\downarrow} \smallsetminus \{a\}^{\downarrow} \neq \varnothing\}$.
(iv) $INDP(\mathbb{K}) = \{A \in \mathcal{P}(\Omega) :\ \forall a \in A\ \ (A \smallsetminus a)^{\downarrow} \smallsetminus \{a\}^{\downarrow} \neq \varnothing\}$.

	a	b	c	d
u_1	0	1	1	1
u_2	1	1	1	0
u_3	0	0	1	1
u_4	0	1	0	0

TABLE 2. Formal Context $\mathbb{K}$ of Example 7.4

Proof. (i): We have that

$$b \in Dc_{\mathbb{K}}(A) \iff b \leftarrow_{\mathbb{K}} A \iff b^{\downarrow} \supseteq A^{\downarrow} \iff u\mathcal{R}b \;\; \forall u \in A^{\downarrow} \iff b \in A^{\downarrow\uparrow},$$

therefore (i) follows.
(ii): It follows immediately by part (i).
(iii): By definition of $Ic_{\mathbb{K}}$ we have that $a \not\leftarrow A \smallsetminus a$, that is equivalent to say that $a^{\downarrow} \not\supseteq (A \smallsetminus a)^{\downarrow}$ by (31). Hence, there exists $u \in (A \smallsetminus a)^{\downarrow}$ such that $u \notin a^{\downarrow}$, i.e. $u\mathcal{R}b$ for any $b \in A \smallsetminus a$ and $u \not\mathcal{R}\, a$. Thus, we conclude that $u \in \bigcap_{b \in A \smallsetminus a}\{b\}^{\downarrow} \smallsetminus \{a\}^{\downarrow}$, i.e. $u \in (A \smallsetminus a)^{\downarrow} \smallsetminus \{a\}^{\downarrow}$.
(iv): It follows immediately by part (iii). ☐

The $\mathbb{K}$-essential subsets are characterized in the next proposition.

Proposition 7.2. *$B \in ESS_{\mathbb{K}}(A)$ if and only if it satisfies the following conditions:*
$(EK1)$ *$B \subseteq A$;*
$(EK2)$ *there exists $b \in B$ such that $(A \smallsetminus B)^{\downarrow} \smallsetminus \{b\}^{\downarrow} \neq \emptyset$;*
$(EK3)$ *$B' \subsetneq B$, $u \in (A \smallsetminus B')^{\downarrow} \implies u \in (B')^{\downarrow}$.*

Proof. By Definition 5.6, we have that $B \leftarrow_{ess} A$ if and only if $(E1), (E2), (E3)$ hold. Conditions $(E1)$ and $(EK1)$ are the same. Let us show that $(EK2)$ is equivalent to $(E2)$. Indeed, condition $(E2)$ means that $B \not\leftarrow_{\mathcal{K}} A \smallsetminus B$, that is $B^{\downarrow} \not\supseteq (A \smallsetminus B)^{\downarrow}$. In other terms, there exists $u \in U$ such that $u\mathcal{R}a$ for each $a \in A \smallsetminus B$ but $u \not\mathcal{R}\, b$ for some $b \in B$, i.e. there exists $b \in B$ such that $(A \smallsetminus B)^{\downarrow} \smallsetminus \{b\}^{\downarrow} \neq \emptyset$. This proves the equivalence between $(E2)$ and $(EK2)$. Finally, condition $(E3)$ requires minimality of B with respect to the condition stated in $(E2)$. Equivalently, we are requiring that if $B' \subsetneq B$, then $B' \leftarrow_{\mathcal{K}} A \smallsetminus B'$, i.e. $(B')^{\downarrow} \supseteq (A \smallsetminus B')^{\downarrow}$. In other terms, if $u\mathcal{R}a$ for each $a \in A \smallsetminus B'$, then $u\mathcal{R}b'$ for any $b' \in B'$. So, $u \in (B')^{\downarrow}$. This shows the equivalence between $(E3)$ and $(EK3)$. ☐

Remark 7.3. *Let $\mathbb{K}$ be a formal context on Ω. By Theorem 3.6, we associate with the dependency relation $\leftarrow_{\mathbb{K}}$ an information table $\mathcal{I}$ such that $\leftarrow_{\mathbb{K}}$ coincides with $\leftarrow_{\mathcal{I}}$. In particular, we have that $RED_{\mathbb{K}}(A) = RED_{\mathcal{I}}(A)$ and $ESS_{\mathbb{K}}(A) = ESS_{\mathcal{I}}(A)$ for any $A \in \mathcal{P}(\Omega)$. Hence, by Theorems 3.14 and 3.15, we deduce that $RED_{\mathbb{K}}(A) = Tr(ESS_{\mathbb{K}}(A))$.*

We now interpret in an example the role of the set operators $Dc_{\mathbb{K}}$ and $Ic_{\mathbb{K}}$ and of the set systems $RED_{\mathbb{K}}(A)$ and $ESS_{\mathbb{K}}(A)$.

Example 7.4. Four students must support four examinations. At the end of the session, the four boys compare the results of each exam. We can modelize the cited situation by means of the formal context $\mathbb{K} = (U, \mathcal{R})$ on the set $\Omega = \{a, b, c, d\}$, where $U = \{u_1, u_2, u_3, u_4\}$ and $u\mathcal{R}a$ if and only if the student u passed the exam a. We set 1 in this case and 0 otherwise. We represent the formal context in Table 2.
For instance, let us fix $A = \{a\}$ and $B = \{a, b, c\}$. Then, by Proposition 7.1 we have $Dc_{\mathbb{K}}(A) = A^{\downarrow\uparrow} = \{a, b, c\}$, $Dc_{\mathbb{K}}(B) = B$, $Ic_{\mathbb{K}}(A) = A$ and $Ic_{\mathbb{K}}(B) = \{a\}$. In the case of $Dc_{\mathbb{K}}(A)$, this means that the set of exams passed by the students that have passed a is $\{a, b, c\}$, while in the case of $Dc_{\mathbb{K}}(B)$, the students that passed a, b and c have not passed the exam d.
On the other hand, in the case of $Ic_{\mathbb{K}}(B)$, we are saying that a is the only exam for which there is a student, namely u_1, that passed the remaining exams of B but not a itself. So, in particular, $Ic_{\mathbb{K}}(A) = A$.
In the remaining part of the example, we will use string notation when we express some set system. It is easy to see that

$$CLOS(\mathbb{K}) = \{\emptyset, b, c, bc, cd, abc, bcd, \Omega\}$$

A subset C belongs to $CLOS(\mathbb{K})$ whenever the students that passed the exams of C have not passed other exams in $\Omega \smallsetminus C$.
Moreover, it is straightforward to see that

$$ESS(\mathbb{K}) = \{a, d\}$$

and
$$RED(\mathbb{K}) = \{ad\}$$
The set system $ESS(\mathbb{K})$ is given by the minimal subsets $B \in \mathcal{P}(\Omega)$ such that $B \not\leftarrow \Omega \smallsetminus B$ or, equivalently, for which there exists $b \in B$ such that $(\Omega \smallsetminus B)^{\downarrow} \smallsetminus \{b\}^{\downarrow} \neq \varnothing$. In this way, we have found the minimal subsets of Ω whose deletion yields subsets of exams that have been passed from at least a student.
Contrariwise, the set system $RED(\mathbb{K})$ is given by the minimal subsets $B \in \mathcal{P}(\Omega)$ such that if $C \leftarrow \Omega$, then $C \leftarrow B$. But, since $\Omega^{\downarrow} = \varnothing$, we deduce that $RED(\mathbb{K})$ must contain all the subsets B of Ω such that $B^{\downarrow} = \varnothing$. Roughly speaking, $RED(\mathbb{K})$ contains the minimal subsets B of exams that have not been passed from all the students.

8. Dependency Relations Induced by Scott's Information Systems

In this section we treat the main example of dependency relation that is localized to a specific set system $\mathcal{F} \in SS(\Omega)$. The structure on Ω which naturally leads to such a localized dependency relation is that of *Scott's information system* [48]. We first recall the classical notion of Scott's information system.

