# KNOWLEDGE PAIRING SYSTEMS IN GRANULAR COMPUTING

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Abstract. In Pawlak's theory of information systems, one has a collection of objects and knows the values of any object with respect a certain class of properties, usually called attributes. An implicit assumption of this theory is that the difference between the nature of objects and attributes is well outlined. In our paper we generalize the concept of information system by analyzing the case in which there is no a priori distinction in the nature of objects and attributes and, hence, both the interpretations are admissible. We call the structure arising in the previous context *knowledge pairing system*. We study the indiscernibility relations induced in both the admissible interpretations by means of up-down operators, in such a way to have a direct analogy with the extent and intent operators used in Formal Context Analysis. In particular, we investigate three models of knowledge pairing systems arising from real contexts and modeled respectively by graphs, digraphs and hypergraphs. We show the real convenience to use the notion of knowledge pairing system focusing on interpretation of this structure and discussing the two admissible perspectives obtained by avoiding the difference between the roles of objects and attributes.

## 1. Introduction

In database theory there is a very frequent need to study finite tables having a very large quantity of data, therefore many researches have been directed towards the purpose of reducing and simplifying the interpretation of these data. With such an aim, Pawlak [47] developed the so called *rough set theory* (abbreviated RST). RST is an elegant and powerful methodology in extracting and minimizing rules from data tables (*information systems* or, also, *information tables* in Pawlak terminology).
However, RST has been considered as a part of the more general emerging methodological paradigm named *granular computing* (abbreviated GrC) [51, 52, 55, 68].
GrC deals with representing and processing information in the form of some type of aggregates. These aggregates are generally called *information granules* or simply granules and they arise in the process of data abstraction and knowledge derivation from data. The scope of GrC covers various fields of study related to knowledge representation and data extraction.
In 1979 the concept of *information granularity* was introduced by Zadeh [78] and it was related to the research on fuzzy sets. Next, the term *granular computing* was introduced again by Zadeh in 1997 (see [79]). Roughly speaking, information granules are collections of entities arranged together due to their similarity, functional or physical adjacency, indistinguishability, and so on.
Since 1979, granular computing has become a very developed area of research in the scope of both applied and theoretical information science [52, 68]. From a methodological perspective, GrC can be considered as an important attempt to investigate several research fields by means of the unifying granularity paradigm: rough set theory [44, 45, 46, 48, 49, 50, 53, 70, 72, 77] and its generalizations [10, 11, 12, 76], mathematical morphology [64], temporal dynamics [8, 9, 13, 14, 16, 29], machine learning [74], formal concept analysis [38, 67, 69], database theory [33, 34, 56], data mining [35, 42, 43, 71], fuzzy set theory [54, 59, 79], interactive computing [61, 62], matroid theory [36, 37, 39, 40, 41, 80], hyergraph theory [15, 65, 66, 17], graph theory [18, 20, 63, 57], discrete dynamical systems [1, 2, 6, 7, 58].
The unifying perspective of GrC provides the useful interpretation tool which allows us to assign the same name for several notions used in different research fields. Under the umbrella of the granular paradigm, RST is developing new investigation potentialities, both a conceptual and computational level (for a detailed and philosophically pregnant discussion on this important topic see [77]). In our paper we try to give a further contribution to develop a conceptual vision of RST [77] within a granular perspective (abbreviated RST-GrC).
The original point of our work arises from the following remark.

**Remark 1.1.** *Starting by the relatively simple notion of information table, we can develop very refined mathematical and conceptual notions without additional hypotheses. In other terms, RST-GrC can be considered a theory that naturally can provide very complex structures by starting from very simple hypotheses.*

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The important aspect we want to underline is the phrase *without additional hypothesis*. In general, when we work within a given mathematical theory, if we adjoin new hypotheses we usually obtain more rich and deeper results. On the other hand, there is a price to pay to obtain a greater conceptual depth: *more hypotheses we adjoin, more specialistic becomes our theory.* As a direct consequence of this very simple consideration, we can observe an increasing proliferation of very specialistic works having no connection each other. Wanting to use a metaphor taken by granular computing, we have an infinity of micro-granular research groups that have poor connections between them.
On the other hand, an information table (in the finite case), or an information system (in the general case), is a structure that is ubiquitous both in mathematics and information sciences. Therefore, by its proper nature, an information system is a *non-specialistic structure*. Many general mathematical structures, such as for example graphs, digraphs, hypergraphs, metric spaces can be re-interpreted (in several different and non equivalent ways) as special types of information systems (for details see [15, 20, 25, 27, 28]). The key notion that strongly intervenes in RST-GrC is that of *indiscernibility with respect to a fixed attribute subset* $A$ of an information system $\mathcal{I}$. Originally, Pawlak's indisceriiblity notion [47] was introduced as a basic tool to extract meaningful information from a data table. In other terms, the origins of the indiscernibility notion are *strictly of practice nature*. On the other hand, a more general meaning can be associated to the indiscernibility notion. Let us assume that we have a given set $\Omega$ of elements. On the previous set we have no structure. Then, any mathematical theory arises by providing the set $\Omega$ of some type of structure. However, also when we introduce some additional hypotheses on $\Omega$, the nature of its elements remain indeterminate, i.e. the elements of $\Omega$ remain simply *symbols*, without a concrete interpretation. Then the real philosophical teaching that we can derive from Pawlak RST is that the elements of $\Omega$ can be viewed as *attributes* of some other *universe set* $U$. In other terms, any element $a \in \Omega$ becomes a function $a : U \to Val_a$, taking values in an appropriate value set $Val_a$. Now, usually, the attention of the RST practitioners is posed basically on the indiscernibility induced by $\Omega$ on $U$. But if we change perspective, we can interpret the indiscernibility induced on $U$ as a way to produce on $\Omega$ itself a type of reflected indiscernibility, that in this paper we will call *indistinguishability*. Roughly speaking, we will say that two subsets $A$ and $A'$ of $\Omega$ are between *indistinguishable*, in symbols $A \approx A'$, if they induce on $U$ the same indiscernibility relation. Accordingly, by starting with any set $\Omega$ we think its elements as attributes of some universe set $U$, and as a consequence of this interpretation we obtain an equivalence relation $\approx$ on the power set $\mathcal{P}(\Omega)$. Thus, we pass from a vision of $\Omega$ as a *non-interpreted set* to a new vision of $\Omega$ as an *attribute set*, or, equivalently, as a *function set*. Such a simple change of outlook enables us to analyze the corresponding quotient structure $\mathcal{P}(\Omega)/\approx$. Then, a natural question arises: *is it an interesting mathematical structure that need a deep investigation?*. In [25, 26] we try to show that the study of $\mathcal{P}(\Omega)/\approx$ provides interesting results both of mathematical and informational type. On the other hand, in our paper we introduce and study a very natural structure that can be used when one has the possibility to consider two given sets $U$ and $\Omega$ both as attribute sets. We call such a structure *knowledge pairing system* and we investigate it on some real models induced by graphs, digraphs and hypergraphs.
A knowledge pairing system is simply a rectangular data table where the entities of both the rows and columns can be considered as potential attributes that suitably induce indiscernibility relations in both the directions. Let us note that in the scope of Pawlak's theory, there is a clear and precise distinction between objects and attributes. Thus, in this case, one is led to analyze a data table in which the role of objects and that of attributes is well defined. Nevertheless, there are several concrete and theoretical contexts in which the entities defining a rectangular data table can assume, at the same time, the nature of objects and attributes. So that, in these contexts, it becomes natural to study the data table by inducing some properties from the rows entities to the columns ones and vice versa. At the same time, it is also natural to investigate the new mathematical tools and their corresponding heuristic interpretations that arise from the interrelations between the "objects-attributes" and the "attributes-objects". As a matter of fact, by assuming a bidirectional investigation perspective in the given data table, it is possible to define two new set operators that Pawlak's theory does not take into account. To be more specific, let $\mathcal{I}$ be a rectangular table whose row entity set and column entity set are respectively denoted by $U$ and $\Omega$ and $F : U \times \Omega \to Val$ is the corresponding information map. Then, in [25] it has been introduced the set operator $\Gamma : \mathcal{P}(U) \to \mathcal{P}(\Omega)$, defined by

$$\Gamma(Z) := \{a \in \Omega : \forall z, z' \in Z,\ F(z, a) = F(z', a)\},$$

for any $Z \subseteq U$. In [25] it has been showed that the set operator $\Gamma$ is an essential tool to test whether a given set partition of $U$ is an indiscernibility partition (see Theorem 6.10 of [25]). Now, if we analyze the rectangular table $\mathcal{I}$ as a knowledge pairing system, the role of $U$ and $\Omega$ is, in both cases, that of an attribute set; therefore, we can also consider the dual operator $\Gamma' : \mathcal{P}(\Omega) \to \mathcal{P}(U)$ given by

$$\Gamma'(A) := \{u \in U : \forall a, a' \in A, \, F(u,a) = F(u,a')\},$$

for any $A \subseteq \Omega$. Then, in a knowledge pairing system (but not in an information table), the previous operators can be considered concurrently in their respective actions. In particular, in a knowledge pairing system, it is possible to study both the compositions $\Gamma \circ \Gamma'$ and $\Gamma' \circ \Gamma$, similarly to what happens in formal context analysis for the set operators $^{\uparrow}$ and $^{\downarrow}$ (see [32]). We analyze both the compositions $\Gamma \circ \Gamma'$ and $\Gamma' \circ \Gamma$ in Section 5 and in some concrete scenarios: in Example 6.2 we compute the aforementioned compositions and then interpret them in relation to the outcome of some universitary exams, while in Example 7.4 we study them in relation to the information we can extract by knowing the flight routes to and from some airports.

In [15, 20, 21, 24, 27, 28] the structure of hypergraphs, graphs and digraphs has been investigated by means of the classical Pawlak's indiscernibility paradigm. Through such a study, several new theoretical results concerning the previous structures have been established. For example, by interpreting a simple graph as an information table, in [20] it has been provided a new classification of subgraphs of the Petersen graph in terms of classical Pawlak's reducts. This is an example of how the RST-GrC paradigm yields new mathematical interpretations in graph theory. In a similar way, in [27] new combinatorial formulas concerning some basic digraph families have been found by interpreting the classical lower and upper approximation functions in terms of out-adjacency. Analogously, the standpoint adopted in [15], that is to see the transpose of the incidence matrix as an information table, brought towards the introduction of new combinatoric formulas concerning the structure of the uniform hypergraphs.

Based on the previous considerations, we can note that the interpretation of digraphs, graphs and hypergraphs as information tables is useful to determine new results in the scope of these theories. Therefore, from this perspective, RST-GrC can be also considered an efficient tool to discover new theorems and propositions within all those specific disciplines that can be embraced by such a general paradigm.

The main aim of our paper consists of delineating a conceptual framework within which Pawlak's indiscernibility can be studied in both directions. From a merely practical standpoint, we provide some possible cases of study. Clearly, in this first work on the topic, we can provide only a limited number of examples. However, by means of the indiscernibility investigation in a double direction, we can analyze graphs, digraphs and hypergraphs by a more complete perspective. In this sense, for example, the notion of maximal separator for a hyperedge family arises by Theorem 6.8 and has an intrinsic combinatorial susceptible to future research. Analogously, in Proposition 6.18, two classical hyperedge subfamilies of the uniform hypergraph $\binom{\hat{n}}{k}$ have been interpreted in terms of discernibility hypergraph of a knowledge pairing system.

We briefly describe the contents of the sections of our paper.

In Section 2 we discuss in detail the theoretical and practical motivations underlying the introduction of these structures. Section 2 can be considered as a type of enlarged introduction of our work. In Section 3 we recall the basic notions and we fix the notations that we will use in the remaining part of the paper. In Section 4 we recall the basic order structures induced on $\mathcal{P}(\Omega)/\approx$ (this part has been developed in detail in [25]). In Section 5 we formally introduce the knowledge pairing systems and we treat all basic induced notions. In Section 6 we compare the knowledge pairing systems induced respectively from hypergraphs and graphs. In Section 7 we treat the knowledge pairing systems induced by directed graphs (briefly digraphs).

## 2. Motivations to Introduce Knowledge Pairing Systems

The basic purpose of our paper is to provide a more flexible interpretation of the usual notion of *information system* and related concepts (see [47, 48, 49, 50]), in order to obtain a tool useful to better investigate the tabular knowledge derived from all those real and abstract cases where at least one of the following two situations occurs.

(A) There is no clear distinction between the nature of objects and that of attributes.
(B) We have two sets of elements and the possibility to interpret one of them as the object set and the other as the attribute set and even, at the same time, the reverse situation.

Practical situations that naturally lead towards one of the previous cases are for example typical real representations described by means of graphs, digraphs and hypergraphs.

By referring to case $(A)$, there are several real situations where the elements of some universe set are between them interrelated without it being possible to distinguish which of these elements are objects and which are attributes of an appropriate information table. A typical case where such a situation occurs is when the elements (vertices) of a given set are interrelated between them as the vertices of

| | $h$ | $s$ |
|---|---|---|
| $O$ | 0 | 1 |
| $P$ | 1 | 0 |
| $Q$ | 1 | 1 |

FIGURE 1. The information table $T[\mathcal{J}]$

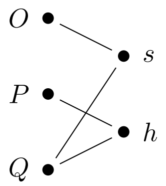



FIGURE 2. The bipartite graph $B_{\mathcal{J}}$

| | $O$ | $P$ | $Q$ | $h$ | $s$ |
|---|---|---|---|---|---|
| $O$ | 0 | 0 | 0 | 0 | 1 |
| $P$ | 0 | 0 | 0 | 1 | 0 |
| $Q$ | 0 | 0 | 0 | 1 | 1 |
| $h$ | 0 | 1 | 1 | 0 | 0 |
| $s$ | 1 | 0 | 1 | 0 | 0 |

FIGURE 3. The adjacency matrix of $B_{\mathcal{J}}$

an undirected graph $G$. In this case, the adjacency matrix $Adj(G)$ of the graph $G$ turns out to be the more natural choice to represent the knowledge induced by the presence of edges between vertices of $G$. In foregoing papers (see [19, 20, 21, 22, 23]), we dealt with the mathematical consequences of the interpretation of the adjacency matrix of a simple graph as a Boolean information table.