Definition 8.1. *A Scott's information system $\mathcal{S}$ on Ω is an ordered pair $(Con, \leftarrow_{\mathcal{S}})$, where $Con \in SS(\Omega)$ and $\leftarrow_{\mathcal{S}} \in BREL(\Omega)$ in such a way that the following conditions are satisfied:*
- $(S1)$ *any $X \in Con$ is a finite subset of Ω;*
- $(S2)$ $\leftarrow_{\mathcal{S}} \subseteq \Omega \times Con$;
- $(S3)$ $a \in X$ *and* $X \in Con \implies a \leftarrow_{\mathcal{S}} X$;
- $(S4)$ $a \leftarrow_{\mathcal{S}} Y$ *and* $y \leftarrow_{\mathcal{S}} X \ \ \forall y \in Y \implies a \leftarrow_{\mathcal{S}} X$;
- $(S5)$ $a \leftarrow_{\mathcal{S}} X \implies X \cup \{a\} \in Con$;
- $(S6)$ $\{a\} \in Con$ *for any* $a \in \Omega$;
- $(S7)$ *if* $X \in Con$ *and* $X' \subseteq X$, *then* $X' \in Con$;

Usually, we use the notation $Y \leftarrow_{\mathcal{S}} X$ in order to say that $y \leftarrow_{\mathcal{S}} X$ for any $y \in Y$. In such a context, the elements of Ω are called tokens *of $\mathcal{S}$.*

In the remaining part of the section, we use the notation $\mathcal{S} = (Con, \leftarrow_{\mathcal{S}})$ to denote a generic Scott's information system on Ω. The results which we establish in this section provide a first connection between the notion of generalized dependency derived by RST developed in the above sections and Scott's information systems.
In the next proposition we show that the binary relation $\leftarrow \mathcal{S}$ is a $[\mathcal{F}]$-dependency relation on Ω when $\mathcal{F}$ coincides with the set system Con given in Definition 8.1.

Proposition 8.2. *$\leftarrow_{\mathcal{S}}$ is a $[Con]$-dependency relation on Ω.*

Proof. Let $X \subseteq X'$, where $X' \in Con$. By $(S3)$, it follows that $x \leftarrow_{\mathcal{S}} X'$ for any $x \in X$, i.e. $X \leftarrow_{\mathcal{S}} X'$.
Let now $Z \leftarrow_{\mathcal{S}} Y$ and $Y \leftarrow_{\mathcal{S}} X$. Clearly, by the definition of $\leftarrow_{\mathcal{S}}$, we have that $z \leftarrow_{\mathcal{S}} Y$ for any $z \in Z$, thus, by $(S4)$, it must necessarily be $z \leftarrow_{\mathcal{S}} X$ for any $z \in Z$, i.e. $Z \leftarrow_{\mathcal{S}} X$. Finally, $(D3)$ is obvious. □

We study now the $(\leftarrow_{\mathcal{S}}, Con)$-Pawlak preclosure operator. To this aim, let us establish the main properties of the set operator $Dc_{Con,\leftarrow_{\mathcal{S}}}$.

Theorem 8.3. *Let $A, B \in \mathcal{P}(\Omega)$. Then:*
(i) $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = \varnothing$ *if* $A \notin Con$;
(ii) *if* $A \in Con$, *then* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = \{a \in \Omega : a \leftarrow_{\mathcal{S}} A\}$;
(iii) *if* $A, B \in Con$ *and* $A \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$, *then* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$;
(iv) *if* $A \in Con$, *then* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \leftarrow_{\mathcal{S}} A$;
(v) *if* $A \in \max(Con)$, *then* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = A$;
(vi) *if Ω is finite and* $A \in Con$, *then* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \in Con$;
(vii) *if Ω is finite and* $A \in Con$, *then* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = Dc^2_{Con,\leftarrow_{\mathcal{S}}}(A)$.

Proof. (i): It follows immediately by the definition of $Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ and by $(S2)$.
(ii): Let $A \in Con$ and $a \leftarrow_{\mathcal{S}} A$. Since by $(S6)$ the singleton $\{a\}$ belongs to Con for any $a \in \Omega$, it easily follows that $a \in \bigcup\{C \in Con : C \leftarrow_{\mathcal{S}} A\} = Dc_{Con,\leftarrow_{\mathcal{S}}}$. On the contrary, let $a \in Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$. Then, there exists $C_a \in Con$ such that $a \in C_a$ and $C_a \leftarrow_{\mathcal{S}} A$. By $(S3)$, it follows that $a \leftarrow_{\mathcal{S}} C_a$ and, by $(D2)$, that $a \leftarrow_{\mathcal{S}} A$. This shows the claim.
(iii): Let $a \in Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$. By part (ii) and by (20), we have that $a \leftarrow_{\mathcal{S}} A \leftarrow_{\mathcal{S}} B$, so $a \leftarrow_{\mathcal{S}} B$ by $(D2)$, i.e. $a \in Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$.
(iv): Just notice that for any $a \in Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$, it results that $a \leftarrow_{\mathcal{S}} A$. Hence, by $(D3)$, we conclude that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \leftarrow_{\mathcal{S}} A$.

(v): Let $A \in \max(Con)$. Assume by contradiction that there exists $b \in \Omega \smallsetminus A$ such that $b \leftarrow_{\mathcal{S}} A$. By $(S5)$, we would have that $A \cup \{b\}$, contradicting the maximality of A. Then, $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = A$.
(vi): Let Ω be a finite set of tokens and $A \in Con$. By part (ii), it follows that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = \{a \in \Omega : a \leftarrow_{\mathcal{S}} A\}$. Now, if $a \leftarrow_{\mathcal{S}} A$ and $b \leftarrow_{\mathcal{S}} A$, then by $(S5)$, it must necessarily be $A \cup \{a\}, A \cup \{b\} \in Con$. But, it is immediate by $(D2)$ to notice that $b \leftarrow_{\mathcal{S}} A \cup \{a\}$ and $a \leftarrow_{\mathcal{S}} A \cup \{b\}$, thus, again by $(S5)$, we have $A \cup \{a, b\} \in Con$. By induction, the claim becomes straightforward to be proved.
(vii): By (i) of Proposition 4.7 and by the above part (vi) of this proposition, we have that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \subseteq Dc^2_{Con,\leftarrow_{\mathcal{S}}}(A)$. On the other hand, let $a \in Dc^2_{Con,\leftarrow_{\mathcal{S}}}(A)$. By part (iii) and by $(D2)$, it follows that $b \leftarrow A$ for any $b \in Dc^2_{Con,\leftarrow_{\mathcal{S}}}(A)$, that is, by $(D3)$, $Dc^2_{Con,\leftarrow_{\mathcal{S}}}(A) \leftarrow A$ or, equivalently, $Dc^2_{Con,\leftarrow_{\mathcal{S}}}(A) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$. This concludes the proof. □

We now characterize to condition to be a reduct through the set operators $Dc_{Con,\leftarrow_{\mathcal{S}}}$ and $Ic_{\leftarrow_{\mathcal{S}}}$. In particular, the statement of the next result is almost identical to that of Theorem 5.5. In this case, our assumptions on Con are weaker, since it usually happens that $Con \notin UCL(\Omega)$. Nevertheless, as a consequence of (iii) of Theorem 8.3, we overcome the difficulty posed by the lack of the union closed condition for Con.

Proposition 8.4. *Let $A \in Con$ and $B \subseteq A$. Then the following conditions are equivalent:*

- (i) $B \in RED_{\leftarrow_{\mathcal{S}}}(A)$.
- (ii) $Dc_{Con,\leftarrow_{\mathcal{S}}}(B) = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ *and* $Dc_{Con,\leftarrow_{\mathcal{S}}}(B') \subsetneq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ *for all* $B' \subsetneq B$.
- (iii) $Dc_{Con,\leftarrow_{\mathcal{S}}}(B) = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ *and* $Ic_{\leftarrow_{\mathcal{S}}}(B) = B$.

Proof. $(i) \implies (ii)$: Let $B \subseteq A$ be minimal with respect to the property that $C \leftarrow_{\mathcal{S}} A \implies C \leftarrow_{\mathcal{S}} B$. By (ii) of Theorem 8.3, this is equivalent to require that B is minimal with respect to the property $C \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \implies C \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$. Now, since $A \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$, we have $A \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$ that, by (iii) of Theorem 8.3, gives $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$. On the other hand, by (i) of Proposition 4.7, we also have that $Dc_{Con,\leftarrow_{\mathcal{S}}}(B) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$. This proves that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$. Finally, let $B' \subsetneq B$ and assume that $C \leftarrow_{\mathcal{S}} A$. By the minimality of B, there exists $c \in C$ such that $c \not\leftarrow B'$. By (ii) of Theorem 8.3 this means that $c \notin Dc_{Con,\leftarrow_{\mathcal{S}}}(B')$. So, $Dc_{Con,\leftarrow_{\mathcal{S}}}(B') \subsetneq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$.
$(ii) \implies (iii)$: Let us assume that $Dc_{Con,\leftarrow_{\mathcal{S}}}(B) = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ and $Dc_{Con,\leftarrow_{\mathcal{S}}}(B') \subsetneq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ for all $B' \subsetneq B$. We now prove that $Ic_{\leftarrow_{\mathcal{S}}}(B) = B$. For, take $B' = B \smallsetminus \{b\}$, for some $b \in B$. Assume by contradiction that $b \leftarrow_{\mathcal{S}} B \smallsetminus b$, i.e. $b \in Dc_{Con,\leftarrow_{\mathcal{S}}}(B')$. In particular, it follows that $B \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B')$ or, equivalently, $B \leftarrow_{\mathcal{S}} B'$. By (iii) of Theorem 8.3, it must be $Dc_{Con,\leftarrow_{\mathcal{S}}}(B) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(B')$. The reverse inclusion trivially holds. Hence, we showed that $Dc_{Con,\leftarrow_{\mathcal{S}}}(B') = Dc_{Con,\leftarrow_{\mathcal{S}}}(B) = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$, contradicting our hypothesis. Then, $b \not\leftarrow_{\mathcal{S}} B'$, that is $b \in Ic_{\leftarrow_{\mathcal{S}}}(B)$. By the arbitrariness of b, we deduce that $Ic_{\leftarrow_{\mathcal{S}}}(B) = B$.
$(iii) \implies (i)$: Let $B \subseteq A$ be such that $Dc_{Con,\leftarrow_{\mathcal{S}}}(B) = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ and $Ic_{\leftarrow_{\mathcal{S}}}(B) = B$. Clearly, $(B1)$ is satisfied. We must prove that B satisfies $(B2)$. To this regard, let $C \leftarrow_{\mathcal{S}} A$. By (ii) of Theorem 8.3, this means that $C \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$, i.e. $C \leftarrow_{\mathcal{S}} B$. Finally, let us show that B satisfies $(B3)$. For, assume by contradiction that there exists $B' \subsetneq B$ satisfying $(B2)$. Then, there would be $b \in B$ such that $B' \subseteq B \smallsetminus b$. Since $B' \subsetneq A$ and $A \in Con$, we would have $B' \leftarrow_{\mathcal{S}} A$. On the other hand, we would also have that $A \leftarrow_{\mathcal{S}} B'$ and, in particular, by $(D3)$, that $b \leftarrow_{\mathcal{S}} B'$. This is absurd, since $Ic_{\leftarrow_{\mathcal{S}}}(B) = B$, i.e. $b \not\leftarrow_{\mathcal{S}} B'$. This proves that B satisfies $(B3)$, i.e. $B \in RED_{\leftarrow_{\mathcal{S}}}(A)$. □

In the next result, we characterize the essential subsets in terms of the set operator $Dc_{Con,\leftarrow_{\mathcal{S}}}$. Also in this case, the claim of the next proposition is almost identical to that of Proposition 5.7, though we do not require that Con is union closed. Indeed, (iii) of Theorem 8.3 provides the tool to weaken our assumptions.

Proposition 8.5. *Let $A \in Con$ and $B \subseteq A$. Then the following conditions are equivalent:*

- (i) $B \in ESS_{\leftarrow_{\mathcal{S}}}(A)$.
- (ii) $Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B) \subsetneq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ *and* $Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B') = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ *for all* $B' \subsetneq B$.

Proof. $(i) \implies (ii)$: Assume that $B \in ESS_{\leftarrow_{\mathcal{S}}}(A)$. Hence, by $(E2)$, it results that $B \not\leftarrow_{\mathcal{S}} A \smallsetminus B$ or, equivalently, there exists $b \in B$ such that $b \not\leftarrow_{\mathcal{S}} A \smallsetminus B$. In particular, by (ii) of Theorem 8.3, $b \notin Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B)$. But, clearly, $b \in Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$. This implies that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B) \subsetneq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$. Now, let us consider $B' \subsetneq B$. By the minimality of B, we have $B' \leftarrow_{\mathcal{S}} A \smallsetminus B'$, i.e. $B' \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B')$. Hence, $A \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B')$ and, by (iii) of Theorem 8.3, we conclude that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B')$. The reverse inclusion clearly holds and the claim has been showed.
$(ii) \implies (i)$: Let $B \subseteq A$ be such that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B) \subsetneq Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ and $Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B') = Dc_{Con,\leftarrow_{\mathcal{S}}}(A)$ for all $B' \subsetneq B$. $(E1)$ trivially holds. Now, let us prove $(E2)$. To this regard, assume

by contradiction that $B \leftarrow_{\mathcal{S}} A \smallsetminus B$. Then, by (ii) of Theorem 8.3 and by (20), it follows that $B \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B)$. Therefore, a fortiori, $A \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B)$ and, again by (iii) of Theorem 8.3, we conclude that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \subseteq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B)$. Clearly, the reverse inclusion also holds, but this contradicts our hypothesis. Hence, $B \not\leftarrow_{\mathcal{S}} A \smallsetminus B$.
We finally prove minimality of B with respect to property $(E2)$. For, let us assume by contradiction that there exists $B' \subsetneq B$ be such that $B' \not\leftarrow_{\mathcal{S}} A \smallsetminus B'$. Thus, there would be $b' \in B'$ such that $b' \not\leftarrow_{\mathcal{S}} A \smallsetminus B'$, i.e. $b' \notin Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B')$. This means that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) \neq Dc_{Con,\leftarrow_{\mathcal{S}}}(A \smallsetminus B')$, contradicting our hypothesis. This proves that $(E3)$ and, in particular, that $B \in ESS_{\leftarrow_{\mathcal{S}}}(A)$. □

In the next examples, we provide some interpretations relative to the set operators $Dc_{Con,\leftarrow_{\mathcal{S}}}$ and $Ic_{\leftarrow_{\mathcal{S}}}$ and to the set systems $RED_{\leftarrow_{\mathcal{S}}}(A)$ and $ESS_{\leftarrow_{\mathcal{S}}}(A)$ when Ω is a vector space and A is any finite subset of Ω and when $\Omega = \mathbb{N}$ and A is again any finite subset of Ω.

Example 8.6. Let us consider a vector space V on a field $\mathbb{K}$ and the associated Scott system $\mathcal{S} := (\mathcal{P}_f(V), \leftarrow_{\mathcal{S}})$, where $a \leftarrow_{\mathcal{S}} Y$ if and only if $a \in \mathrm{Span}(Y)$. Let $A \subseteq_f \Omega$. By (ii) of Theorem 8.3, it can be easily shown that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = \mathrm{Span}(A)$. In particular, there exists no subset $A \in Con$ such that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = A$.
Furthermore, it is straightforward to show that $Ic_{\leftarrow_{\mathcal{S}}}(A) = A$ if and only if the vectors of A are linearly independent. In fact, let $Ic_{\leftarrow_{\mathcal{S}}}(A) = A$, then for any $a \in A$ it results that $a \notin \mathrm{Span}(A \smallsetminus a)$, so a cannot be obtained as linear combination of vectors of $\mathrm{Span}(A \smallsetminus a)$, so it is linearly independent with the vectors of $A \smallsetminus a$. On the other hand, assume that the vectors of A are mutually linearly independent and let $a \in A$. Clearly, $a \notin \mathrm{Span}(A \smallsetminus a)$, so $a \in Ic_{\leftarrow_{\mathcal{S}}}(A)$.
Let us now provide an interpretation for dependency reducts and essential subsets. By (iii) of Proposition 8.4, we have that $B \in RED_{\leftarrow_{\mathcal{S}}}(A)$ if and only if $\mathrm{Span}(A) = \mathrm{Span}(B)$ and B is constituted by minimal subsets of linearly independent vectors of A. On the contrary, by Proposition 8.5, $B \in ESS_{\leftarrow_{\mathcal{S}}}(A)$ if and only if its deletion from A gives rise to a finite set $A \smallsetminus B$ whose span does not coincide with the vector space spanned by A, while the deletion of any of its proper subsets B' from A yields exactly $\mathrm{Span}(A)$. In particular, given a finite vector subset A, the deletion of an element that is linearly independent vector always yields a vector subspace different from $\mathrm{Span}(A)$. For example, if the vectors of A are pairwise linearly independent, then $ESS_{\leftarrow_{\mathcal{S}}}(A) = \{\{v\} : v \in A\}$.