To be more detailed, we interpreted any simple undirected graph $G = (V(G), E(G))$ as the Boolean information system $\mathcal{I}[G] = \langle U, Att, F\rangle$, where $U := V(G)$, $Att := V(G)$ and $F(u, a) = 1$ if and only if there is an edge in $G$ with ends $u$ and $a$. The interpretation of a graph as a particular type of Boolean information system might seem to produce a very particular type of Boolean information systems. However, we now show that this first impression is unfounded and, moreover, we also show that our interpretation can be considered as a type of generalization of any Boolean information system $\mathcal{I} = \langle U, Att, F\rangle$ such that $Att \cap U = \emptyset$.
Let us consider the Boolean information system $\mathcal{J} = \langle U, Att, F\rangle$, where $U = \{O, P, Q\}$, $Att = \{h, s\}$ and $F$ is described in the information table $T[\mathcal{J}]$ given in Figure 1.
We take now the bipartite graph $B_{\mathcal{J}} = (V, E)$ such that $V = \{O, P, Q, h, s\}$ and

$$E = \{\{O, s\}; \{P, h\}; \{Q, h\}; \{Q, s\}\},$$

that is represented in Figure 2.
Now, if we consider the Boolean information system $\mathcal{I}_{adj}[B_{\mathcal{J}}]$, the information table associated to the information system is exactly the adjacency matrix of $B_{\mathcal{J}}$, that is represented in Figure 3.
We can then note that the information table of the original information system $\mathcal{J}$ (see Fig.1) is the sub-table located in the upper right corner of the information table of $\mathcal{I}_{adj}[B_{\mathcal{J}}]$ given in Fig. 3. On the other hand, it is clear that the additional information contained in the information table of Fig. 3 are redundant, therefore we can consider equivalent the two information systems $\mathcal{J}$ and $\mathcal{I}_{adj}[B_{\mathcal{J}}]$. This simple example shows that we can always see any Boolean information system as a Boolean information system induced from a bipartite graph.
Now the natural question is : *what means, in terms of information systems, to study a Boolean information system induced from a non necessarily bipartite graph?*
In order to answer the previous question, let us consider a potential situation pre-existing to the information system described in the bipartite graph of Fig.2. Assume that in such a hypothetical pre-existing situation we have more information with respect to the case given in Fig.2; however, we also suppose that our knowledge is less accurate than that provided in Fig.2. In this pre-existing situation we know only

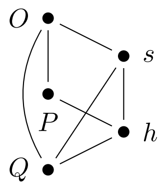



FIGURE 4. The graph $G_R$

| | $O$ | $P$ | $Q$ | $h$ | $s$ |
|---|---|---|---|---|---|
| $O$ | 0 | 1 | 1 | 0 | 1 |
| $P$ | 1 | 0 | 0 | 1 | 0 |
| $Q$ | 1 | 0 | 0 | 1 | 1 |
| $h$ | 0 | 1 | 1 | 0 | 1 |
| $s$ | 1 | 0 | 1 | 1 | 0 |

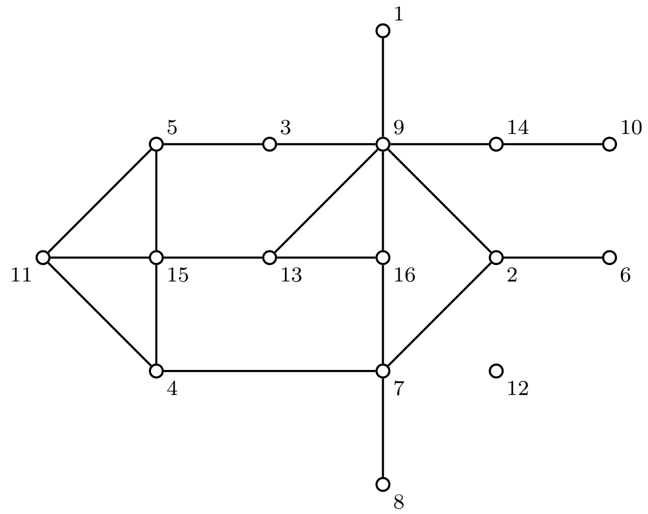
FIGURE 5. The adjacency matrix of $G_R$

that there is a generic binary *symmetric* relation, namely $R$, concerning some pair of elements belonging to the set $\{O, P, Q, h, s\}$. We don't know a priori if such a relation $R$ can generate new and smaller binary relations $R_1$, $R_2, \ldots$, in the set $\{O, P, Q, h, s\}$ that may be of more interest for us with respect to $R$. Therefore, in this situation of partial indetermination, the only reasonable thing to do is to connect with an undirected edge any two elements of $\{O, P, Q, h, s\}$ which are in $R$-relation between them. For example, let us suppose that $R$ is the following binary symmetric relation on $\{O, P, Q, h, s\}$:

$$R = \{\{O, s\}; \{P, h\}; \{Q, h\}; \{Q, s\}; \{O, P\}; \{O, Q\}; \{s, h\}\}.$$

We represent therefore this knowledge with the graph $G_R$ given in Fig.4.
When we examine the graph in Fig.4, we may not be able to discern which are potential "objects" and potential "attributes" that form an information system useful for our aims. Because of the indetermination of the situation at issue, it seems then reasonable to treat all the elements in the set $\{O, P, Q, h, s\}$ as they were both objects and attributes of a Boolean information system $\mathfrak{I}_R$ determined from the relation $R$. This means that the information table of $\mathfrak{I}_R$ is exactly the adjacency matrix (see Figure 5) of the graph $G_R$.
Hence, by using our initial notation, we have that $\mathfrak{I}_R = \mathcal{I}_{adj}[G_R]$. Furthermore, we can use several techniques used in rough set theory (attribute reduction, similarity analysis etc), in order to refine the rough information provided from the relation $R$. Let us note that if we are only interested to the more particular knowledge provided in Fig.1, this knowledge is present in the table in Fig. 5, but it is hidden. We should then be able to select it using usual reduction techniques or by obtaining new external information.
The previous discussion shows that we can think the study of the Boolean information system $\mathcal{I}_{adj}[G]$ induced from a fixed simple undirected graph $G$ as an examination of a "potentially redundant" information system obtained from "limited" or "little" information . This preliminary examination can be useful in order to determine a smaller sub-system more functional for our specific objectives. This interpretation opens the following new research perspective: to study a graph $G$ as if it were a Boolean information system $\mathcal{I}_{adj}[G]$ having a surplus of "limited" information which must be subsequently reduced and selected. Let us note that the hypothesis of simplicity for $G$ is not necessary for the definition of $\mathcal{I}_{adj}[G]$, however, since in the paper we focus our attention on the Boolean systems and their links with the formal context theory, we deal with graph without multiple edges. The hypothesis that $G$ has no loops, in terms of information systems, means that any our objects can not be in relation to itself. This hypothesis is not substantially restrictive because when a user will get the smaller reduced information system, the universe of the objects will be disjoint from the attribute set, therefore reflexive relations will be of no interest. It is thus evident that our choice to consider graphs as knowledge pairing systems has as main characteristic the indistinguishability of the natures of objects and attributes.

**Example 2.1.** A very used example (see for instance [31]) in social network analysis concerns the marriage ties of noble families in Florence in the 15th century. The vertices of the simple undirected graph $G$ in Figure 6 (adapted from [31]) represent the 16 elite Florentine families in the 15th century. Moreover, two vertices are connected with an edge if the two corresponding families are linked by a




| | | | |
|---|---|---|---|
| 1. | Acciaiuoli | 9. | Medici |
| 2. | Albizzi | 10. | Pazzi |
| 3. | Barbadori | 11. | Peruzzi |
| 4. | Bischeri | 12. | Pucci |
| 5. | Castellani | 13. | Ridolfi |
| 6. | Ginori | 14. | Salviati |
| 7. | Guadagni | 15. | Strozzi |
| 8. | Lamberteschi | 16. | Tornabuoni |

FIGURE 6. The elite families in Florence in the 15th century

marriage tie. Then, if we take for example the vertex subset $A = \{9, 13, 16\}$, it is easy to verify that the indiscernibility partition induced by $A$ is $\pi_G(A) = 123(14)|4568(10)(11)(12)|7|9|(13)|(15)|(16)$. In this real case, each block of $\pi_G(A)$ is a group of families having all a similar marriage relation with respect to the families in $A$: Medici, Ridolfi, Tornabuoni. Therefore, the vertices $1, 2, 3, 14$ are the families having a marriage relation with some member of the Medici family and no other marriage relation with members of the families Ridolfi or Tornabuoni. Analogously, the vertices $4, 5, 6, 8, 10, 11, 12$ are the families that have not marriage ties with no member of the families Medici, Ridolfi and Tornabuoni.

However, the specific situation occurring in a real situation described by means of an undirected graph $G$ is that a distinction between objects and attributes is not important because we have a symmetric tabular knowledge where objects and attributes are identified. On the other hand, we can deal with many real situations in which direct relations between elements of some set are defined; we can modelize this general context by means of digraphs. The study of digraphs as information tables has been started in [27]. Even in the digraphs case, the most natural means to describe the knowledge tabular induced by them is the adjacency matrix $Adj[\vec{D}]$ of the digraph $\vec{D}$. Nevertheless, in this situation, there are two different ways to interpret attributes, depending on whether they are settled as columns or rows of the adjacency table.
In the next example, we discuss a real case, that can be induced by an information table. We will see that this case presents characteristic features both of the situation (A) that of the situation (B).

**Example 2.2.** Let us consider the following six European airports:

$a.$ Amsterdam
$b.$ Bruxelles
$c.$ Copenaghen
$d.$ Dublin
$e.$ Edinburgh
$f.$ Florence

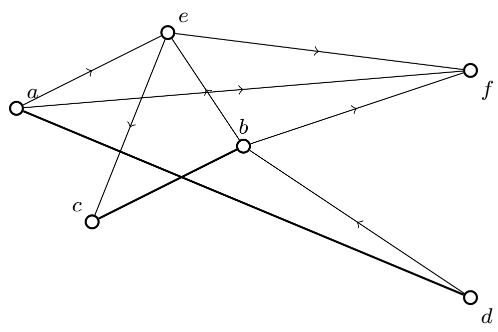



FIGURE 7. Flights in six European airports.

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $b$ | 0 | 0 | 1 | 0 | 1 | 1 |
| $c$ | 0 | 1 | 0 | 0 | 0 | 0 |
| $d$ | 1 | 1 | 0 | 0 | 0 | 0 |
| $e$ | 0 | 0 | 1 | 0 | 0 | 1 |
| $f$ | 1 | 0 | 0 | 0 | 0 | 0 |

FIGURE 8. The adjacency matrix of $\vec{D}$

We can think of the airports as vertices of a digraph $\vec{D}$. We put an arc from a city $v$ to a city $w$ if there is a flight from $v$ to $w$. If there are flights in both directions, we put a line with no arrow between the two cities. In Figure 7, we see an example of the situation occurring.
In Figure 8 we represent the adjacency matrix of $\vec{D}$.
In this case the more natural tabular knowledge induced from the directed arcs of $D$ is that induced from the usual adjacency matrix $Adj(\vec{D})$, that is (in our example) a $6\times 6$ Boolean non-symmetric matrix having 1 in the place $(i, j)$ if there is a flight from $i$-th city to the $j$-th, otherwise there is a 0 in that position. It is clear then that if we choose as objects the rows of our information table $Adj(\vec{D})$, we obtain a type of interpretation for the corresponding indiscernibility partition. For example, if we choose as attribute subset the column subset $A = \{b, c\}$, the corresponding indiscernibility partition is $\pi_{col}(A) = af|be|cd$. Hence, each block of $\pi_{col}(A)$ consists of the cities having the same flight tables with respect to Bruxelles and Copenaghen. In fact, Amsterdam and Florence have no direct flights to Bruxelles and Copenaghen, while starting from Bruxelles and Edinburgh there are flights directed to Copenaghen and starting from Copenaghen and Dublin there are directed flights towards Bruxelles.
On the other hand, the situation changes completely if we choose $A = \{b, c\}$ as a row attribute subset. In this case we obtain the indiscernibility partition $\pi_{row}(A) = ad|b|cef$, that is different from the previous $\pi_{col}(A)$ and whose blocks also have a different mean with respect to the above case. For example, the vertices corresponding to Amsterdam and Dublin form an indiscernibility block of $\pi_{row}(A)$ because there is no direct flight starting from Bruxelles or Copenaghen and arriving to one of them. In other terms, when we fix a column subset $A$ of $Adj(\vec{D})$, we analyze the flight flows towards the cities of $A$ while, if we fix a row subset $A$ of $Adj(\vec{D})$, we are interested in the flight flows starting from the cities of $A$.

At the beginning of our discussion, we said that there are several real situations in which there could be the possibility to interpret a set of elements as objects or attributes. We now present two concrete examples in which there is no a priori choice concerning the nature of the objects and of the attributes and we will see that the resulting interpretation is correct and satisfactory in both cases.

**Example 2.3.** Let Adam, Bill, Carol, Dana and Eve be 5 students chosen in a math class and suppose that we want to classify their attitudes according to the exams they passed. Let Algebra, Geometry, Analysis and Computer Science be the courses they attended during the first semester. Let $A_L = \{Adam, Carol, Eve\}$, $G_E = \{Adam, Dana\}$, $A_N = \{Bill, Eve, Dana, Carol\}$, $C_S = \{Bill, Eve, Dana\}$ be the sets of students that passed respectively Algebra, Geometry, Analysis and Computer Science. Then we can consider the hypergraph $H$ having the five students as vertices and $A_L, G_E, A_N, C_S$ as hyperedges. The incidence matrix $Inc(H)$ is a Boolean table where in the place $(i, j)$ there is a 1 if the

| | $A_L$ | $G_E$ | $A_N$ | $C_S$ |
|---|---|---|---|---|
| Adam | 1 | 1 | 0 | 0 |
| Bill | 0 | 0 | 1 | 1 |
| Carol | 1 | 0 | 1 | 0 |
| Dana | 0 | 1 | 1 | 1 |
| Eve | 1 | 0 | 1 | 1 |

FIGURE 9. The incidence matrix of the student hypergraph $H$

| | Adam | Bill | Carol | Dana | Eve |
|---|---|---|---|---|---|
| $A_L$ | 1 | 0 | 1 | 0 | 1 |
| $G_E$ | 1 | 0 | 0 | 1 | 0 |
| $A_N$ | 0 | 1 | 1 | 1 | 1 |
| $C_S$ | 0 | 1 | 0 | 1 | 1 |

FIGURE 10. The transposed matrix of the student hypergraph $H$

$i$-th student stays in the $j$-th hyperedge, otherwise there will be 0 in that place. In Figure 9 we represent $Inc(H)$.
For example, if we fix $K = \{A_L, G_E\}$ as attribute set, we obtain

$$\pi_{row}(K) = Adam|Bill|Dana|Carol, Eve,$$

in fact, Adam is the unique student that passed both Algebra and Geometry, Bill did not pass any of them, Dana is the unique student that passed Geometry but not Algebra, while Carol and Eve are the students that passed Algebra but not Geometry. In this way, we can conclude that Adam has competences in both Algebra and Geometry, Bill is not good at any of the two subjects, Dana prefers Geometry to Algebra while Carol and Eve prefer Algebra to Geometry.
On the other hand, we can consider $A_L, G_E, A_N, C_S$ as they were objects and the five students as attributes. Then we have the tabular representation of this new interpretation in Figure 10.
Let $A = \{Adam, Bill\}$. Then

$$\pi_{col}(A) = A_L G_E | A_N C_S.$$

The block $A_L G_E$ is the set of exams that have been passed only by Adam while the block $A_N C_S$ is the set of exams that have been passed only by Bill. Thus the set partition $\pi_{col}(A)$ can be seen as a partition of all the possible results of the exams for the students in $A$. The immediate interpretation in this case is that one can analyze the statistical trend of the exams of the semester. Therefore, we see that both the two interpretation are satisfactory and can be useful in order to extract informations from a concrete situation.