Example 8.7. Let us consider $\Omega = \mathbb{N}$ and the associated Scott system $\mathcal{S} := (\mathcal{P}_f(\mathbb{N}), \leftarrow_{\mathcal{S}})$, where $n \leftarrow_{\mathcal{S}} A$ if and only if there exists $a \in A$ such that $n \leq a$. Let $A \subseteq_f \Omega$. We now evaluate the set operators $Dc_{Con,\leftarrow_{\mathcal{S}}}$ and $Ic_{\leftarrow_{\mathcal{S}}}$. To this regard, let us note that $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = \{0, \dots, \max(A)\}$.
On the other hand, it follows that $Ic_{\leftarrow_{\mathcal{S}}}(A) = \{\max(A)\}$. Let us show that $Ic_{\leftarrow_{\mathcal{S}}}(A) = A$ if and only if $A = \varnothing$ or $|A| = 1$. In fact, if $A = \varnothing$ or $|A| = 1$, it is obvious that A is fixed point for the set operator $Ic_{\leftarrow_{\mathcal{S}}}$. On the contrary, let us assume that $Ic_{\leftarrow_{\mathcal{S}}}(A) = A$ and $A \neq \varnothing$. This means that $a \not\leftarrow_{\mathcal{S}} A \smallsetminus a$ for each $a \in A$, that is $a > a'$ for any $a' \in A \smallsetminus a$, i.e. $a = \max(A)$ for any $a \in A$. By the uniqueness of the maximum element, we conclude that $A = \{\max(A)\}$.
By (iii) of Proposition 8.4 and by what we have shown above, it results that $B \in RED_{\leftarrow_{\mathcal{S}}}(A)$ if and only if $B = \{\max(A)\}$. As a matter of fact, just observe that if $B \subseteq A$ and $Dc_{Con,\leftarrow_{\mathcal{S}}}(A) = Dc_{Con,\leftarrow_{\mathcal{S}}}(B)$, then $\max(A) = \max(B)$. Moreover, since $Ic_{\leftarrow_{\mathcal{S}}}(B) = B$, then $|B| = 1$, so $B = \{\max(A)\}$.
Finally, let us find $ESS_{\leftarrow_{\mathcal{S}}}(A)$. To this regard, let $B \in ESS_{\leftarrow_{\mathcal{S}}}(A)$. By definition of essential subset, we have that $B \not\leftarrow_{\mathcal{S}} A \smallsetminus B$. In particular, B must necessarily contain $\max(A)$. Now, let us take $B' \subsetneq B$. We distinguish two cases: if $b' := \max(A) \notin B'$, then it is obvious that $B' \leftarrow_{\mathcal{S}} A \smallsetminus B'$; whereas if $b' \in B'$, then for any $a \in A \smallsetminus B'$ we would have $a < b'$, i.e. $B' \not\leftarrow_{\mathcal{S}} A \smallsetminus B'$, contradicting the definition of essential subset. This shows that $ESS_{\leftarrow_{\mathcal{S}}}(A) = \varnothing$.

9. Dependency Relations Induced by Possibility Measures

Possibility theory (PT) is a mathematical theory whose basic aim is to investigate some types of uncertainty. Possibility theory was introduced by Zadeh in [61] and it can be considered an alternative to probability theory (for further studies see [29, 30]).
In this section we study a dependency relation which can be naturally defined in PT and, in this case, we also provide a description for the basic set systems and set operators described in Section 4 and Section 5.
In PT, our starting set Ω is interpreted as a *universe of discourse*. The basic notion of PT is that of *possibility measure* on Ω (see [29, 61]), which now we recall.

Definition 9.1. *A* possibility measure *on Ω is a map $\Pi : \mathcal{P}(\Omega) \to [0, 1]$ such that:*

$(P1)$ $\Pi(\Omega) = 1$*;*

(P2) $\Pi(\emptyset) = 0$;
(P3) *if* $B_i \in \mathcal{P}(\Omega)$ *for* $i \in I$, *then* $\Pi(\bigcup_{i\in I} B_i) = \sup_{i\in I} \Pi(B_i)$.

Let us note that by $(P3)$ it follows that

$$\Pi(B) = \sup\{\Pi(b) : b \in B\}, \tag{32}$$

for any $B \in \mathcal{P}(\Omega)$.
Let Π be a fixed possibility measure on Ω. We define then the following binary relation between elements of $\mathcal{P}(\Omega)$. For any $A, B \in \mathcal{P}(\Omega)$, we set

$$B \leftarrow_\Pi A :\Longleftrightarrow \Pi(B) \le \Pi(A)$$

Then we have the following result.

Proposition 9.2. $\leftarrow_\Pi$ *is a dependency relation on* Ω.

Proof. Let $B \subseteq A$. Then, by $(P3)$, $\Pi(B) = \sup\{\Pi(b) : b \in B\} \le \sup\{\Pi(a) : a \in A\}$, so that $B \leftarrow_\Pi A$. Therefore $\leftarrow_\Pi$ is inclusive. The transitivity is obvious. Finally, let I be an arbitrary index set and let $B_i \leftarrow_\Pi A$ for any $i \in I$, that is $\Pi(B_i) \le \Pi(A)$ for all $i \in I$. Then, again by $(P3)$, we have that $\Pi(\bigcup_{i\in I} B_i) = \sup_{i\in I} \Pi(B_i) \le \Pi(A)$, i.e. $\bigcup_{i\in I} B_i \leftarrow_\Pi A$. Then the thesis follows by Proposition 2.6. □

As in the previous sections, we will use simply the symbol Π instead of $\leftarrow_\Pi$ to denote the set operators and related set systems on Ω which depend on the dependency relation $\leftarrow_\Pi$.
By (21), it follows that the Π-Pawlak's dependency closure operator can be described by

$$Dc_\Pi(A) = \{b \in \Omega : \Pi(b) \le \Pi(A)\}, \tag{33}$$

for any $A \in \mathcal{P}(\Omega)$.
Therefore we obtain the following characterizations for the Π-Pawlak's closed subsets.

Proposition 9.3. *Let* $A, A' \in \mathcal{P}(\Omega)$. *Then:*

(i) $A \in CLOS(\Pi)$ *if and only if for any* $b \in \Omega \smallsetminus A$ *we have* $\Pi(b) > \Pi(A)$;
(ii) *if* $A, A' \in CLOS(\Pi)$, *we have that* $A \subseteq A'$ *or* $A' \subseteq A$, *in other terms, the lattice* $(CLOS(\pi), \subseteq)$ *is totally ordered.*

Proof. (i): Let $A \in CLOS(\Pi)$. Then, by definition, $A = Dc_\Pi(A)$. Therefore, if $b \in \Omega \smallsetminus A$, by (33) we have that $\Pi(b) > \Pi(A)$. Vice versa, assume that $\Pi(b) > \Pi(A)$ for all $b \in \Omega \smallsetminus A$. Let $b \in Dc_\pi(A)$. By (33), we have $\Pi(b) \le \Pi(A)$, therefore $b \in A$.
(ii): Let $A, A' \in CLOS(\Pi)$ such that $A \nsubseteq A'$. Therefore there exists $a \in A$ such that $a \notin A'$. Since A' is Π-dependency closed, by previous part (i), we have that $\Pi(a) > \Pi(A')$. Let now $a' \in A'$, and we assume by absurd that $a' \notin A$. Again, since $A \in CLOS(\Pi)$, by (i) we have that $\Pi(a') > \Pi(A) \ge \Pi(a) > \Pi(A')$. Hence we obtain an element $a' \in A'$ such that $a' \not\leftarrow_\Pi A'$, that is in contrast with the inclusive property of $\leftarrow_\Pi$. This shows that $A' \subseteq A$. □

Analogously, by (24) we obtain the following description for the Π-Pawlak's independency core operator:

$$Ic_\Pi(A) = \{a \in A : \Pi(a) > \Pi(A \smallsetminus a)\}, \tag{34}$$

for any $A \in \mathcal{P}(\Omega)$.
Let us note that as a direct consequence of (33) and (34) we have that

$$Ic_\Pi(A) = \{a \in A : a \notin Dc_\Pi(A \smallsetminus a)\} \tag{35}$$

For the Π-Pawlak's independent subset family we have the following very simple characterization.