**Example 2.4.** Suppose we want to buy a car. We choose between four models $m_1, m_2, m_3, m_4$. We are interested in the following properties (attributes): speed, color and roadholding. To regard the previous attributes, we can summarize the information at issue in the table given Figure 11.

| $\mathcal{C}$ | *Speed* | *Color* | *RoadH* |
|---|---|---|---|
| $m_1$ | Medium | Green | Medium |
| $m_2$ | Medium | Blue | High |
| $m_3$ | Medium | Blue | Medium |
| $m_4$ | High | Green | Poor |

FIGURE 11. Car Information Table

Let us fix, for example, $A = \{Speed, Color\}$ as attribute set, we obtain

$$\pi_{row}(A) = m_1|m_2 m_3|m_4,$$

since $m_1$ is the unique green car whose speed is medium, $m_2$ and $m_3$ are both blue and have medium speed while $m_4$ is the unique green car with high speed.
On the other hand, we can consider *Speed*, *Color*, *RoadHolding* as they were objects and the four models as attributes. Then, the representing table can be obtained by transposing the matrix of Figure 11.

Let $W = \{m_1, m_3\}$. Then

$$\pi_{col}(W) = Speed, RoadHolding|Color.$$

this means that the block $Speed, RoadHolding$ consists of all car properties whose values are exactly the same for both $m_1$ and $m_3$.

In all above examples we have seen that in many cases described by means of an information table, there are present one or both the possibilities contemplated in (A) and (B). This remark is the basic motivation leading us to introduce a more refined interpretation of the investigation potentiality derived from the study of a data rectangular table. The leading point of our analysis starts with the introduction of a data table in which, simply, we consider both the row elements and the column elements as potential attributes. We will call such a data table *knowledge pairing system*. Secondly, we associate to any knowledge pairing system $\mathcal{K}$ a new very rich mathematical structure that we will call *indistinguishability space* of $\mathcal{K}$. The idea is similar, although formally different, with respect to Pawlak's idea to associate a family of knowledge spaces $(U, \equiv_A)$ to any information table $\mathcal{I} = \langle U, Att, Val, F\rangle$, where $U$ is the object set, $Att$ the attribute set, $Val$ the value set, $F$ the information map and $\equiv_A$ the indiscernibility relation on $U$ induced by an attribute subset $A \subseteq Att$. In our context, an indistinguishability space is naturally induced by a knowledge pairing system, and its definition takes into account the possibility to use both the indiscernibility relations induced by rows and columns of our data table.

## 3. Recalls and Notations

3.1. **Notations.** In this paper we deal with finite sets. Let $X$ be a set, then we denote by $\mathcal{P}(X)$ the power set of $X$ and by $|X|$ the number of elements in $X$. We also denote by $X \setminus A$ the complement of a subset $A$ while we use the notation $A^c$ if $X$ is clear from the context. If $A$ and $B$ are two sets, we denote by $A \triangle B$ their symmetric difference. If $k$ is a non-negative integer and $X$ is a finite set such that $|X| \geq k$, we denote by $\binom{X}{k}$ the family of all the subsets of $X$ with exactly $k$ elements (i.e. the so-called $k$-subsets of $X$). If $n$ is a positive integer, we denote by $\hat{n}$ the set $\{1, \dots, n\}$. Finally, we denote by $I_n$ the $n \times n$ identity matrix and by $J_n$ the $n \times n$ matrix having 1 in all its entries.

3.2. **Set Partitions.** Let $X$ be a set, a *set-partition* on $X$ is a not empty family $\pi = \{B_i : i \in I\}$ such that $\bigcup_{i \in I} B_i = X$ and $B_i \cap B_j = \emptyset$ for all $i \neq j$. The subsets $B_i$ are called *blocks* of $\pi$. Since $X$ is a finite set, a set partition $\pi$ of $X$ has only a finite number of blocks $B_1, \dots, B_M$. We use the standard notation $\pi := B_1|\dots|B_M$ to denote the set partition having blocks $B_1, \dots, B_M$. If $\pi$ is a set partition with $M$ distinct blocks, we set $|\pi| := M$. If $Y \subseteq X$ and $Y \subseteq B_i$, for some index $i \in I$, we say that $Y$ is a *sub-block* of $\pi$ and we write $Y \preccurlyeq \pi$. If $x \in X$, we denote by $\pi(x)$ the (unique) block of $\pi$ containing the element $x$.

We denote by $\Pi(X)$ the set of all set-partitions of $X$. It is well known that on the set $\Pi(X)$ we can consider a partial order $\preceq$ defined as follows: if $\pi, \pi' \in \Pi(X)$, then

$$\pi \preceq \pi' :\Longleftrightarrow (\forall B \in \pi)\,(\exists B' \in \pi') \,:\, B \subseteq B' \tag{1}$$

A useful and immediate characterization of the partial order in (1) is the following:

$$\pi \preceq \pi' \Longleftrightarrow (\forall x \in X)\,(\pi(x) \subseteq \pi'(x)) \tag{2}$$

The pair $\mathbb{P}(X) = (\Pi(X), \preceq)$ is a complete lattice which is called *partition lattice* of the set $X$ [5].

3.3. **Graphs.** For general notions on graph theory we refer the reader to Diestel book [30], however in this section we recall some basic facts concerning the undirected graphs which we will use in this paper.

**Definition 3.1.** *We always denote by $G = (V(G), E(G))$ a finite simple (i.e. with no loops and no multiple edges) undirected graph, with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. If $v_i, v_j \in V(G)$, we will write $v_i \sim v_j$ if $\{v_i, v_j\} \in E(G)$ and $v_i \nsim v_j$ otherwise. The* adjacency matrix *of $G$ is the $n \times n$ matrix $Adj(G) := (a_{ij})$ such that $a_{ij} := 1$ if $v_i \sim v_j$ and $a_{ij} := 0$ otherwise. If $v \in V(G)$, we set $N_G(v) := \{w \in V(G) : v \sim w\}$, that is usually called neighborhood of $v$ in $G$. If $A \subseteq V(G)$ we set*

$$N_G(A) := \bigcup_{v \in A} N_G(v) \tag{3}$$

Three important classes of simple graphs that will be used in the following are

- The *complete graph* on $n$ vertices, denoted by $K_n$ and such that $v_i \sim v_j$ for any pair of indexes $i \neq j$.
- The *bipartite graph* that is a graph with $V(G) = B_1 \cup B_2$, $B_1 \cap B_2 = \emptyset$ and edges having a vertex in $B_1$ and the other in $B_2$. In this case the pair $(B_1, B_2)$ is called a bipartition of $G$ and we write $G = (B_1|B_2)$.

- The $(p,q)$-*complete bipartite graph* $K_{p,q}$, is a bipartite graph with $|B_1| = p$, $|B_2| = q$ and edge set $E(G) := \{\{v_i, v_j\} : v_i \in B_1, v_j \in B_2\}$.

3.4. **Digraphs.** For all general notions concerning digraphs we refer the reader to [3]. Here we recall only some basic notions which we will use in the sequel. A *digraph* (see [3]) is a couple $D = (V(D), Arc(D))$, where $V(D) = \{v_1, \dots, v_n\}$ and $Arc(D) = \{e_1, \dots, e_m\}$ are both finite sets. The elements of $V(D)$ are called vertices of $D$ and the elements of $Arc(D)$ are called arcs of $D$. For each arc $e \in Arc(D)$ there are two distinct vertices $v_i, v_j$ in $V(D)$ such that $e_k = (v_i, v_j)$. In this case we say that $v_i$ is the tail of $e$ and that $v_j$ is the head of $e$ (to denote this fact we sometimes write $v_i := t(e)$ and $v_j := h(e)$). If the ordered pair $(v, w)$ is an arc of $D$ we also write $v \to w$.
As it is well known, a digraph $D$ is uniquely determined by its *adjacency matrix* $Adj(D)$, that is the $n \times n$ matrix $(a_{ij})$ defined by $a_{ij} := 1$ if $(v_i, v_j) \in Arc(D)$ and $a_{ij} := 0$ otherwise. The *converse digraph* of a digraph $D$, denoted by $D^*$, is defined by $V(D^*) = V(D)$ and $(v_i, v_j) \in Arc(D^*)$ iff $(v_j, v_i) \in Arc(D)$.
If $v \in V(D)$, we set

$$N_D^+(v) := \{z \in V(D) : (v,z) \in Arc(D)\},\ N_D^-(v) := \{u \in V(D) : (u,v) \in Arc(D)\}. \tag{4}$$

Usually $N_D^+(v)$ is called the *out-neighborhood* of $v$ in $D$ and $N_D^-(v)$ is called the *in-neighborhood* of $v$ in $D$.

3.5. **Hypergraphs.** We recall now the basic notion of hypergraph (see [4] for definition and basic results concerning the hypergraphs).

**Definition 3.2.** *An* hypergraph *is a pair* $H = (V(H), E(H))$, *where* $V(H) = \{v_1, \dots, v_n\}$ *is an arbitrary finite set and* $E(H) = \{Y_1, \dots, Y_m\}$ *is a non-empty family of subsets* $Y_1, \dots, Y_m$ *of* $V(H)$. *The elements* $v_1, \dots, v_n$ *are called* vertices *of* $H$ *and the subsets* $Y_1, \dots, Y_m$ *are called* hyperedges *of* $H$. *An* hypergraph *on* $V(H)$ *is a hypergraph having* $V(H)$ *as vertex set. If* $v \in V(H)$, *we set* $H(v) := \{Y \in E(H) : v \in Y\}$. *Particular classes of hypergraphs can be introduces as follows*

- *If* $|Y_1| = \cdots = |Y_m| = k$ *we say that* $H$ *is a* $k$-uniform *hypergraph.*
- *When* $V(H) = \hat{n}$, $0 \le k \le n$ *and* $E = \binom{\hat{n}}{k}$, *we call the hypergraph* $H := (\hat{n}, \binom{\hat{n}}{k})$ *the* complete $(n,k)$-*uniform hypergraph and (with abuse of notation) we denote it simply by* $\binom{\hat{n}}{k}$.

**Definition 3.3.** *Let* $H = (V(H), E(H))$ *be a given hypergraph. We call the* $n \times m$ *matrix* $Inc(H) = (b_{ij})$ *such that* $b_{ij} := 1$ *if* $v_i \in Y_j$ *and* $b_{ij} := 0$ *otherwise the* incidence matrix *of* $H$.

Obviously $H$ is uniquely determined by its incidence matrix.

**Definition 3.4.** *Let* $H = (V(H), E(H))$ *be a given hypergraph, with* $V(H) = \{v_1, \dots, v_n\}$ *and* $E(H) = \{Y_1, \dots, Y_m\}$. *We call the hypergraph* $H^* = (V(H^*), E(H^*))$, *where* $V(H^*) := \{Y_1, \dots, Y_m\}$, $E(H^*) := \{H(v_1), \dots, H(v_n)\}$, *the* dual hypergraph *of* $H$, *therefore the* $i$-*th hyperedge of* $H^*$ *is* $H(v_i) := \{Y \in E(H) : v_i \in Y\}$. *In this case,* $Inc(H^*)$ *is the transposed matrix of* $Inc(H)$.

**Remark 3.5.**
*(i) In this paper we use the letter* $G$ *(instead of* $H$*) in order to denote an arbitrary finite undirected simple graph (in the sequel briefly simple graph). Moreover, if* $G = (V(G), E(G))$ *is a simple graph, the elements of* $E(G)$ *are usually called* edges *instead of hyperedges. Finally, we denote the edges of* $G$ *with the letters* $e_1, \dots, e_m$ *instead of* $Y_1, \dots Y_m$.
*(ii) If* $G = (V(G), E(G))$ *is a simple graph, each edge* $e_k \in E(G)$ *coincides with a* 2*-subset* $\{v_i, v_j\}$ *of distinct vertices, usually called the* ends *of* $e_k$.
*(iii) If* $Y_k$ *is a hyperedge having elements* $y_1, \dots, y_l$, *we use the classical notation* $Y = \{y_1, \dots, y_l\}$, *but several times we also use the type of string notation* $Y = y_1 \dots y_l$.

## 4. Order Structures Induced by Information Tables

The basic notion in RST is that of information table, whose formalization is the following.

**Definition 4.1.** [47] *An* information table *is a structure* $\mathcal{I} = \langle U, Att, Val, F\rangle$, *where* $U = \{u_1, u_2, \dots, u_m\}$ *is a non-empty finite set whose elements are called* objects, $Att = \{a_1, a_2, \dots, a_n\}$ *is a non-empty finite set whose elements are called* attributes, $Val$ *is a non-empty finite set whose elements are called* values *and* $F : U \times Att \to Val$ *is an application called* information map. *In particular, if* $Val = \{0,1\}$, $\mathcal{I}$ *is called a* Boolean information table.

Based on the previous definition, one defines the following $A$-*indiscernibility relation* on the object set $U$ when $A \subseteq Att$: if $u, u' \in U$ then

$$u \equiv_A u' :\Longleftrightarrow F(u,a) = F(u',a), \forall a \in A. \tag{5}$$

If $u \in U$, the equivalence class $[u]_A$ of $u$ with respect to $\equiv_A$ is called the A-*indiscernibility class* of $u$ and a subset $C \subseteq U$ is called an $A$-*indiscernibility class* whenever there exists some $u \in U$ such that $C = [u]_A$. If $C_1, \dots, C_n$ are all the distinct $A$-indiscernibility classes in $U$, the set partition of $U$

$$\pi_{\mathcal{I}}(A) := \{[u]_A : u \in U\} = C_1 | \dots | C_n \tag{6}$$

is called the $A$-*indiscernibility partition* of $\mathcal{I}$.
In database theory and RST a relevant role is played by the following indiscernibility partition family [73]:

$$\Pi_{ind}(\mathcal{I}) := \{\pi_{\mathcal{I}}(A) : A \subseteq Att\}.$$

In [25] the order structure

$$\mathbb{P}_{ind}(\mathcal{I}) := (\Pi_{ind}(\mathcal{I}), \preceq) \tag{7}$$

has been studied also when both $U$ and $Att$ are arbitrary (not necessarily finite) sets and it has been shown that it is always a complete lattice, named *indiscernibility partition lattice*.
On the other hand, we can also consider the following equivalence relation $\approx$ on $\mathcal{P}(Att)$: if $A, A' \subseteq Att$ we set

$$A \approx A' :\Longleftrightarrow \pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(A'). \tag{8}$$

We denote by $[A]_{\approx}$ the equivalence class of $A$ with respect to $\approx$. It is immediate to see that $[A]_{\approx}$ is union-closed (see [25] for details). Therefore $[A]_{\approx}$ contains a maximum member $M_{\mathcal{I}}(A)$ with respect to the set-theoretic inclusion. Furthermore, in [25] it has been shown that the attribute subset family

$$MAXP(\mathcal{I}) := \{M_{\mathcal{I}}(A) : A \subseteq Att\} \tag{9}$$

induces a poset structure

$$\mathbb{M}(\mathcal{I}) := (MAXP(\mathcal{I}), \subseteq^*) \tag{10}$$

that is a complete lattice order isomorphic to $\mathbb{P}_{ind}(\mathcal{I})$. The elements of $MAXP(\mathcal{I})$ are called *maximum partitioners* of $\mathcal{I}$ and the lattice $\mathbb{M}(\mathcal{I})$ is called *maximum partitioners lattice* of $\mathcal{I}$ (for details see [25]).
To obtain a more explicit analogy between the indiscernibility partition lattice of an information system and the concept lattice of a formal context [32], we also introduce the following set:

$$Gran(\mathcal{I}) := \{(M_{\mathcal{I}}(A), \pi_{\mathcal{I}}(A)) : A \subseteq Att\}. \tag{11}$$

On the previous set we consider the following order structure:

$$\mathbb{G}(\mathcal{I}) := (Gran(\mathcal{I}), \subseteq^* \times \preceq) \tag{12}$$

where $\subseteq^*$ is the dual inclusion order and $\subseteq^* \times \preceq$ is the direct product order of $\subseteq^*$ and $\preceq$. In [25] it has been shown that $\mathbb{G}(\mathcal{I})$ is isomorphic to $\mathbb{M}(\mathcal{I})$. So, we introduce the following definition.