Proposition 9.4. $INDP(\Pi) = \{\emptyset\} \cup \{\{a\} : a \in \Omega, \Pi(a) > 0\}$.

Proof. Clearly, $\emptyset \in INDP(\Pi)$. Let now A be a non-empty element of $INDP(\Pi)$. Hence $Ic_\Pi(A) = A$, that is, for any $a \in A$, it results that $\Pi(a) > \Pi(A \smallsetminus a)$. Assume that there are two distinct elements $a, b \in Ic_\Pi(A)$, i.e. $a \notin Dc_\Pi(A \smallsetminus a)$ and $b \notin Dc_\Pi(A \smallsetminus B)$. By (ii) of Proposition 9.3, we have $Dc_\Pi(A \smallsetminus a) \subseteq Dc_\Pi(A \smallsetminus b)$ or $Dc_\Pi(A \smallsetminus b) \subseteq Dc_\Pi(A \smallsetminus a)$. Suppose the the former holds (the other case is symmetric). Then $b \in A \smallsetminus a$, but $b \notin Dc_\Pi(A \smallsetminus a)$ and this is an absurd. By (34), it is now straightforward to see that A must necessarily be a singleton $\{a\}$ such that $\Pi(a) > 0$. □

In the next result, we explicitly express the conditions satisfied by $B \subseteq A$ in order to be a dependency reduct of A. In what follows, we use the notation $\Im(\Omega) := \{\Pi(b) : b \in \Omega\}$.

Proposition 9.5. *Let* $B \subseteq A$. *Then* $B \in RED_\Pi(A)$ *if and only if* $|B| \le 1$ *and* $]\,\Pi(B), \Pi(A)] \cap \Im(\Omega) = \emptyset$.

Proof. Let $B \subseteq A$. By (iii) of Theorem 5.5, we have that $B \in RED_{\Pi}(A)$ if and only if $Dc_{\Pi}(B) = Dc_{\Pi}(A)$ and $Ic_{\Pi}(B) = B$. The latter condition implies that $B = \varnothing$ or $B = \{b\}$ and $\Pi(b) > 0$. On the other hand, the condition $Dc_{\Pi}(B) = Dc_{\Pi}(A)$ implies that $\Pi(c) \leq \Pi(B) \leq \Pi(A)$ for any $c \in Dc_{\Pi}(A)$, i.e. $]\,\Pi(B), \Pi(A)] \cap \Im(\Omega) = \varnothing$. $\square$

We finally give a characterization for the essential subsets of A.

Proposition 9.6. *Let $B \subseteq A$. Then $B \in ESS_{\Pi}(A)$ if and only if the two following conditions hold:*
- (i) $]\,\Pi(A \smallsetminus B), \Pi(A)] \cap \Im(\Omega) \neq \varnothing$;
- (ii) *for any $B' \subsetneq B$, we have that* $]\,\Pi(A \smallsetminus B'), \Pi(A)] \cap \Im(\Omega) = \varnothing$

Proof. By Proposition 5.7, we have that $B \in ESS_{\Pi}(A)$ if and only if $Dc_{\Pi}(A \smallsetminus B) \subsetneq Dc_{\Pi}(A)$ and $Dc_{\Pi}(A \smallsetminus B') = Dc_{\Pi}(A)$ for any $B' \subsetneq B$. In other terms, there exists $c \in \Omega$ such that $\Pi(c) \leq \Pi(A)$ but $\Pi(c) > \Pi(A \smallsetminus B)$ and, moreover, for any $B' \subsetneq B$ and each $c \in Dc_{\Pi}(A)$, we have that $\Pi(c) \leq \Pi(A \smallsetminus B') \leq \Pi(A)$. In particular, the previous conditions are equivalent to require that both $]\,\Pi(A \smallsetminus B), \Pi(A)] \cap \Im(\Omega) \neq \varnothing$ and $]\,\Pi(A \smallsetminus B'), \Pi(A)] \cap \Im(\Omega) = \varnothing$, for any $B' \subsetneq B$, hold. $\square$

10. $[\mathcal{F}]$-Dependency Envelopes

In this last section we investigate the problem of *generating* generalized dependency relations. More in detail, starting with a fixed set system $\mathcal{F}$ and taking any ordered pair system $\mathcal{D} \in BREL(\Omega)$, we consider the smallest $[\mathcal{F}]$-dependency relation $\mathcal{D}^+_{\mathcal{F}}$ on Ω which contains all ordered pairs $(X, Y) \in \mathcal{D}$. We call $\mathcal{D}^+_{\mathcal{F}}$ the $[\mathcal{F}]$-dependency envelope of $\mathcal{D}$ and we will see how to generate it recursively when a condition of stabilization holds. This recursive process can be considered a formalization and a generalization of the deductive inference by means of the Armstrong's rules (for details see [49]).
To start, it is immediate to verify that the intersection of $[\mathcal{F}]$-dependency relations is again a $[\mathcal{F}]$-dependency relation. Hence we have:

Proposition 10.1. *If $\{\mathcal{D}_i : i \in I\} \subseteq DREL(\mathcal{F}|\Omega)$, then $\bigcap_{i \in I} \mathcal{D}_i \in DREL(\mathcal{F}|\Omega)$.*

Let now $\mathcal{D} \in BREL(\Omega)$. We set

$$\mathcal{D}^+_{\mathcal{F}} := \bigcap \{\mathcal{E} \in DREL(\mathcal{F}|\Omega) : \mathcal{D} \subseteq \mathcal{E}\} \tag{36}$$

and, in particular,

$$\mathcal{D}^+ := \mathcal{D}^+_{\mathcal{P}(\Omega)} = \bigcap \{\mathcal{E} \in DREL(\Omega) : \mathcal{D} \subseteq \mathcal{E}\} \tag{37}$$

Then we have the following result.

Proposition 10.2. *$\mathcal{D}^+_{\mathcal{F}}$ is the smallest $[\mathcal{F}]$-dependency relation on Ω containing $\mathcal{D}$ as a subrelation and the set operator $Env_{\mathcal{F}} : \mathcal{D} \in BREL(\Omega) \mapsto \mathcal{D}^+_{\mathcal{F}} \in DREL(\mathcal{F}|\Omega)$ is a closure operator on $\mathcal{P}(\Omega)^2$. Moreover, the closure system induced by the closure operator $Env_{\mathcal{F}}$ is $DREL(\mathcal{F}|\Omega)$.*

Proof. Let us observe that the set $\{\mathcal{E} \in DREL(\mathcal{F}|\Omega) : \mathcal{D} \subseteq \mathcal{E}\}$ is non-empty since $\mathcal{P}(\Omega)^2 \in DREL(\mathcal{F}|\Omega)$. Thus, by Proposition 10.1, it follows that $\mathcal{D}^+_{\mathcal{F}} \in DREL(\mathcal{F}|\Omega)$. Moreover, it is obvious that $\mathcal{D}^+$ is the smallest dependency relation on Ω containing $\mathcal{D}$.
We now prove that the set operator $Env_{\mathcal{F}} : \mathcal{D} \mapsto \mathcal{D}^+_{\mathcal{F}}$ is a closure operator on $\mathcal{P}(\Omega)^2$. Let $\mathcal{D} \in BREL(\Omega)$. Then $\mathcal{D} \subseteq \mathcal{D}^+_{\mathcal{F}}$. On the other hand, if $\mathcal{D} \subseteq \mathcal{G}^+_{\mathcal{F}}$, it is immediate to see that $\mathcal{D}^+_{\mathcal{F}} \subseteq \mathcal{G}^+_{\mathcal{F}}$ since the $[\mathcal{F}]$-dependency relations containing $\mathcal{G}^+_{\mathcal{F}}$ also contain $\mathcal{D}$. Finally, since $\mathcal{D}^+_{\mathcal{F}}$ is a $[\mathcal{F}]$-dependency relation, we have $(\mathcal{D}^+_{\mathcal{F}})^+_{\mathcal{F}} = \mathcal{D}^+_{\mathcal{F}}$ and the claim has been proved.
Finally, if $\mathcal{D} \in DREL(\mathcal{F}|\Omega)$, then $\mathcal{D} = \mathcal{D}^+_{\mathcal{F}} := Env_{\mathcal{F}}(\mathcal{D})$, so that $DREL(\mathcal{F}|\Omega) = Fix(Env_{\mathcal{F}})$ is the closure system induced by $Env_{\mathcal{F}}$. $\square$

Since $Env_{\mathcal{F}}$ is a closure operator on $\mathcal{P}(\Omega)^2$, by Theorem 4.8 it follows that

$$Inc(Env_{\mathcal{F}}) = \{(\mathcal{D}', \mathcal{D}) \in BREL(\Omega)^2 : \mathcal{D}' \subseteq Env_{\mathcal{F}}(\mathcal{D}) = \mathcal{D}^+_{\mathcal{F}}\}$$

is a dependency relation on $\mathcal{P}(\Omega)^2$. We denote such a relation by $\leftsquigarrow_{\mathcal{F}}$. So that, by using this notation, we have that $\leftsquigarrow_{\mathcal{F}} \in DREL(\mathcal{P}(\Omega)^2)$ and

$$\mathcal{D}' \leftsquigarrow_{\mathcal{F}} \mathcal{D} :\Longleftrightarrow \mathcal{D}' \subseteq \mathcal{D}^+_{\mathcal{F}},$$

for any $\mathcal{D}, \mathcal{D}' \in BREL(\Omega)$.

Definition 10.3. *We call:*
- *$\mathcal{D}^+_{\mathcal{F}}$ the $[\mathcal{F}]$-*dependency envelope *of $\mathcal{D}$ and, in particular, $\mathcal{D}^+$ the* dependency envelope *of $\mathcal{D}$;*

- $Env_{\mathcal{F}}$ *the* $[\mathcal{F}]$-dependency envelope closure operator *on* Ω *and, in particular,* $Env := Env_{\mathcal{P}(\Omega)}$ *the* dependency envelope closure operator *on* Ω*;*
- $\leftsquigarrow_{\mathcal{F}}$ *the* $[\mathcal{F}]$-dependency envelope relation *on* Ω *and, in particular,* $\leftsquigarrow := \leftsquigarrow_{\mathcal{P}(\Omega)}$ *the* dependency envelope relation *on* Ω.

Remark 10.4. *A priori, we cannot exclude the possibility that the notion of* $[\mathcal{F}]$*-dependency relation becomes useful on a computational level. For example, it cannot be excluded that the use of some restrictions to specific set systems could be helpful even in the attribute reduction problem. The underlying issue concerns the link between the study of the attribute reduction and the determination of dependency envelopes* $\mathcal{D}^+$*, for specific relations* $\mathcal{D} \in BREL(\Omega)$ *based on the Armstrong's rules (for recent studies on such topics see for example* [51, 54]*). To this regard, it is possible (but this is currently only an open research perspective) that the determination of specific smaller relations* $\mathcal{D}^+_{\mathcal{F}}$ *can lead towards an approximation methodology for the computation of the whole* $\mathcal{D}^+$*. Nevertheless, since in this paper we introduced the notion of* $[\mathcal{F}]$*-dependency relation and examined how it relates many branches of computer science, we prefer to investigate the first basic properties. In this perspective, the analysis of computational consequences goes beyond the aims of the paper and we postpone the previous and other further issues to future researches.*

We set now

$$\mathcal{D}^0_{\mathcal{F}} := \mathcal{D} \cup \{(B,A) \in \hat{\Omega}_{tr} : A \in \mathcal{F}\} = \mathcal{D} \cup \{(B,A) \in \mathcal{P}(\Omega)^2 : B \subseteq A \in \mathcal{F}\}$$

Next, if $k \geq 0$, we inductively define $\mathcal{D}^{k+1}_{\mathcal{F}}$ as follows. A pair (B,A) belongs to $\mathcal{D}^{k+1}_{\mathcal{F}}$ if and only if one of the following condition holds:

$(P1)$ there exists $C \in \mathcal{P}(\Omega)$ such that (B,C), $(C,A) \in \mathcal{D}^k_{\mathcal{F}}$.
$(P2)$ there exists $\{B_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ such that $(B_i, A) \in \mathcal{D}^k_{\mathcal{F}}$ for each $i \in I$ and $B = \bigcup_{i \in I} B_i$.

In view of $(P2)$, we have the following ascending chain

$$\mathcal{D} \subseteq \mathcal{D}^0_{\mathcal{F}} \subseteq \mathcal{D}^1_{\mathcal{F}} \subseteq \cdots \subseteq \mathcal{D}^k_{\mathcal{F}} \subseteq \cdots, \tag{38}$$

and we set

$$\mathcal{D}^{\diamond}_{\mathcal{F}} := \bigcup_{k \geq 0} \mathcal{D}^k_{\mathcal{F}}$$

Definition 10.5. *Let* $k \geq 0$*. We say that* $\mathcal{D} \in BREL(\Omega)$ *is* $(k,\mathcal{F})$-stabilizing *if*

$$\mathcal{D}^0_{\mathcal{F}} \subsetneq \mathcal{D}^1_{\mathcal{F}} \subsetneq \cdots \subsetneq \mathcal{D}^{k-1}_{\mathcal{F}} \subsetneq \mathcal{D}^k_{\mathcal{F}} = \mathcal{D}^{k+1}_{\mathcal{F}} = \cdots \tag{39}$$

We denote by $SREL_k(\mathcal{F}|\Omega)$ *the collection of all* $(k,\mathcal{F})$*-stabilizing binary relations* $\mathcal{D} \in BREL(\Omega)$*, and we also set* $SREL(\mathcal{F}|\Omega) := \bigcup_{k \geq 0} SREL_k(\mathcal{F}|\Omega)$*. We say that* $\mathcal{D}$ *is* $\mathcal{F}$-stabilizing *if* $\mathcal{D} \in SREL(\mathcal{F}|\Omega)$*. In particular, we say that* $\mathcal{D} \in BREL(\Omega)$ *is* k-stabilizing *if it is* $(k,\mathcal{F})$*-stabilizing and that* $\mathcal{D}$ *is* stabilizing *if it is* $\mathcal{P}(\Omega)$*-stabilizing.*

- When $\mathcal{F}$ coincides with the whole $\mathcal{P}(\Omega)$ we omit the symbol $\mathcal{F}$ from all previous notations.

Let us also notice that

- $k \neq k' \implies SREL_k(\mathcal{F}|\Omega) \cap SREL_{k'}(\mathcal{F}|\Omega) = \emptyset$;
- $DREL(\mathcal{F}|\Omega) \subseteq SREL_0(\mathcal{F}|\Omega)$, i.e. if $\mathcal{D} \in DREL(\mathcal{F}|\Omega)$ then $\mathcal{D} = \mathcal{D}^0_{\mathcal{F}} = \mathcal{D}^1_{\mathcal{F}} = \cdots = \mathcal{D}^{\diamond}_{\mathcal{F}}$;
- if Ω is finite, then $BREL(\Omega) = SREL(\mathcal{F}|\Omega)$.

We now provide a simple example of stabilizing binary relation.

Example 10.6. Let $\mathbb{N} = \{0,1,2,\dots\}$ and $\mathcal{D} = \{(0,1)\} \in BREL(\mathbb{N})$. Then $\mathcal{D} \in SREL_2(\mathbb{N})$. In fact, we have that

$$\mathcal{D}^k = \hat{\mathbb{N}}_{tr} \cup \{(B,A) \in \hat{\mathbb{N}}_{ntr} : B \setminus \{0\} \subseteq A\},$$

for all $k \geq 2$.