**Definition 4.2.** *We call* $\mathbb{G}(\mathcal{I})$ *the* granular partition lattice *of* $\mathcal{I}$.

We observe now that the real theoretical significance of the previous isomorphism is the possibility to consider equivalent each other the study of the family of all the $A$-indiscernibliity relations on $U$ and the study of the equivalence relation $\approx$ on $\mathcal{P}(Att)$. We call $\approx$ the *indistinguishability relation* on $\mathcal{I}$ and $[A]_{\approx}$ the *indistinguishability class* of $A$ (we also call it *local indistiguishability poset* of $A$). From this perspective, we can use the universe set $U$ simply as an intermediate tool to work on attribute subsets. Then, two attribute subsets $A$ and $A'$ are indistinguishable when they induce the same set partition on the universe set $U$. Moreover, another important fact is that the equivalence relation $\approx$ induces a very rich mathematical structure on the attribute power set $\mathcal{P}(Att_{\mathcal{I}})$, and the richness of this structure is a consequence of the way in which the indiscernibility partitions $\pi_{\mathcal{I}}(A)$ are related each other. In order to better understand the importance of the indistinguishability relation, we will introduce now two lattice associated to any information table $\mathcal{I}$. Let $MAXP(\mathcal{I}) = \{C_1, \dots, C_k\}$ and let $B_{1j}, \dots, B_{p_j j}$ be the distinct blocks of the indiscernibility partition of $\pi_{\mathcal{I}}(C_j)$ for $1 \leq j \leq k$. Let $C_j = \{c_{1j}, \dots, c_{q_j j}\}$. Fix $1 \leq j \leq k$. For any $1 \leq i \leq p_j$ and $1 \leq k \leq q_j$ we define the following map

$$\phi_j(B_{ij}, c_{kj}) := F_{\mathcal{I}}(u, c_{kj}), \tag{13}$$

where $u$ is any element of $B_{ij}$. This map is well defined because, by definition of indiscernibility, the information map $F_{\mathcal{I}}$ is constant on the elements belonging to same indiscernibility block.

**Definition 4.3.** *We denote by $S_j(\mathcal{I})$ the information table having objects $B_{1j}, \dots, B_{p_j j}$, attributes $c_{1j}, \dots, c_{q_j j}$ and information map $\phi_j$ given in* (13). *We call $S_j(\mathcal{I})$ the $j^{th}$-*indiscernibility sub-table *of $\mathcal{I}$.*

We set now

$$ISUB(\mathcal{I}) := \{S_1(\mathcal{I}), \dots, S_k(\mathcal{I})\}, \tag{14}$$

and

$$PART(\mathcal{I}) := \{[C_1]_{\approx}, \dots, [C_k]_{\approx}\}. \tag{15}$$

Since the elements of $PART(\mathcal{I})$ are also a set partition of the power set $\mathcal{P}(Att_{\mathcal{I}})$ we also use the following set partition notation:

$$\Lambda_{\mathcal{I}} := [C_1]_{\approx}| \dots |[C_k]_{\approx} \tag{16}$$

At this point we introduce the following partial orders $\sqsubseteq$ and $\sqsubseteq'$, respectively on the sets $ISUB(\mathcal{I})$ and $PART(\mathcal{I})$.
If $S_j(\mathcal{I}), S_l(\mathcal{I}) \in ISUB(\mathcal{I})$ and $[C_l]_{\approx}, [C_j]_{\approx} \in PART(\mathcal{I})$ we define

$$S_l(\mathcal{I}) \sqsubseteq S_j(\mathcal{I}), \text{ or equivalently } [C_l]_{\approx} \sqsubseteq' [C_j]_{\approx}, \text{ if } C_l \subseteq^* C_j \tag{17}$$

and

$$\mathbb{S}(\mathcal{I}) := (ISUB(\mathcal{I}), \sqsubseteq), \tag{18}$$

$$\mathbb{I}(\mathcal{I}) := (PART(\mathcal{I}), \sqsubseteq'). \tag{19}$$

We have then the following immediate result

**Theorem 4.4.** *$\mathbb{S}(\mathcal{I})$ and $\mathbb{I}(\mathcal{I})$ are two lattices that are both order isomorphic to the maximum partitioner lattice $\mathbb{M}(\mathcal{I})$.*

*Proof.* The thesis follows immediately from the definition of the sets $ISUB(\mathcal{I})$, $PART(\mathcal{I})$ and by (17). □

**Definition 4.5.** *We call $\mathbb{S}(\mathcal{I})$ the* indiscernibility sub-table lattice *of $\mathcal{I}$ and $\mathbb{I}(\mathcal{I})$ the* indistinguishability lattice *of $\mathcal{I}$.*

From this perspective, it's clear that the maximum partitioners and the minimum ones can be chosen canonically within any local indistinguishability poset. Concerning the macro-granular structure, the previous two lattices provide the same information. The real difference between them is in the micro-granular level: in fact, $\mathbb{I}(\mathcal{I})$ detects the inner structure of any local indistinguishability poset while $\mathbb{S}(\mathcal{I})$ provides informations on how each maximum partitioner is linked to the values appearing on the blocks of the corresponding indiscernibility partition that it induces.

## 5. Knowledge Pairing Systems and Related Notions

In this section we will reinterpret and generalize Pawlak's information systems in the light of the indistiguishability relation.

**Definition 5.1.** *A* knowledge pairing system *is a structure $\mathcal{K} := \langle U, \Omega, Val, F\rangle$ such that:*

- *$U$ and $\Omega$ are two finite sets whose elements are called respectively* row attributes *and* column attributes*;*
- *$Val$ is a set of values;*
- *$F : U \times \Omega \to Val$, assigning to any pair $(u, a) \in U \times \Omega$ the value $F(u, a)$, is called* knowledge map.

*We denote by $T[\mathcal{K}]$ the rectangular table having the elements of $U$ on the rows, those of $\Omega$ on the columns and the value $F(u_i, a_j)$ in the place $(i, j)$. We call $T[\mathcal{K}]$ the* knowledge table *of $\mathcal{K}$.*

**Remark 5.2.** *In many concrete situations or for knowledge pairing systems induced by specific kinds of mathematical structures, such as graphs, digraphs and hypergraphs, row and column attributes can be a priori interrelated via an additional structure or not. Clearly, both cases are covered by Definition 5.1.*

**Remark 5.3.** *Any classical information system (objects-attributes) can be seen as a knowledge pairing system where* objects=row attributes *and* attributes=column attributes*. Nevertheless, as we will see below, to interpret a classical information system as a knowledge pairing system refines our investigation, since indiscernibility is induced not only by attributes on objects (as in an usual information system) but also by objects on attributes. This explain why we introduce the term pair* (row attribute, column attribute) *instead of* (object, attribute).

In a knowledge pairing system, we introduce the following two indiscernibility relations.
If $A \subseteq \Omega$ and $W \subseteq U$, we set

$$u \equiv_A^{\downarrow} u' :\Longleftrightarrow F(u,a) = F(u',a),\ \forall a \in A, \tag{20}$$

and

$$a \equiv_W^{\uparrow} a' :\Longleftrightarrow F(u,a) = F(u,a'),\ \forall u \in W. \tag{21}$$

If $u \in U$ and $a \in \Omega$, we denote respectively by $[u]_A^{\downarrow}$ and $[a]_W^{\uparrow}$ the equivalence classes of $u$ with respect to $\equiv_A^{\downarrow}$ and of $a$ with respect to $\equiv_W^{\uparrow}$.
If $A \subseteq \Omega$ and $W \subseteq U$, we set

$$\pi_{\mathcal{K}}^{\downarrow}(A) := \{[u]_A^{\downarrow} : u \in U\} \tag{22}$$

and

$$\pi_{\mathcal{K}}^{\uparrow}(W) := \{[a]_W^{\uparrow} : a \in \Omega\}. \tag{23}$$

If $A, A' \subseteq \Omega$, we set

$$A \approx^{\downarrow} A' :\Longleftrightarrow\ \pi_{\mathcal{K}}^{\downarrow}(A) = \pi_{\mathcal{K}}^{\downarrow}(A') \tag{24}$$

and we denote by $[A]_{\approx}^{\downarrow}$ the equivalence class of $A$ with respect to $\approx^{\downarrow}$ and by $M_{\mathcal{K}}^{\downarrow}(A)$ its maximum member. We denote by $\min([A]_{\approx}^{\downarrow})$ the minimal elements of the poset $([A]_{\approx}^{\downarrow}, \subseteq)$.
Analogously, if $W, W' \subseteq U$, we set

$$W \approx^{\uparrow} W' :\Longleftrightarrow\ \pi_{\mathcal{K}}^{\uparrow}(W) = \pi_{\mathcal{K}}^{\uparrow}(W') \tag{25}$$

and we denote by $[W]_{\approx}^{\uparrow}$ the equivalence class of $W$ with respect to $\approx^{\uparrow}$ and by $M_{\mathcal{K}}^{\uparrow}(W)$ its maximum member. We denote by $\min([W]_{\approx}^{\uparrow})$ the minimal elements of the poset $([W]_{\approx}^{\uparrow}, \subseteq)$.

**Remark 5.4.** *In an usual information system one has a unique way to define the indistinguishability relation* $\approx$ *that corresponds to* $\approx^{\downarrow}$. *As a matter of fact, in this case, one must consider only the indiscernibility induced by attributes on objects. On the other hand, in a knowledge pairing system, one has also to analyze the additional indistinguishability relation* $\approx^{\uparrow}$, *defined via the indiscernibility induced by row attributes on column attributes.*

We also set:

$$\Pi_{ind}^{\downarrow}(\mathcal{K}) := \{\pi_{\mathcal{K}}^{\downarrow}(A) :\ A \subseteq \Omega\},\ \ \mathbb{P}_{ind}^{\downarrow}(\mathcal{K}) := (\Pi_{ind}^{\downarrow}(\mathcal{K}), \preceq),$$
$$MAXP^{\downarrow}(\mathcal{K}) := \{M_{\mathcal{K}}^{\downarrow}(A),\ A \subseteq \Omega\},\ \ \mathbb{M}^{\downarrow}(\mathcal{K}) := (MAXP^{\downarrow}(\mathcal{K}), \subseteq^{*}),$$
$$Gran^{\downarrow}(\mathcal{K}) := \{(M_{\mathcal{K}}^{\downarrow}(A), \pi_{\mathcal{K}}^{\downarrow}(A)) : A \subseteq \Omega\},\ \ \mathbb{G}^{\downarrow}(\mathcal{K}) := (Gran^{\downarrow}(\mathcal{K}), \subseteq^{*}).$$

and

$$MINP^{\downarrow}(\mathcal{K}) := \bigcup\{\min([A]_{\approx}^{\downarrow}) : A \in MAXP^{\downarrow}(\mathcal{K})\}.$$

In [26] it has been shown that $MINP^{\downarrow}(\mathcal{K})$ is an abstract simplicial complex, i.e. it is a set family containing the empty set and such that if $X \in MINP^{\downarrow}(\mathcal{K})$ and $Y \subseteq X$ then $Y \in MINP^{\downarrow}(\mathcal{K})$. This implies that all subsets of $\Omega$ that are minimal with respect to the property of inducing a down-indiscernibility partition form a family invariant with respect to deletion of elements. In other terms, if $X \in MINP^{\downarrow}(\mathcal{K})$, then $X \setminus \{x\}$ induces less knowledge than $X$ for any $x \in X$ and, however, the knowledge provided by $X \setminus \{x\}$ is minimal in the down-indistinguishability class $[X \setminus \{x\}]_{\approx}^{\downarrow}$.
Let $MAXP^{\downarrow}(\mathcal{K}) = \{C_1, \dots, C_k\}$ and let $B_{1j}^{\downarrow}, \dots, B_{p_j j}^{\downarrow}$ be the distinct blocks of the indiscernibility partition of $\pi_{\mathcal{K}}^{\downarrow}(C_j)$ for $1 \le j \le k$. Let $C_j = \{c_{1j}, \dots, c_{q_j j}\}$. Fix $1 \le j \le k$. For any $1 \le i \le p_j$ and $1 \le k \le q_j$ we define the following map

$$\phi_j^{\downarrow}(B_{ij}^{\downarrow}, c_{kj}) := F_{\mathcal{K}}(u, c_{kj}), \tag{26}$$

where $u$ is any element of $B_{ij}^{\downarrow}$.
We denote by $S_j^{\downarrow}(\mathcal{K})$ the information table having objects $B_{1j}^{\downarrow}, \dots, B_{p_j j}^{\downarrow}$, attributes $c_{1j}, \dots, c_{q_j j}$ and information map $\phi_j^{\downarrow}$ given in (26).
Furthermore, we set:

$$ISUB^{\downarrow}(\mathcal{K}) := \{S_1^{\downarrow}(\mathcal{K}), \dots, S_k^{\downarrow}(\mathcal{K})\},\quad PART^{\downarrow}(\mathcal{K}) := \{[C_1]_{\approx}^{\downarrow}, \dots, [C_k]_{\approx}^{\downarrow}\}$$

and

$$\mathbb{S}^{\downarrow}(\mathcal{K}) := (ISUB^{\downarrow}(\mathcal{K}), \sqsubseteq),$$
$$\mathbb{I}^{\downarrow}(\mathcal{K}) := (PART^{\downarrow}(\mathcal{K}), \sqsubseteq').$$

Analogously:

$$\Pi^{\uparrow}_{ind}(\mathcal{K}) := \{\pi^{\uparrow}_{\mathcal{J}}(W) : W \subseteq U\},\ \mathbb{P}^{\uparrow}_{ind}(\mathcal{K}) := (\Pi^{\uparrow}_{ind}(\mathcal{K}), \preceq),$$
$$MAXP^{\uparrow}(\mathcal{K}) := \{M^{\uparrow}_{\mathcal{K}}(W), W \subseteq U\},\ \mathbb{M}^{\uparrow}(\mathcal{K}) := (MAXP^{\uparrow}(\mathcal{K}), \subseteq^*),$$
$$Gran^{\uparrow}(\mathcal{K}) := \{(M^{\uparrow}_{\mathcal{K}}(W), \pi^{\uparrow}_{\mathcal{K}}(W)) : W \subseteq U\},\ \mathbb{G}^{\uparrow}(\mathcal{K}) := (Gran^{\uparrow}(\mathcal{K}), \subseteq^*).$$

and

$$MINP^{\uparrow}(\mathcal{K}) := \bigcup\{\min([A]^{\uparrow}_{\approx}) : A \in MAXP^{\uparrow}(\mathcal{K})\}.$$

Let $MAXP^{\uparrow}(\mathcal{K}) = \{W_1, \dots, W_k\}$ and let $B^{\uparrow}_{1j}, \dots, B^{\uparrow}_{p_j j}$ be the distinct blocks of the indiscernibility partition of $\pi^{\uparrow}_{\mathcal{K}}(W_j)$ for $1 \le j \le k$. Let $W_j = \{u_{1j}, \dots, u_{q_j j}\}$. Fix $1 \le j \le k$. For any $1 \le i \le p_j$ and $1 \le k \le q_j$ we define the following map

$$\phi^{\uparrow}_j(B^{\uparrow}_{ij}, u_{kj}) := F_{\mathcal{K}}(u_{kj}, a), \tag{27}$$

where $a$ is any element of $B^{\uparrow}_{ij}$.
We denote by $S^{\uparrow}_j(\mathcal{K})$ the information table having objects $B^{\uparrow}_{1j}, \dots, B^{\uparrow}_{p_j j}$, attributes $u_{1j}, \dots, u_{q_j j}$ and information map $\phi^{\uparrow}_j$ given in (27).
Furthermore, we set:

$$ISUB^{\uparrow}(\mathcal{K}) := \{S^{\uparrow}_1(\mathcal{K}), \dots, S^{\uparrow}_k(\mathcal{K})\},\quad PART^{\uparrow}(\mathcal{K}) := \{[W_1]^{\uparrow}_{\approx}, \dots, [W_k]^{\uparrow}_{\approx}\}$$

and

$$\mathbb{S}^{\uparrow}(\mathcal{K}) := (ISUB^{\uparrow}(\mathcal{K}), \sqsubseteq),$$
$$\mathbb{I}^{\uparrow}(\mathcal{K}) := (PART^{\uparrow}(\mathcal{K}), \sqsubseteq').$$

Based on the previous indistinguishability relations we also introduce the following terminology.

**Definition 5.5.** *We call:*

- $\equiv^{\downarrow}_A$ *the* $A$-down indiscernibility relation *and* $\equiv^{\uparrow}_W$ *the* $W$-up indiscernibility relation*;*
- $\pi^{\downarrow}_{\mathcal{K}}(A)$ *the* $A$-down indiscernibility partition *and* $\pi^{\uparrow}_{\mathcal{K}}(W)$ *the* $W$-up indiscernibility partition*;*
- $\approx^{\downarrow}$ *the* down-indistinguishability relation *and* $\approx^{\uparrow}$ *the* up-indistinguishability relation*;*
- down-maximum partitioner *of* $\mathcal{K}$ *any member of* $MAXP^{\downarrow}(\mathcal{K})$ *and* up-maximum partitioner *of* $\mathcal{K}$ *any member of* $MAXP^{\uparrow}(\mathcal{K})$*;*
- down-minimal partitioner *of* $\mathcal{K}$ *any member of* $MINP^{\downarrow}(\mathcal{K})$ *and* up-minimal partitioner *of* $\mathcal{K}$ *any member of* $MINP^{\uparrow}(\mathcal{K})$*;*
- down-indiscernibility partition lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{P}^{\downarrow}_{ind}(\mathcal{K})$ *and* up-indiscernibility partition lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{P}^{\uparrow}_{ind}(\mathcal{K})$*;*
- down-maximum partitioner lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{M}^{\downarrow}(\mathcal{K})$ *and* up-maximum partitioner lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{M}^{\uparrow}(\mathcal{K})$*;*
- down-granular partitioner lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{G}^{\downarrow}(\mathcal{K})$ *and* up-granular partitioner lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{G}^{\uparrow}(\mathcal{K})$*;*
- down-indiscernibility sub-table lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{S}^{\downarrow}(\mathcal{K})$ *and* up-indiscernibility sub-table lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{S}^{\uparrow}(\mathcal{K})$*;*
- down-indistinguishability lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{I}^{\downarrow}(\mathcal{K})$ *and* up-indistinguishability lattice *of* $\mathcal{K}$ *the lattice* $\mathbb{I}^{\uparrow}(\mathcal{K})$.

Finally, we define the basic notion of this paper.

**Definition 5.6.** *Let* $\mathcal{K}$ *be a knowledge pairing system. We call* $(\mathcal{K}, \approx^{\downarrow}, \approx^{\uparrow})$ *the* indistinguishability space *of* $\mathcal{K}$

One of the most relevant aspects when we consider a knowledge pairing system and its related indistinguishability spaces is the possibility to investigate these structures in a similar way to the formal context analysis methods (see [32]). In fact, similarly to FCA where the two classical *intent* and *extent* set operators are defined, also in our context we can define two analogous set operators. In order to proceed on this research line, we introduce the following notions (for some results concerning a particular case of these notions, see also [25, 27]).

Let $\mathcal{K} = \langle U, \Omega, Val, F\rangle$ be a knowledge pairing system and let $Z \subseteq U$ and $A \subseteq \Omega$.
We set then

$$\Gamma^{\uparrow}(Z) := \{a \in \Omega\ :\ \forall\ z, z' \in Z, F(z, a) = F(z', a)\}, \tag{28}$$

and

$$\Gamma^{\downarrow}(A) := \{z \in U \ : \ \forall\, a, a' \in A, F(z, a) = F(z, a')\}. \tag{29}$$

Let us observe that $\Gamma^{\uparrow} : \mathcal{P}(U) \to \mathcal{P}(\Omega)$ and $\Gamma^{\downarrow} : \mathcal{P}(\Omega) \to \mathcal{P}(U)$.

**Definition 5.7.** *We call*

- $\Gamma^{\uparrow}(Z)$ *the* up-indiscernibility closure *of* $Z$*;*
- $\Gamma^{\downarrow}(A)$ *the* down-indiscernibility closure *of* $A$.

The relevance of the previous set operators $\Gamma^{\uparrow}$ and $\Gamma^{\downarrow}$ is due to the following result.

**Theorem 5.8.** *(i) Let* $\pi = B_1|\dots|B_M \in \Pi(U)$ *and let* $A = \bigcap_{i=1}^{M} \Gamma^{\uparrow}(B_i)$*. Then*

$$\pi \in \Pi^{\downarrow}_{ind}(\mathcal{K}) \iff \pi = \pi^{\downarrow}_{\mathcal{K}}(A), \tag{30}$$

*and in this case it results that* $A = M^{\downarrow}_{\mathcal{K}}(A)$.
*(ii) Let* $\pi' = B'_1|\dots|B'_N \in \Pi(\Omega)$ *and let* $Z = \bigcap_{i=1}^{N} \Gamma^{\downarrow}(B'_i)$*. Then*

$$\pi' \in \Pi^{\uparrow}_{ind}(\mathcal{K}) \iff \pi' = \pi^{\uparrow}_{\mathcal{K}}(Z), \tag{31}$$

*and in this case it results that* $Z = M^{\uparrow}_{\mathcal{K}}(Z)$.

*Proof.* The proof of both $(i)$ and $(ii)$ is similar to the proof of Theorem 5.4 in [28]. □

We can use the previous two maps in order to connect in a non-trivial way the relations induced by $\approx^{\uparrow}$ and $\approx^{\downarrow}$.

**Proposition 5.9.** *Let* $\mathcal{K}$ *be a knowledge pairing system.*

(i) *If* $Z \in \mathcal{P}(U)$*, then* $\Gamma^{\uparrow}(Z) \in MAXP^{\downarrow}(\mathcal{K})$*;*
(ii) *If* $A \in \mathcal{P}(\Omega)$*, then* $\Gamma^{\downarrow}(A) \in MAXP^{\uparrow}(\mathcal{K})$.

*Proof.* The proofs of both $(i)$ and $(ii)$ are similar to that of Proposition 6.3 in [25]. □

By analogy with the extent and the intent operators of FCA [32], if $Z \subseteq U$ and $A \subseteq \Omega$, we set

$$\Gamma^{\uparrow\downarrow}(Z) := \Gamma^{\downarrow}(\Gamma^{\uparrow}(Z)) \ \text{ and } \ \Gamma^{\downarrow\uparrow}(A) := \Gamma^{\uparrow}(\Gamma^{\downarrow}(A)). \tag{32}$$

We obtain then the following immediate result

**Proposition 5.10.** *Both the set operators*

$$\Gamma^{\uparrow\downarrow} := \Gamma^{\downarrow} \circ \Gamma^{\uparrow} : \mathcal{P}(U) \to \mathcal{P}(U) \ \text{ and } \ \Gamma^{\downarrow\uparrow} := \Gamma^{\uparrow} \circ \Gamma^{\downarrow} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega) \tag{33}$$

*are inclusion-preserving.*

We consider now the following restricted maps:

$$\gamma^{\uparrow\downarrow} := \Gamma^{\uparrow\downarrow}_{|MAXP^{\downarrow}(\mathcal{K})} \ \text{ and } \ \gamma^{\downarrow\uparrow} := \Gamma^{\downarrow\uparrow}_{|MAXP^{\uparrow}(\mathcal{K})} \tag{34}$$

Then we obtain the following property for the above set operators.

**Proposition 5.11.** *With the previous notations:*

- $\gamma^{\uparrow\downarrow}$ *is an inclusion-preserving set operator from* $MAXP^{\uparrow}(\mathcal{K})$ *to* $MAXP^{\uparrow}(\mathcal{K})$*;*
- $\gamma^{\downarrow\uparrow}$ *is an inclusion-preserving set operator from* $MAXP^{\downarrow}(\mathcal{K})$ *to* $MAXP^{\downarrow}(\mathcal{K})$.

*Proof.* It follows immediately by Proposition 5.9. □

In the next section we discuss, on the specific example of the student hypergraph, the interpretation of all the above notions introduced in this section.

5.1. **Other Classical Notions of RST in Knowledge Pairing Systems.** The usual typical notions of RST derived from the discernibility matrix can be carry out naturally in any knowledge pairing system. Let $\mathcal{K} = \langle U, \Omega, F, Val\rangle$ be a knowledge pairing system.

**Definition 5.12.**

- *Let* $u, u' \in U$ *we set*

$$\Delta^{\downarrow}_{\mathcal{H}}(u, u') := \{a \in \Omega : F(u, a) \neq F(u', a)\}. \tag{35}$$

*We denote by* $\Delta^{\downarrow}[\mathcal{H}]$ *the* $|U| \times |U|$ *matrix having the subset* $\Delta^{\downarrow}_{\mathcal{H}}(u, u')$ *in the* $(u, u')$*-entry. We call* $\Delta^{\downarrow}[\mathcal{H}]$ *the* down-discernibility matrix *of* $\mathcal{H}$*. We denote by* $DISC^{\downarrow}(\mathcal{K}) := \{\Delta^{\downarrow}_{\mathcal{H}}(u, u') : u, u' \in U\}$ *the* down-discernibility hypergraph*.*

| | $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|---|---|---|---|---|
| $m_1$ | $\emptyset$ | CR | C | SR |
| $m_2$ | * | $\emptyset$ | R | SCR |
| $m_3$ | * | * | $\emptyset$ | SCR |
| $m_4$ | * | * | * | $\emptyset$ |

Table 1. Down-discernibility matrix $\Delta^{\downarrow}[\mathcal{C}]$

- *Let $a, a' \in \Omega$ we set*

$$\Delta^{\uparrow}_{\mathcal{H}}(a, a') := \{u \in U : F(u, a) \neq F(u, a')\}. \tag{36}$$

*We denote by $\Delta^{\uparrow}[\mathcal{H}]$ the $|\Omega| \times |\Omega|$ matrix having the subset $\Delta^{\uparrow}_{\mathcal{H}}(a, a')$ in the $(a, a')$-entry. We call $\Delta^{\uparrow}[\mathcal{H}]$ the* up-discernibility matrix *of $\mathcal{H}$. We denote by $DISC^{\uparrow}(\mathcal{K}) := \{\Delta^{\uparrow}_{\mathcal{H}}(a, a') : a, a' \in \Omega\}$ the* up-discernibility hypergraph.

The following result relates the entries of the down-discernibility matrix to the down-indiscernibility relation. Obviously, a similar result holds also for the case up, but we omit the corresponding statement.

**Proposition 5.13.** [19] *Let $D \subseteq \Omega$ and $v, w \in U$. Then:*
*(i) $D = \Delta^{\downarrow}_{\mathcal{H}}(v, w) \Longrightarrow v \equiv^{\downarrow}_{\Omega \setminus D} w$;*
*(ii) $v \equiv^{\downarrow}_{\Omega \setminus D} w \Longrightarrow \Delta^{\downarrow}_{\mathcal{H}}(v, w) \subseteq D$;*
*(iii) Let $C \subseteq \Omega$. Then $\Delta^{\downarrow}_{\mathcal{H}}(v, w) \cap C = \emptyset \Longleftrightarrow v \equiv^{\downarrow}_{C} w$.*

The discernibility matrix is at the basis of the *core* and the *reducts* computation so it has a special role inside the rough set theory (see [47]). A reduct of an information table $\mathcal{I}$ can be considered as a subset of all attributes of $\mathcal{I}$ sufficient to provide the same knowledge of the whole attribute set. The core of $\mathcal{I}$ is the subset of all attributes of $\mathcal{I}$ whose elimination causes a substantial change in the knowledge induced from $\mathcal{I}$. In our context we have the following notions (we omit to write explicitly the up case).

**Definition 5.14.** *An element $c \in \Omega$ is called* down-indispensable *if $\pi^{\downarrow}_{\mathcal{K}}(\Omega) \neq \pi^{\downarrow}_{\mathcal{K}}(\Omega \setminus \{c\})$. The subset of all down-indispensable elements of $\Omega$ is called* down-core *of $\mathcal{K}$ and it is denoted by $CORE^{\downarrow}(\mathcal{K})$. A subset $C \subseteq \Omega$ is said a* down-reduct *of $\mathcal{H}$ if:*
*(i) $\pi^{\downarrow}_{\mathcal{K}}(\Omega) = \pi^{\downarrow}_{\mathcal{K}}(C)$;*
*(ii) $\pi^{\downarrow}_{\mathcal{K}}(\Omega) \neq \pi^{\downarrow}_{\mathcal{K}}(C \setminus \{c\})$ for all $c \in C$;*
*We denote by $RED^{\downarrow}(\mathcal{K})$ the family of all down-reducts of $\mathcal{K}$.*

**Remark 5.15.** *From a computational point of view, if the time complexities to compute the reduct families $RED^{\uparrow}(\mathcal{K})$ and $RED^{\downarrow}(\mathcal{K})$ are respectively $O(f(x))$ and $O(g(x))$, then the time complexity to determine both the up and down reducts of $\mathcal{K}$ is $\max\{O(f(x)), O(g(x))\}$.*

The notion of core can be generalized in the following way (see [20] for details).

**Definition 5.16.** *We say that a subset $C \subseteq \Omega$ is* down-essential *if:*
*(i) $\pi^{\downarrow}_{\mathcal{K}}(\Omega \setminus C) \neq \pi^{\downarrow}_{\mathcal{K}}(\Omega)$;*
*(ii) for all $D \subsetneq C$ we have that $\pi^{\downarrow}_{\mathcal{K}}(\Omega \setminus D) = \pi^{\downarrow}_{\mathcal{K}}(\Omega)$.*
*We denote by $ESS^{\downarrow}(\mathcal{K})$ the family of all the down-essential subsets.*

**Example 5.17.** By referring to Example 2.4, we represent in Table 1 the down-discernibility matrix $\Delta^{\downarrow}[\mathcal{C}]$ of the knowledge representation system $\mathcal{C}$. We use $S$, $C$ and $R$ to denote respectively Speed, Color and Roadholding.
By applying Definition 5.16, it is immediate to see that $ESS^{\downarrow}(\mathcal{C}) = \{\{C\}, \{R\}\}$. This means that $\pi^{\downarrow}_{\mathcal{C}}(\Omega \setminus \{C\}) \neq \pi^{\downarrow}_{\mathcal{C}}(\Omega)$ and $\pi^{\downarrow}_{\mathcal{C}}(\Omega \setminus \{R\}) \neq \pi^{\downarrow}_{\mathcal{C}}(\Omega)$. So, the deletion of $C$ or $R$ makes more difficult the choice between the four models of cars, since it involves a lost of informations. The previous two subsets are the only subsets behaviouring in this way. Moreover, it is easy to verify by Definition 5.14 that $CORE^{\downarrow}(\mathcal{C}) = \{C, R\}$ and $RED^{\downarrow}(\mathcal{C}) = \{\{C, R\}\}$, so $C$ and $R$ are fundamental to ensure the same degree of knowledge of $\Omega$.