In the part (i) of the next result, we show how to construct the $[\mathcal{F}]$-dependency envelope of a $\mathcal{F}$-stabilizing binary relation $\mathcal{D}$ by using the chain given in (38).

Theorem 10.7. *If* $\mathcal{D} \in SREL(\mathcal{F}|\Omega)$*, then:*

(i) $\mathcal{D}^{\diamond}_{\mathcal{F}} = \mathcal{D}^+_{\mathcal{F}}$*;*
(ii) $DP(\mathcal{D}^+_{\mathcal{F}}) = DP(\mathcal{D})$*;*
(iii) $DP_{\mathcal{F}}(\mathcal{D}) = CLOS(\mathcal{D}^+_{\mathcal{F}}) \cap \mathcal{F}$*;*
(iv) *if* $A \in DP(\mathcal{D})$*, then* $Dc_{\mathcal{F},\leftarrow}(A) \subseteq A$.

Proof. (i) : We first prove that $\mathcal{D}^{\diamond}_{\mathcal{F}} \in DREL(\mathcal{F}|\Omega)$. Let $B \subseteq A \in \mathcal{F}$. Then $(B,A) \in \mathcal{D}^0_{\mathcal{F}}$ and hence $(B,A) \in \mathcal{D}^{\diamond}_{\mathcal{F}}$.
Let now $(C,D),(D,E) \in \mathcal{D}^{\diamond}_{\mathcal{F}}$. By (38), there exists an index j such that (C,D), $(D,E) \in \mathcal{D}^j_{\mathcal{F}}$. Then, by applying $(P1)$ to the previous pairs, we have that $(C,E) \in \mathcal{D}^{j+1}_{\mathcal{F}} \subseteq \mathcal{D}^{\diamond}_{\mathcal{F}}$.
On the other hand, let $\{B_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ such that (B_i, C) in $\mathcal{D}^{\diamond}_{\mathcal{F}}$ for any $i \in I$. By hypothesis, there exists $k \geq 0$ such that $\mathcal{D} \in SREL_k(\mathcal{F}|\Omega)$. Then, by (39) we deduce that $(B_i, C) \in \mathcal{D}^k_{\mathcal{F}}$ for each $i \in I$. By applying $(P2)$ to the previous pairs, by definition of $\mathcal{D}^{k+1}_{\mathcal{F}}$ we have that $(\bigcup_{i\in I} B_i, C) \in \mathcal{D}^{k+1}_{\mathcal{F}} = \mathcal{D}^k_{\mathcal{F}} = \mathcal{D}^{\diamond}_{\mathcal{F}}$. This shows that $\mathcal{D}^{\diamond}_{\mathcal{F}} \in DREL(\mathcal{F}|\Omega)$.
Finally, since $\mathcal{D} \subseteq \mathcal{D}^{\diamond}_{\mathcal{F}}$, we have $\mathcal{D}^{+}_{\mathcal{F}} \subseteq \mathcal{D}^{\diamond}_{\mathcal{F}}$. Vice versa, if $(A,B) \in \mathcal{D}^{\diamond}_{\mathcal{F}}$ and $\mathcal{E} \in DREL(\mathcal{F}|\Omega)$ is such that $\mathcal{D} \subseteq \mathcal{E}$, by definition of $(P1)$ and $(P2)$ it is clear that $(A,B) \in \mathcal{E}$. Therefore $(A,B) \in \mathcal{D}^{+}_{\mathcal{F}}$, and hence the thesis follows.
(ii) : By (3), we have that $DP(\mathcal{D}^{+}_{\mathcal{F}}) \subseteq DP(\mathcal{D})$. Conversely, let $A \in DP(\mathcal{D})$ and let $(B,C) \in \mathcal{D}^{+}_{\mathcal{F}}$ such that $C \subseteq A$. We have to prove that $B \subseteq A$. Then, if $(B,C) \in \mathcal{D}^0_{\mathcal{F}}$ we have two possibilities. If $(B,C) \in \mathcal{D}$, since $A \in DP(\mathcal{D})$ and $C \subseteq A$, we also have $B \subseteq A$. Otherwise, by definition of $\mathcal{D}^0_{\mathcal{F}}$, we have $B \subseteq C \in \mathcal{F}$.
Assume now that $(B,C) \in \mathcal{D}^{+}_{\mathcal{F}} \smallsetminus \mathcal{D}^0_{\mathcal{F}}$. So, $(B,C) \in \mathcal{D}^k_{\mathcal{F}}$, for some $k \geq 1$. Then, by previous part (i), the pair (B,C) can be obtained by applying $(P1)$ or $(P2)$ to some pairs belonging to $\mathcal{D}^{k-1}_{\mathcal{F}}$. Thus, we must show our claim inductively.
To this regard, let $(B,C) \in \mathcal{D}^1_{\mathcal{F}}$ and assume that there exists $D \in \mathcal{P}(\Omega)$ such that $(B,D),(D,C) \in \mathcal{D}^0_{\mathcal{F}}$. Now, since $A \in DP(\mathcal{D})$ and $C \subseteq A$, it follows that $D \subseteq A$, and therefore $B \subseteq A$.
On the other hand, if there exist $\{B_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ for which $(B_i, C) \in \mathcal{D}^0_{\mathcal{F}}$ for any $i \in I$ and $B = \bigcup_{i\in I} B_i$, then (as before) $B_i \subseteq A$ for any $i \in I$, so that $B = \bigcup_{i\in I} B_i \subseteq A$. This proves that $DP(\mathcal{D}) = DP(\mathcal{D}^0_{\mathcal{F}}) = DP(\mathcal{D}^1_{\mathcal{F}})$.
Assume now that $DP(\mathcal{D}^{j-1}_{\mathcal{F}}) = DP(\mathcal{D}^j_{\mathcal{F}})$ for any $j \geq k$. Let us prove that $DP(\mathcal{D}^k_{\mathcal{F}}) = DP(\mathcal{D}^{k+1}_{\mathcal{F}})$.
By (3), we have only to show that $DP(\mathcal{D}^k_{\mathcal{F}}) \subseteq DP(\mathcal{D}^{k+1}_{\mathcal{F}})$. Let $(B,C) \in DP(\mathcal{D}^{k+1}_{\mathcal{F}})$ and $A \in DP(\mathcal{D}^k_{\mathcal{F}})$ be such that $C \subseteq A$. If there exists $D \in \mathcal{P}(\Omega)$ such that $(B,D),(D,C) \in \mathcal{D}^k_{\mathcal{F}}$, then $D \subseteq A$ and, then, $B \subseteq A$.
On the other hand, suppose that there exist $\{B_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ for which $(B_i, C) \in \mathcal{D}^k_{\mathcal{F}}$ for any $i \in I$ and $B = \bigcup_{i\in I} B_i$. Then $B_i \subseteq A$ for any $i \in I$, i.e. $B = \bigcup_{i\in I} B_i \subseteq A$. This proves that $DP(\mathcal{D}^k_{\mathcal{F}}) = DP(\mathcal{D}^{k+1}_{\mathcal{F}})$ for any $k \geq 0$ and concludes the proof.
(iii): Let $\mathcal{D} \in SREL(\mathcal{F}|\Omega)$. By part (ii), we have that $DP(\mathcal{D}) = DP(\mathcal{D}^{+}_{\mathcal{F}})$ so, a fortiori, $DP_{\mathcal{F}}(\mathcal{D}) = DP_{\mathcal{F}}(\mathcal{D}^{+}_{\mathcal{F}})$. By Proposition 4.9, we have $DP_{\mathcal{F}}(\mathcal{D}^{+}_{\mathcal{F}}) = CLOS(\mathcal{D}^{+}_{\mathcal{F}}) \cap \mathcal{F}$. This shows the claim.
(iv): Let us denote by $\leftarrow$ the dependency relation $\mathcal{D}^{+}_{\mathcal{F}}$. Let $a \in Dc_{\mathcal{F},\leftarrow}(A)$, then there exists $B_a \in \mathcal{F}$ such that $a \in B_a$ and $B_a \leftarrow A$. By both $(D1\mathcal{F})$ and $(D2)$, it follows that $a \leftarrow A$. Now, by part (ii), we have that $A \in DP(\mathcal{D}^{+}_{\mathcal{F}})$. Thus, it must necessarily be $a \in A$. □