## 6. Knowledge pairing systems induced by graphs and hypergraphs

In this section we show how graphs and hypergraphs can be seen as basic models of knowledge pairing systems. More in detail, we are interested to examine the knowledge pairing system associated to a hypergraph $H$ and that associated to a simple graph $G$.

**Definition 6.1.** *Let $H = (V(H), E(H))$ be a given hypergraph. We associate to $H$ the following knowledge pairing system:*

$$\hat{H} := \langle U, \Omega, Val, F \rangle,$$

*where:*

- $U = V(H)$;
- $\Omega = E(H)$;
- $Val = \{0, 1\}$
- $F : V(H) \times E(H) \to \{0, 1\}$ *is defined by* $F(u, Y) := 1$ *if* $u \in Y$ *and* $F(u, Y) := 0$ *otherwise, for any* $(u, Y) \in V(H) \times E(H)$

In other terms, we associate to $H$ the knowledge pairing system $\hat{H}$ whose row attributes are exactly the vertices of $H$, whereas the column attributes are its hyperedges. Accordingly, we observe that, in some cases, the elements of $U$ and $\Omega$ can take different natures, depending on the context of definition of the knowledge pairing system.

**Example 6.2.** In reference to the student hypergraph $H$ introduced in Example 2.3 of the introductory section, we have the following down-indistinguishability classes of the knowledge pairing system $\hat{H}$:

$$[\emptyset]^{\downarrow}_{\approx} = \{\emptyset\},\ [A_L]^{\downarrow}_{\approx} = \{A_L\},\ [G_E]^{\downarrow}_{\approx} = \{G_E\},\ [A_N]^{\downarrow}_{\approx} = \{A_N\},\ [C_S]^{\downarrow}_{\approx} = \{C_S\},$$

$$[A_LA_N]^{\downarrow}_{\approx} = \{A_LA_N\},\ [A_LC_S]^{\downarrow}_{\approx} = \{A_LC_S\},\ [G_EA_N]^{\downarrow}_{\approx} = \{G_EA_N\},\ [A_NC_S]^{\downarrow}_{\approx} = \{A_NC_S\},$$

$$[A_LA_NC_S]^{\downarrow}_{\approx} = \{A_LA_NC_S\},\ [A_LG_EA_N]^{\downarrow}_{\approx} = \{A_LG_EA_N, A_LG_E\},\ [G_EA_NC_S]^{\downarrow}_{\approx} = \{G_EA_NC_S, G_EC_S\},$$

$$[A_LG_EA_NC_S]^{\downarrow}_{\approx} = \{A_LG_EA_NC_S, A_LG_EC_S\}.$$

Therefore the down-maximum partitioners of $\hat{H}$ are

$$\emptyset, A_L, G_E, A_N, C_S, A_LA_N, A_LC_S, G_EA_N, A_NC_S, A_LA_NC_S, A_LG_EA_N, G_EA_NC_S, A_LG_EA_NC_S,$$

whereas the down-minimal partitioners of $\hat{H}$ are the following:

$$\emptyset, A_L, G_E, A_N, C_S, A_LA_N, A_LC_S, G_EA_N, A_NC_S, A_LA_NC_S, A_LG_E, G_EC_S, A_LG_EC_S.$$

Let us observe that the down-minimal partitioners are exactly the minimal subsets of $E(H)$ inducing all possible set partition of $V(H)$.

In Figure 12 we represent the down-granular partition lattice of $\hat{H}$ (we use lowercase initial letters of the name of the students).

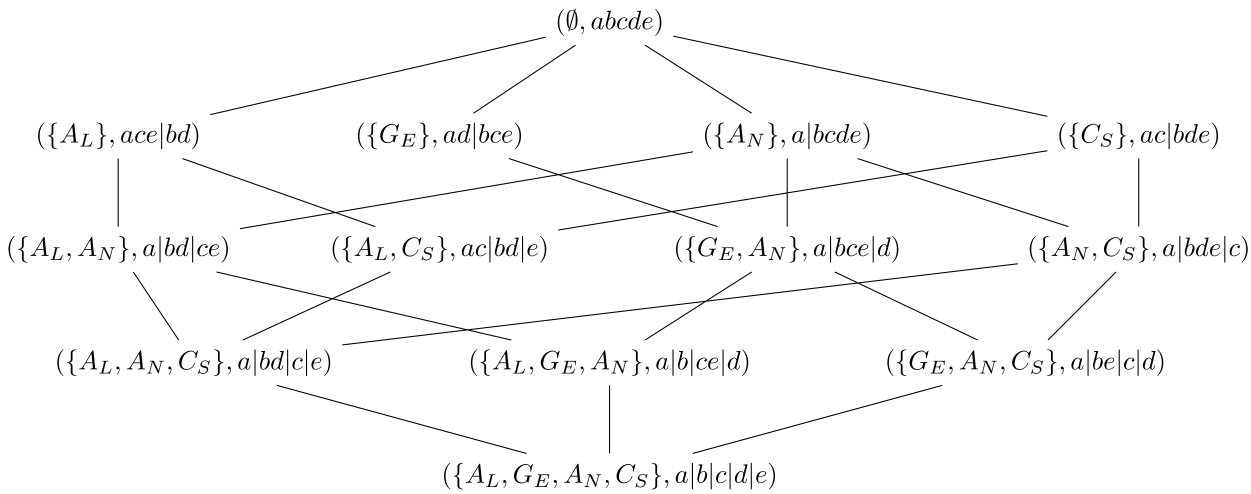



FIGURE 12. The Down-Granular Partition Lattice $\mathbb{G}^{\downarrow}(\hat{H})$

In Figure 13 we give the micro-granular representation of the down-granular partition lattice.

In Figure 14 we represent the down-indiscernibility sub-table lattice.

Furthermore, we have the following up-indistinguishability classes of the knowledge pairing system $\hat{H}$:

$$[\emptyset]^{\uparrow}_{\approx} = \{\emptyset\},\ [c]^{\uparrow}_{\approx} = \{c\},\ [d]^{\uparrow}_{\approx} = \{d\},\ [e]^{\uparrow}_{\approx} = \{e\},$$

$$[ab]^{\uparrow}_{a,b,ab},\ [cd]^{\uparrow}_{\approx} = \{cd\},\ [ce]^{\uparrow}_{\approx} = \{ce\},\ [abde]^{\uparrow}_{\approx} = \{ad, ae, bd, be, de, abd, abe, ade, bde\},$$

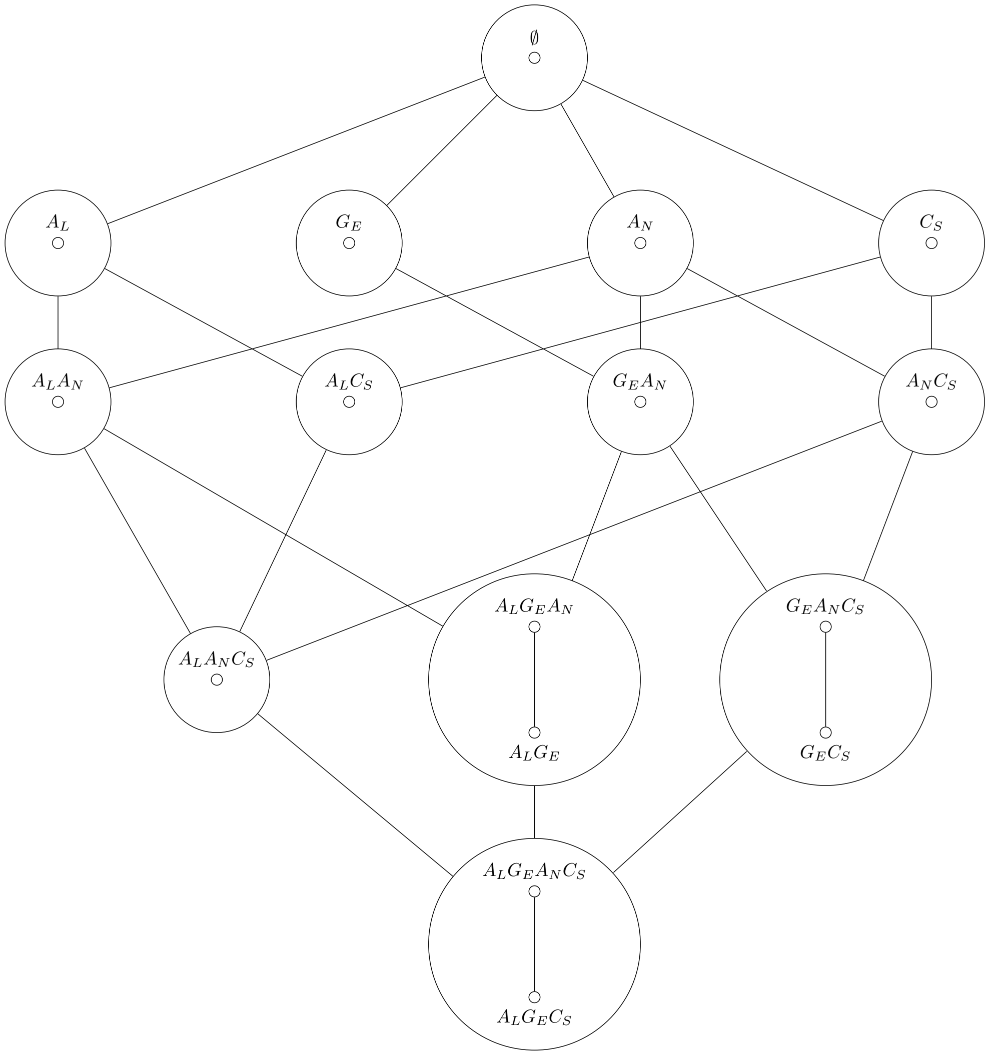



FIGURE 13. Micro granular representation of $\mathbb{M}^{\downarrow}(\hat{H})$.

$$[abcde]^{\updownarrow}_{\approx} = \{ac, bc, abc, acd, ace, bcd, bc, e, cde, abcde, abce, acde, bcde, abcde\}.$$

Therefore the up-maximum partitioners of $\hat{H}$ are

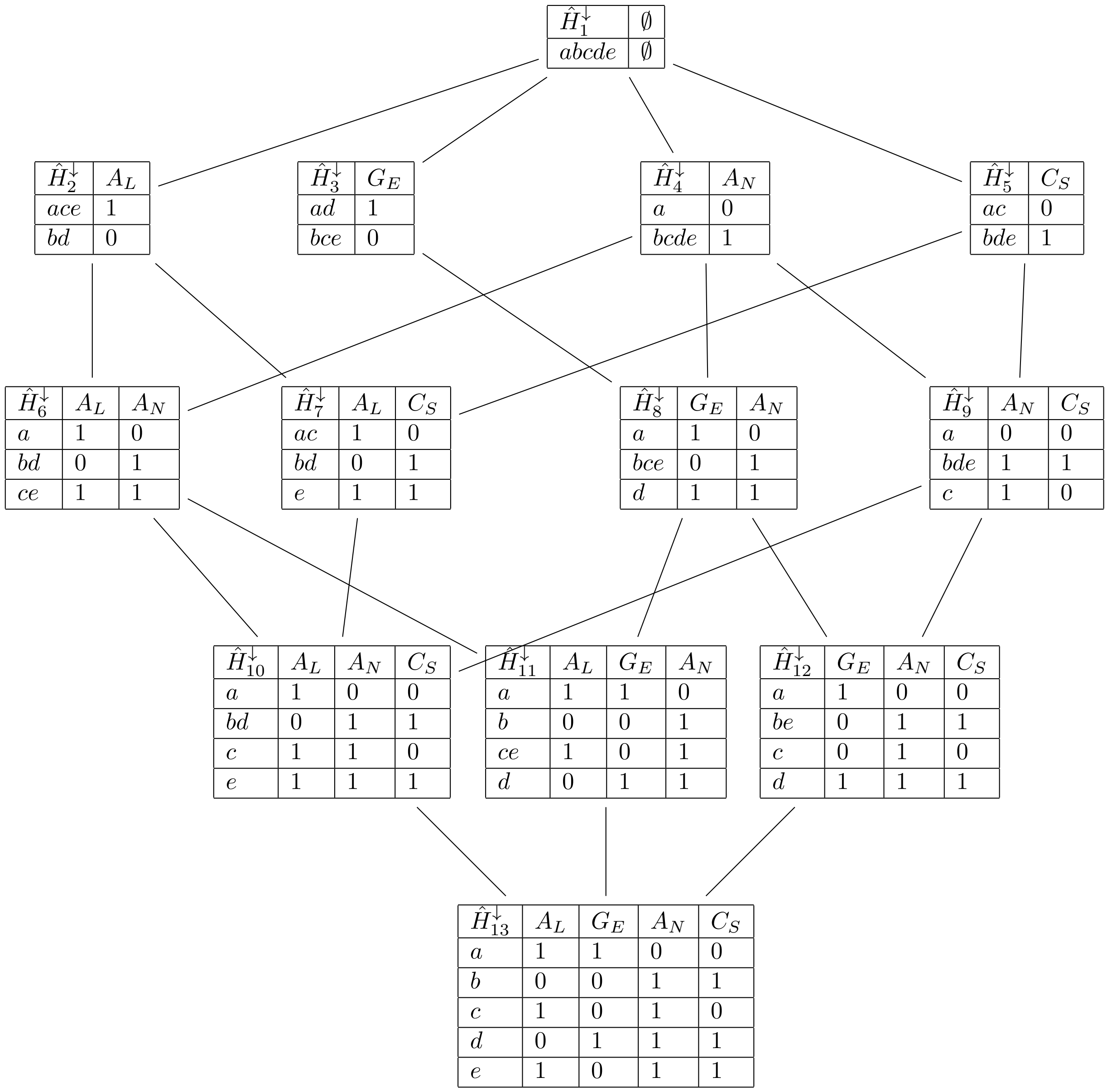
$$\emptyset, c, d, e, ab, cd, ce, abde, abcde,$$

whereas the up-minimal partitioners of $\hat{H}$ are the following:

$$\emptyset, a, b, c, d, e, cd, ce, ad, ae, bd, be, de, ac, bc, cde.$$

We represent the up-granular partition lattice $\mathbb{G}^{\uparrow}(\hat{H})$ in Figure 15.
In Figure 16 we give the micro-granular representation of the up-granular partition lattice.
In Figure 17 we represent the up-indiscernibility sub-table lattice.
We now interpret the role assumed by the maps $\Gamma^{\uparrow}$ and $\Gamma^{\downarrow}$. Let us fix the subset of students $Z = \{b, d\}$. Then, by (28), it is immediate to see that $\Gamma^{\uparrow}(Z) = \{A_L, A_N, C_S\}$. By referring to Figure 9 of Example 2.3, we note that $\Gamma^{\uparrow}(Z)$ consists of all exams whose result had been the same for both Bill and Dana. Hence, in general, $\Gamma^{\uparrow}(Z)$ consists of the biggest exams subset in which the students of $Z$ achieved the same result.
On the other hand, let us fix a subject subset $A = \{A_L, A_N, C_S\}$. Then, by (29), it is immediate to see that $\Gamma^{\downarrow}(A) = \{e\}$. In fact, as we can see in Figure 10 of Example 2.3, Eve is the only student who passed all the three exams of the set $A$. Accordingly, we deduce that $\Gamma^{\downarrow}(A)$ is the biggest student set that achieved the same result in all the exams of the set $A$.
We now compute and interpret $\gamma^{\uparrow\downarrow}(Z)$ and $\gamma^{\downarrow\uparrow}(A)$. Let now $Z = \{c, d\}$. It is immediate to verify that

$$\gamma^{\uparrow\downarrow}(Z) = \{b, c, d, e\}.$$



| $\hat{H}_1^{\downarrow}$ | $\emptyset$ |
|---|---|
| $abcde$ | $\emptyset$ |

| $\hat{H}_2^{\downarrow}$ | $A_L$ |
|---|---|
| $ace$ | 1 |
| $bd$ | 0 |

| $\hat{H}_3^{\downarrow}$ | $G_E$ |
|---|---|
| $ad$ | 1 |
| $bce$ | 0 |

| $\hat{H}_4^{\downarrow}$ | $A_N$ |
|---|---|
| $a$ | 0 |
| $bcde$ | 1 |

| $\hat{H}_5^{\downarrow}$ | $C_S$ |
|---|---|
| $ac$ | 0 |
| $bde$ | 1 |

| $\hat{H}_6^{\downarrow}$ | $A_L$ | $A_N$ |
|---|---|---|
| $a$ | 1 | 0 |
| $bd$ | 0 | 1 |
| $ce$ | 1 | 1 |

| $\hat{H}_7^{\downarrow}$ | $A_L$ | $C_S$ |
|---|---|---|
| $ac$ | 1 | 0 |
| $bd$ | 0 | 1 |
| $e$ | 1 | 1 |

| $\hat{H}_8^{\downarrow}$ | $G_E$ | $A_N$ |
|---|---|---|
| $a$ | 1 | 0 |
| $bce$ | 0 | 1 |
| $d$ | 1 | 1 |

| $\hat{H}_9^{\downarrow}$ | $A_N$ | $C_S$ |
|---|---|---|
| $a$ | 0 | 0 |
| $bde$ | 1 | 1 |
| $c$ | 1 | 0 |

| $\hat{H}_{10}^{\downarrow}$ | $A_L$ | $A_N$ | $C_S$ |
|---|---|---|---|
| $a$ | 1 | 0 | 0 |
| $bd$ | 0 | 1 | 1 |
| $c$ | 1 | 1 | 0 |
| $e$ | 1 | 1 | 1 |

| $\hat{H}_{11}^{\downarrow}$ | $A_L$ | $G_E$ | $A_N$ |
|---|---|---|---|
| $a$ | 1 | 1 | 0 |
| $b$ | 0 | 0 | 1 |
| $ce$ | 1 | 0 | 1 |
| $d$ | 0 | 1 | 1 |

| $\hat{H}_{12}^{\downarrow}$ | $G_E$ | $A_N$ | $C_S$ |
|---|---|---|---|
| $a$ | 1 | 0 | 0 |
| $be$ | 0 | 1 | 1 |
| $c$ | 0 | 1 | 0 |
| $d$ | 1 | 1 | 1 |

| $\hat{H}_{13}^{\downarrow}$ | $A_L$ | $G_E$ | $A_N$ | $C_S$ |
|---|---|---|---|---|
| $a$ | 1 | 1 | 0 | 0 |
| $b$ | 0 | 0 | 1 | 1 |
| $c$ | 1 | 0 | 1 | 0 |
| $d$ | 0 | 1 | 1 | 1 |
| $e$ | 1 | 0 | 1 | 1 |

FIGURE 14. Hasse Diagram of the down-indiscernibility sub-table lattice.

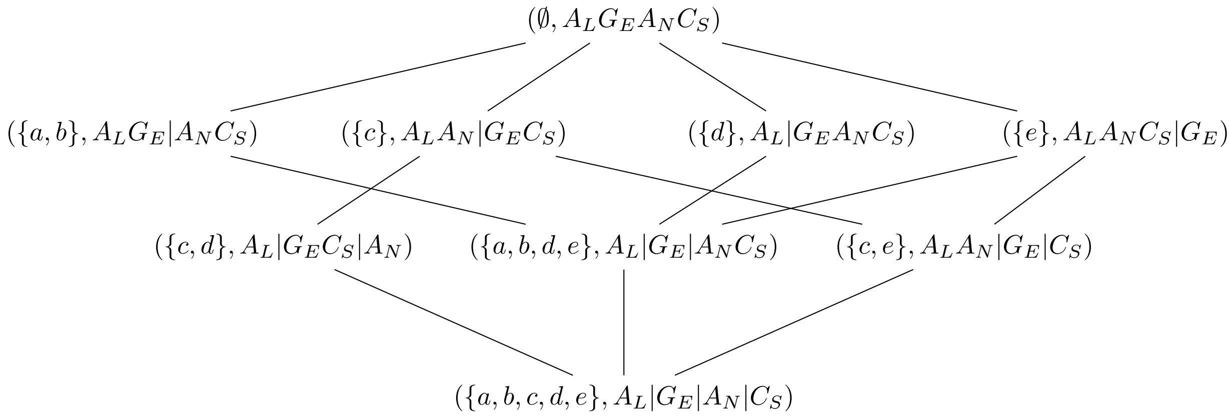



FIGURE 15. The Up-Granular Partition Lattice $\mathbb{G}^{\uparrow}(\hat{H})$

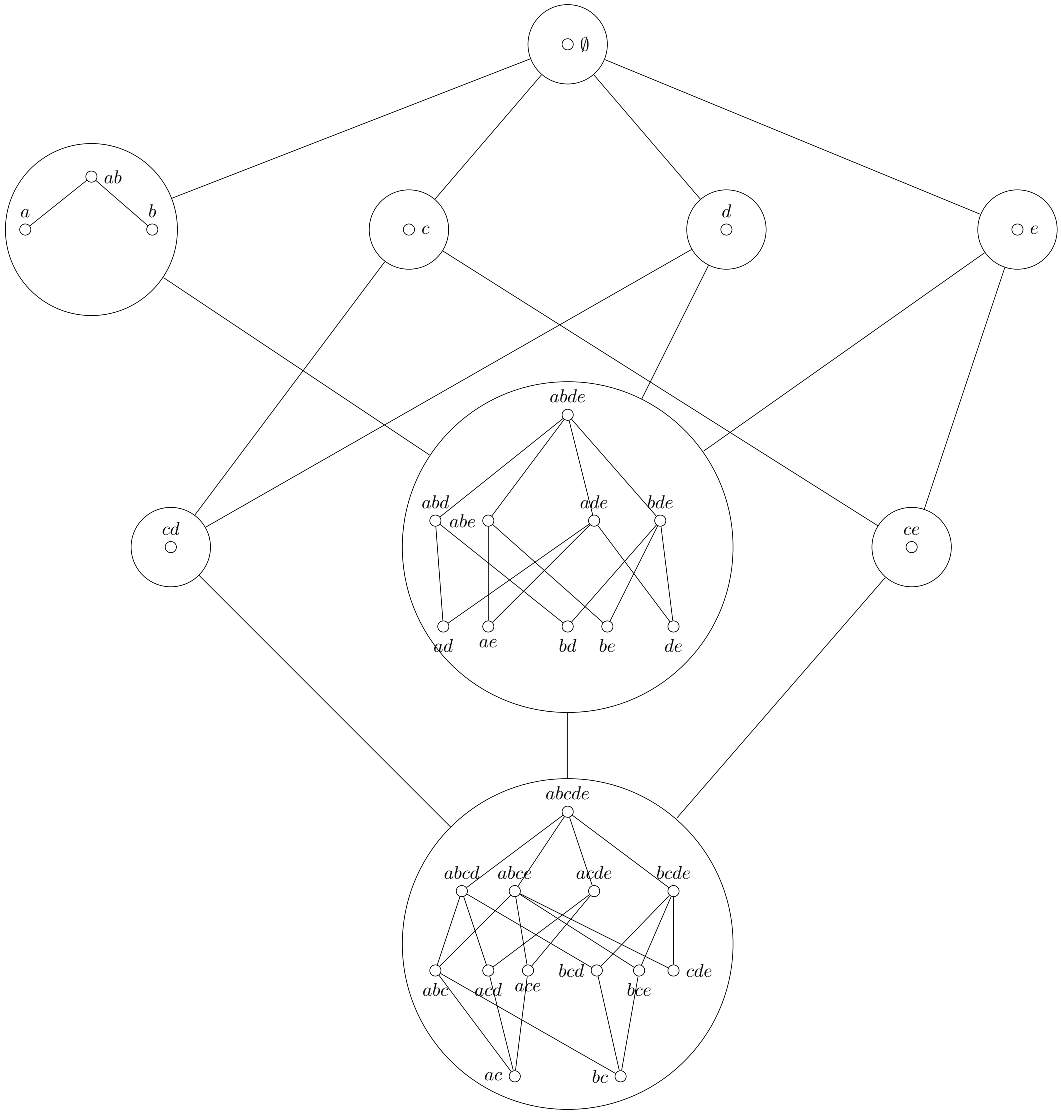


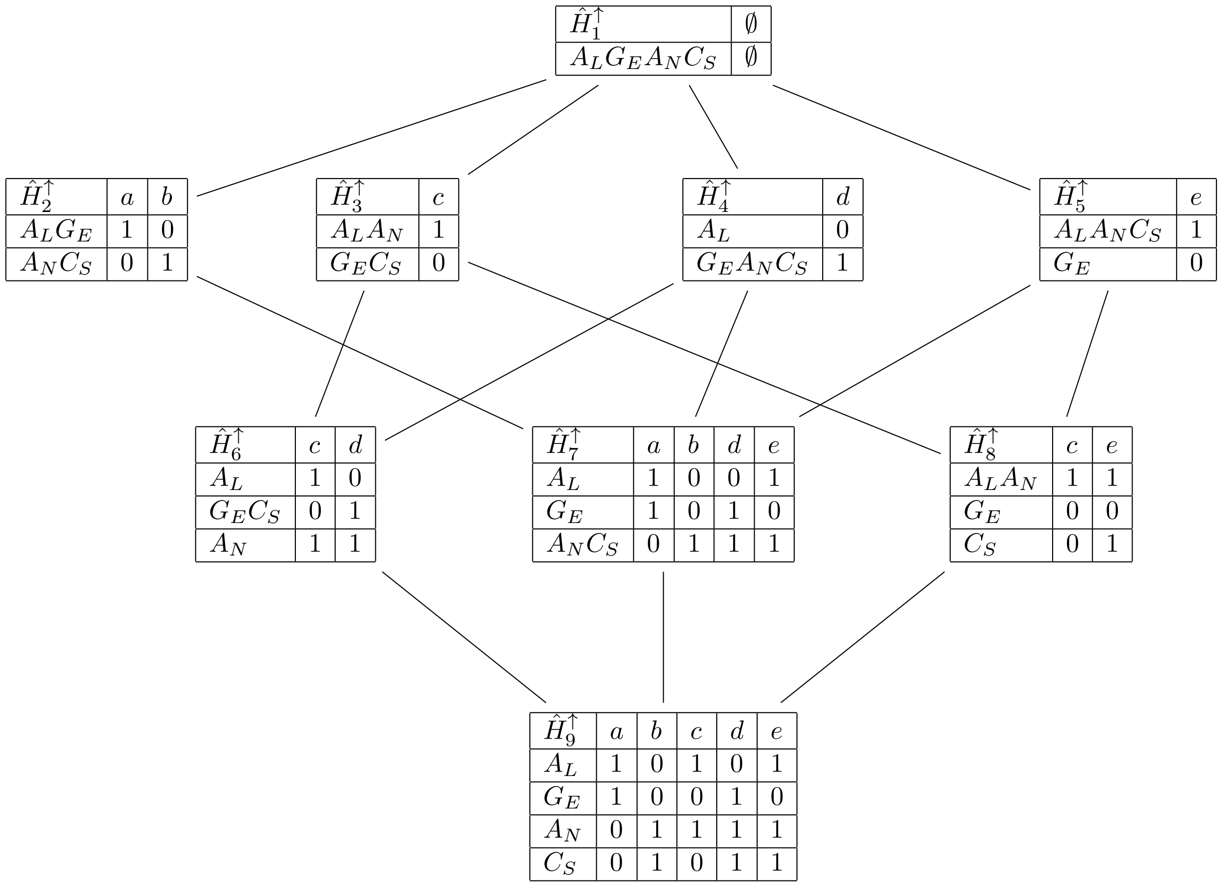

FIGURE 16. Micro granular representation of $\mathbb{M}^{\uparrow}(\hat{H})$.

We deduce that if we fix a subset of students $Z$, then $\gamma^{\uparrow\downarrow}(Z) = \{b, c, d, e\}$ is the biggest set of students that have passed or failed all the exams with respect to the students of $Z$ that have achieved the same result.
Finally, let us fix $A = \{A_L, A_N\}$. Since by Table 10 of Example 2.3 we have $\Gamma^{\downarrow}(A) = \{b, e\}$ and by Table 9 of the same example $\Gamma^{\uparrow}(\{b, e\}) = \{A_L, G_E, A_N\}$, it follows immediately that

$$\gamma^{\downarrow\uparrow}(A) = \{A_L, G_E, A_N\}.$$

In other terms, Bill and Eve achieved the same outcome in Algebra and Analysis, but these are not the only exams in which it is true; in fact, neither of them have passed Geometry, hence we must add Geometry to the list of exams in which they have the same result. Thus, we can see $\gamma^{\downarrow\uparrow}(A)$ as the full list of exams whose outcome has been the same for both Bill and Eve. In general, we have fixed a list of exams $A$ and found the biggest set of students achieving the same result in any of them. Afterwards, we have found the set $\gamma^{\downarrow\uparrow}(A)$ of all exams in which these students obtained the same outcome.