Corollary 10.8. *If Ω is a finite set, then $\mathcal{D}^{\diamond}_{\mathcal{F}} = \mathcal{D}^{+}_{\mathcal{F}}$ and $DP(\mathcal{D}^{+}_{\mathcal{F}}) = DP(\mathcal{D})$, for any $\mathcal{D} \in BREL(\Omega)$.*

We conclude this section by constructing, for any given dependency relation $\mathcal{E}$ on Ω, a specific sub-relation $\mathcal{E}_{ker}$ whose dependency envelope coincides with $\mathcal{E}$. The particular characteristic of such a sub-relation is that all its pairs have a singleton as first component.
Let therefore $\mathcal{E} \in DREL(\Omega)$. We set

$$\mathcal{E}_{re} := \{(b,A) \in \mathcal{E} \cap \hat{\Omega}_{ntr} : \; A' \subsetneq A \implies (b,A') \notin \mathcal{E}\} \tag{40}$$

and

$$\mathcal{E}_{ker} := \{(b,A) \in \mathcal{E}_{re} : \; (b,b') \in \mathcal{E}_{re} \implies (b',A) \notin \mathcal{E} \;\vee\; b' \in A\} \tag{41}$$

In the next result, we will prove that the dependency envelope of $\mathcal{E}_{ker}$ coincides with $\mathcal{E}$ for a given dependency relation $\mathcal{E}$ on a finite set Ω.

Theorem 10.9. *Let Ω be a finite set. Then $\mathcal{E}^{+}_{ker} = \mathcal{E}$.*

Proof. By the fact that $\mathcal{E}_{ker} \subseteq \mathcal{E}$ and $\mathcal{E} \in DREL(\Omega)$, we have that $\mathcal{E}^{+}_{ker} \subseteq \mathcal{E}$. We now show the reverse inclusion. To this regard, let $(B,A) \in \mathcal{E}$. By $(D1)$ and $(D2)$, we have that $(b,A) \in \mathcal{E}$ for any $b \in B$. Choose from the previous a non-trivial pair (b,A). Let us take the set system $\mathcal{F}_{b,A} := \{C \subseteq A : \; (b,C) \in \mathcal{E}\}$. Clearly, $\mathcal{F}_{b,A} \neq \emptyset$ since $A \in \mathcal{F}_{b,A}$. Therefore, it contains a minimal element, that we denote by C. This means that for any $C' \subsetneq C$, it must necessarily be $(b,C') \notin \mathcal{E}$. Now, by applying $(P1)$ and $(P2)$ to the pairs $(b,C) \in \mathcal{E}_{re}$, $(C,A) \in \mathcal{E}^{+}_{re}$ we obtain that $(b,A) \in \mathcal{E}^{+}_{re}$. By the arbitrariness of $b \in B$, by $(P2)$ it follows that $(B,A) \in \mathcal{E}^{+}_{re}$. In other terms, we found a pair $(b,C) \in \mathcal{E}_{re}$ yielding (B,A) by successive applications of $(P1)$ and $(P2)$ on $(\mathcal{E}^{*}_{re})_0$. Let us note that if $C = \emptyset$, then $(b,\emptyset) \in \mathcal{E}_{ker}$. Now, let $C \neq \emptyset$. Assume that $(b,C) \in \mathcal{E}_{re} \smallsetminus \mathcal{E}_{ker}$. This means that there exists $b' \neq b \in \Omega \smallsetminus C$ such that $(b,b') \in \mathcal{E}_{re}$ and $(b',C) \in \mathcal{E}$. Clearly, we have that $(b',C) \in \mathcal{E}_{re}$, otherwise, there would be $C' \subsetneq C$ such that $(b',C') \in \mathcal{E}$; thefore, by

	1	2	3	4
u_1	1	1	1	1
u_2	1	1	2	2
u_3	1	2	3	3
u_4	2	3	4	4
u_5	3	4	4	5
u_6	4	5	5	6
u_7	5	6	6	6

FIGURE 1. Information Table $\mathcal{I}$ of Example 10.10

$(P1)$ applied to the pairs (b, b') and (b', C'), we would have $(b, C') \in \mathcal{E}$, contradicting the minimality of C. Hence, $(b, C) \in \mathcal{E}_{re} \setminus \mathcal{E}_{ker}$ has been obtained through $(D2)$ applied to the pairs $(b, b') \in \mathcal{E}_{re}$ and $(b', C) \in \mathcal{E}_{re}$. Proceeding in this way on these two pairs, we must necessarily obtain (b, C) by successive applications of $(D2)$ to pairs of $\mathcal{E}_{ker}$. This proves that $\mathcal{E}_{ker}$ generates $\mathcal{E}$. $\square$

Now we provide a specific example of $\mathcal{E}_{ker}$ when $\mathcal{E}$ is the dependency relation induced by an information table.

Example 10.10. Let us consider the information table $\mathcal{I}$ on $\Omega = \{1, 2, 3, 4\}$ given in Figure 1. It can be easily seen that

$$CLOS(\mathcal{I}) = \{\varnothing, \{1\}, \{3\}, \{4\}, \{1, 2\}, \{1, 2, 3, 4\}\}$$

Moreover, one can also verify that

$$\mathcal{E}_{ker} = \{1 \leftarrow_{\mathcal{I}} 2, 2 \leftarrow_{\mathcal{I}} 13, 2 \leftarrow_{\mathcal{I}} 14, 2 \leftarrow_{\mathcal{I}} 34, 3 \leftarrow_{\mathcal{I}} 14, 3 \leftarrow_{\mathcal{I}} 24, 4 \leftarrow_{\mathcal{I}} 13, 4 \leftarrow_{\mathcal{I}} 23\}$$

11. CONCLUSIONS

In this paper, we introduced an abstract and generalized version of the *dependency relation* between subsets of a given arbitrary set Ω. The relation we gave in this work provides a paradigm general enough to include basic notions of rough set theory, formal context analysis, domain theory and possibility theory. Starting with this relation we firstly characterized the basic set operators and set systems naturally associated with any generalized dependency relation and, next, we provided an interpretation in various branches of computer science. In particular, we showed that dependency relations appear whenever we deal with closure operators, if $\mathcal{F} = \mathcal{P}(\Omega)$. On the other hand, if $\mathcal{F} \subsetneq \mathcal{P}(\Omega)$, we established the basic properties of the set operators $Dc_{\mathcal{F},\leftarrow}$ and $Ic_{\mathcal{F},\leftarrow}$ in relation to the specific conditions given on the system $\mathcal{F}$. The underlying ideas of our paper are placed within an attempt to provide an abstract and generalized notion of *dependency* (and its first main properties), above all when we restrict the generalized reflexivity of the given relation only to set systems strictly contained in $\mathcal{P}(\Omega)$. In this way, many classical A-RST notions have been interpreted and investigated in terms of this generalized dependency; furthermore, several structures appearing in various disjoint branches of computer science (such as O-RST, FCA, DT and PT) have been traced back to the investigation of a unique relation. In particular, we tried to understand which are the fundamental properties that a set system $\mathcal{F}$ has to satisfy so that the above set operators and set systems assume a mathematical and interpretative meaning.
Furthermore, an axiomatic approach to the notion of dependency in the spirit of this paper can be useful also in mathematics, since structures such as metric spaces, group actions and graphs can be described in terms of a generalized version of Pawlak's information tables [25].
Finally, another open research line has been traced in Section 10, where we dealt with *generation* of dependency relations. In fact, in some recent researches a link between Armstrong's rules (and the related methodologies) and the study of reducts in rough set theory has been highlighted [51, 54]. For instance, the so-called *association reducts* introduced and studied by Slezak in [51] provide an example in this perspective.
The previous argument enables us to surmise that the determination of specific smaller relations $\mathcal{D}^+_{\mathcal{F}}$ should lead towards an approximation methodology for the computation of the whole $\mathcal{D}^+$. However, many problems of computational nature arise from this study; therefore, we believe that a thorough investigation of the previous issue can be the starting point of new researches.

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