We firstly study the down-indiscernibility and the up-indiscernibility relations in the knowledge pairing system $\hat{H}$ with respect to a fixed attribute subset $A$. From the previous definitions, for a given hypergraph $H$, an attribute subset $A$ of $\hat{H}^{\downarrow}$ is a subset of $E(H)$, while an attribute subset $W$ of $\hat{H}^{\uparrow}$ is a subset of $V(H)$. In the next result we describe the indiscernibility classes of $\hat{H}^{\uparrow}$, for an arbitrary hypergraph $H$ (see [15]).



| $\hat{H}_1^{\uparrow}$ | $\emptyset$ |
|---|---|
| $A_L G_E A_N C_S$ | $\emptyset$ |

| $\hat{H}_2^{\uparrow}$ | $a$ | $b$ |
|---|---|---|
| $A_L G_E$ | 1 | 0 |
| $A_N C_S$ | 0 | 1 |

| $\hat{H}_3^{\uparrow}$ | $c$ |
|---|---|
| $A_L A_N$ | 1 |
| $G_E C_S$ | 0 |

| $\hat{H}_4^{\uparrow}$ | $d$ |
|---|---|
| $A_L$ | 0 |
| $G_E A_N C_S$ | 1 |

| $\hat{H}_5^{\uparrow}$ | $e$ |
|---|---|
| $A_L A_N C_S$ | 1 |
| $G_E$ | 0 |

| $\hat{H}_6^{\uparrow}$ | $c$ | $d$ |
|---|---|---|
| $A_L$ | 1 | 0 |
| $G_E C_S$ | 0 | 1 |
| $A_N$ | 1 | 1 |

| $\hat{H}_7^{\uparrow}$ | $a$ | $b$ | $d$ | $e$ |
|---|---|---|---|---|
| $A_L$ | 1 | 0 | 0 | 1 |
| $G_E$ | 1 | 0 | 1 | 0 |
| $A_N C_S$ | 0 | 1 | 1 | 1 |

| $\hat{H}_8^{\uparrow}$ | $c$ | $e$ |
|---|---|---|
| $A_L A_N$ | 1 | 1 |
| $G_E$ | 0 | 0 |
| $C_S$ | 0 | 1 |

| $\hat{H}_9^{\uparrow}$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $A_L$ | 1 | 0 | 1 | 0 | 1 |
| $G_E$ | 1 | 0 | 0 | 1 | 0 |
| $A_N$ | 0 | 1 | 1 | 1 | 1 |
| $C_S$ | 0 | 1 | 0 | 1 | 1 |

FIGURE 17. Hasse Diagram of the up-indiscernibility sub-table lattice.

**Theorem 6.3.** *Let $H$ be a hypergraph, $A \subseteq V(H)$ and $Y, Z \in E(H)$. Then, $Y \equiv_A^{\uparrow} Z \Longleftrightarrow Y \cap A = Z \cap A$.*

6.1. **Characterization of $\hat{H}^{\downarrow}$.** In order to better describe the indiscernibility relation in $\hat{H}^{\downarrow}$, for a generic hypergraph $H$, we introduce now the notion of separator.

**Definition 6.4.** *Let $X$ be a finite non-empty set, $S \subseteq X$ and $\mathcal{F}$ a family (possibly empty) of subsets of $X$. We say that $S$ is a* separator *of $\mathcal{F}$ if for all $Y \in \mathcal{F}$ it results that $S \cap Y = \emptyset$ or $S \subseteq Y$. We also say that $S$ is a* maximal separator *of $\mathcal{F}$ if $S$ is a separator of $\mathcal{F}$ and, for all $x \in S^c$, the subset $S \cup \{x\}$ is not a separator of $\mathcal{F}$. In particular, if $H = (V(H), E(H))$ is a given hypergraph and a subset $S \subseteq V(H)$ is a maximal separator of the whole hyperedge family $E(H)$, we say that $S$ is a* maximal separator *of the hypergraph $H$. We denote by $\mathcal{S}(H)$ the family of all maximal separators of $H$.*

**Remark 6.5.** *In Definition 6.4 the subset family $\mathcal{F}$ can also be empty. In this case each subset $S$ of $X$ is a separator of $\mathcal{F}$ because the condition $\forall\, Y \in \mathcal{F}, S \cap Y = \emptyset \vee S \subseteq Y$ is an empty condition, hence it is verified for all subsets $S$. In particular, if $\mathcal{F}$ is empty the unique maximal separator of $\mathcal{F}$ is the whole set $X$.*

We have then the following result.

**Theorem 6.6.** *Let $H$ be a hypergraph, $v, v' \in V(H)$ and $A \subseteq E(H)$. Then the following conditions are equivalent:*
*(i) $v \equiv_A^{\downarrow} v'$;*
*(ii) $\{v, v'\}$ is a separator of $A$.*

*Proof.* By definition of $\equiv_A^{\downarrow}$ in the information table $\hat{H}^{\downarrow}$ we have that

$$v \equiv_A^{\downarrow} v' \iff F(v, Y) = F(v', Y) \;\; \forall\, Y \in A. \tag{37}$$

Let now $Y \in A$. In (37) we must distinguish two possible cases. In the first, the equality $F(v, Y) = F(v', Y) = 1$ is equivalent to $\{v, v'\} \subseteq Y$. In the other case, the equality $F(v, Y) = F(v', Y) = 0$ is equivalent to $\{v, v'\} \cap Y = \emptyset$. Therefore, the right part of (37) is equivalent to say that $\{v, v'\}$ is a separator of $A$. □

Since a simple graph $G$ is a special case of hypergraph, we deduce the following result.

**Corollary 6.7.** *Let $G$ be a simple graph, $v \neq v' \in V(G)$ and $A \subseteq E(G)$. Then the following conditions are equivalent:*
*(i) $v \equiv_A^{\downarrow} v'$;*
*(ii) If $e \in A$ is incident with $v$ or $v'$, then $e = vv'$.*

In the next result we completely characterize the indiscernibility classes in terms of maximal separators.

**Theorem 6.8.** *Let $H$ be a hypergraph, $A \subseteq E(H)$ and $S$ a non-empty subset of $V(H)$. Then the following conditions are equivalent:*
*(i) $S$ is a block of the indiscernibility partition $\pi_{\hat{H}}^{\downarrow}(A)$.*
*(ii) $S$ is a maximal separator of $A$.*

*Proof.* If $A$ is an empty family of hyperedges then $\pi_{\hat{H}}^{\downarrow}(A) = V(H)$, therefore the unique block of $\pi_{\hat{H}}^{\downarrow}(A)$ is $V(H)$. On the other hand, by Remark 6.5, if $A$ is empty then the unique maximal separator of $A$ is again $V(H)$. Hence the thesis is true when $A$ is the empty family. We can assume therefore that $A$ is a not empty family of hyperedges.
$(i) \Longrightarrow (ii)$ : Let $S$ be a block of the indiscernibility partition $\pi_{\hat{H}}^{\downarrow}(A)$. Let $Y \in A$. In order to prove that $S$ is a separator of $A$ we must show that $S \cap Y = \emptyset$ or $S \subseteq Y$. We assume that $S \cap Y \neq \emptyset$. Let $v \in S \cap Y$. If $S$ is a singleton it is obviously a separator of $A$, otherwise we take an arbitrary element $v' \in S \setminus \{v\}$. Since $S$ is a block of $\pi_{\hat{H}}^{\downarrow}(A)$ we have that $v \equiv_A^{\downarrow} v'$, so $\{v, v'\}$ is a separator of $A$, by Theorem 6.6. Hence $\{v, v'\} \cap Y = \emptyset$ or $\{v, v'\} \subseteq Y$, by definition of separator of $A$ and due to $Y \in A$. Therefore, since $v \in Y$, we get $\{v, v'\} \subseteq Y$; thus $v' \in Y$. Since $v'$ is arbitrary, it follows that $S \subseteq Y$, hence $S$ is a separator of $A$. We prove now that $S$ is also maximal. Let $v' \in V(H) \setminus S$ fixed. We must show that $S \cup \{v'\}$ is not a separator of $A$. Let $v \in S$ arbitrary. Since $S$ is a block of $\pi_{\hat{H}}^{\downarrow}(A)$ it results that $v \not\equiv_A^{\downarrow} v'$ so, by Theorem 6.6, $\{v, v'\}$ is not a separator for $A$. Therefore there exists a hyperedge $Y \in A$ such that $\{v, v'\} \cap Y \neq \emptyset$ and also $\{v, v'\} \nsubseteq Y$. Now, if $v \in Y$ and $v' \notin Y$, it follows that $S \cup \{v'\} \cap Y \neq \emptyset$ and $S \cup \{v'\} \nsubseteq Y$. On the other hand, if $v \notin Y$ and $v' \in Y$, we deduce $S \cup \{v'\} \nsubseteq Y$ and $S \cup \{v'\} \cap Y \neq \emptyset$. Hence in both cases it results that $S \cup \{v'\}$ is not a separator of $A$.
$(ii) \Longrightarrow (i)$ : Let $S$ be a maximal separator of $A$. We choose at first $v, v' \in S$. Since $S$ is a separator of $A$, we have that $S \cap Y = \emptyset$ or $S \subseteq Y$ for all $Y \in A$. Hence, we also have $\{v, v'\} \cap Y = \emptyset$ or $\{v, v'\} \subseteq Y$ for all $Y \in A$, i.e. $\{v, v'\}$ is a separator of $A$. By Theorem 6.6 we deduce then $v \equiv_A^{\downarrow} v'$. We take now $v \in S$, $v' \notin S$ and we prove that $v \not\equiv_A^{\downarrow} v'$. Let us assume (by absurd) that $v \equiv_A^{\downarrow} v'$. By Theorem 6.6 it follows that $\{v, v'\}$ is a separator of $A$. In order to obtain an absurd we will prove that $S \cup \{v'\}$ is a separator of $A$, that is contrary to our assumption that $S$ is a maximal separator of $A$. Let $Y \in A$ arbitrary. Since $\{v, v'\}$ is a separator of $A$, we have that $\{v, v'\} \cap Y = \emptyset$ or $\{v, v'\} \subseteq Y$. If $\{v, v'\} \cap Y = \emptyset$, then $S \nsubseteq Y$, otherwise $v \in S \subseteq Y$, that is a contradiction. Since $S$ is a separator of $A$ it results then $S \cap Y = \emptyset$, so $S \cup \{v'\} \cap Y = \emptyset$ because $v' \notin Y$. On the other hand, if $\{v, v'\} \subseteq Y$, then $v \in S \cap Y$, therefore $S \cap Y \neq \emptyset$. This implies that $S \subseteq Y$ because $S$ is a separator of $A$. Hence $S \cup \{v'\} \subseteq Y$. This shows that $S \cup \{v'\}$ is a separator of $A$. We have proved therefore that if $v, v' \in S$ then $v \equiv_A^{\downarrow} v'$, while if $v \in S$ and $v' \notin S$ then $v \not\equiv_A^{\downarrow} v'$. This obviously is equivalent to say that $S$ is a block of the indiscernibility partition $\pi_{\hat{H}}^{\downarrow}(A)$. □

**Corollary 6.9.** *If $H = (V(H), E(H))$ is a given hypergraph, then $\mathcal{S}(H)$ is a set partition of $V(H)$ that coincides with $\pi_{\hat{H}}^{\downarrow}(E(H))$.*

*Proof.* By definition, the maximal separators of $H$ are the maximal separators of the hyperedge family $E(H)$, hence we obtain the thesis if we take $A = E(H)$ in Theorem 6.8. □

**Example 6.10.** Let $H = (\hat{9}, E(H))$, where $E(H) = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7\}$ and $Y_1 = \emptyset$, $Y_2 = \{1, 2\}$, $Y_3 = \{3, 4\}$, $Y_4 = \{1, 2, 3\}$, $Y_5 = \{4, 8, 9\}$, $Y_6 = \{2, 4, 6, 7, 9\}$, $Y_7 = \{2, 8\}$. Let $A = \{Y_2, Y_3, Y_4, Y_5\}$; then we have that the maximal separators are $\{1, 2\}$, $\{3\}$, $\{4\}$, $\{5, 6, 7\}$ and $\{8, 9\}$. Hence, we have $\pi_{\hat{H}}^{\downarrow}(A) = 12|3|4|567|89$.

**Example 6.11.** We will find the $A-$indiscernibility partition of the incidence information table $INC(\binom{\hat{6}}{3})$. Let $A = \{Y_1, Y_2, Y_3\}$, where $Y_1 = \{1, 2, 3\}$, $Y_2 = \{1, 2, 4\}$ and $Y_3 = \{2, 3, 4\}$. Then we have that all the singletons and $\{5, 6\}$ are separators of $A$. In particular, the maximal separators are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5, 6\}$; therefore $\pi_{\hat{H}}^{\downarrow}(A) = 1|2|3|4|56$.

In next two results, we give the form of two particular maximal separators of any fixed hyperedge subset $A$.

**Corollary 6.12.** *Let $H$ be a hypergraph and $A \subseteq E(H)$ and $T = \bigcup\{Y : Y \in A\}$. If $T^c \neq \emptyset$, then $T^c$ is a block of $\pi^{\downarrow}_{\hat{H}}(A)$.*

*Proof.* Let $Y \in A$. Since $Y \subseteq T$, we have $T^c \cap Y = \emptyset$, therefore $T^c$ is a separator of $A$. Moreover, if $v \notin T^c$ then $v \in T$, therefore there exists a hyperedge $Y \in A$ such that $v \in Y$. Then $T^c \cup \{v\} \cap Y \neq \emptyset$ because $v \in Y$, and $T^c \cup \{v\} \nsubseteq Y$, otherwise $T^c \subset T^c \cup \{v\} \subseteq Y \subseteq T$ (a contradiction since $T^c \neq \emptyset$). Hence $T^c \cup \{v\}$ is not a separator of $A$, and by the arbitrarity of $v$ we conclude that $T^c$ is a maximal separator of $A$. By Theorem 6.8 we deduce then that $T^c$ is a block of $\pi^{\downarrow}_{\hat{H}}(A)$. □

**Corollary 6.13.** *Let $H$ be a hypergraph, $A \subseteq E(H)$ and $S = \bigcap\{Y : Y \in A\}$. If $S \neq \emptyset$, then $S$ is a block of $\pi^{\downarrow}_{\hat{H}}(A)$.*

*Proof.* By definition of $S$, we have that $S \subseteq Y$ for all $Y \in A$, therefore $S$ is a separator of $A$. Now, if $v \notin S$, again by definition of $S$ there exists a hyperedge $Y_0 \in A$ such that $v \notin Y_0$. Hence $S \cup \{v\} \nsubseteq Y_0$ and also $S \cup \{v\} \cap Y_0 = S \neq \emptyset$. Thus $S$ is a maximal separator of $A$, and the thesis follows by Theorem 6.8. □

We now take into account the case where $G$ is a simple graph.

**Proposition 6.14.** *Let $G$ be a simple graph and $A \subseteq E(G)$. Let $T = \bigcup\{e : e \in A\}$, i.e., $T$ is the subset of all vertices of $V(G)$ belonging to the edges in $A$. Then:*
*(i) if $e_1 = v_1v_2$ and $e_2 = v_2v_3$ are two distinct edges in $A$ having the common vertex $v_2$, then each singleton $\{v_i\}$ (i=1,2,3) is a block of $\pi^{\downarrow}_{\hat{H}}(A)$.*
*(ii) If $e = vv' \in A$ and $e \cap e' = \emptyset$ for all $e' \in A \setminus \{e\}$, then $\{v, v'\}$ is a block of $\pi^{\downarrow}_{\hat{H}}(A)$.*
*(iii) If $B$ is a block of $\pi^{\downarrow}_{\hat{G}}(A)$ such that $B \subseteq T$, then $B$ is a single vertex or $B$ is an edge of $A$.*
*(iv) If $T^c$ is not empty then $T^c$ is a block of $\pi^{\downarrow}_{\hat{G}}(A)$.*

*Proof.* ($i$) By Theorem 6.8 it is sufficient to show that each $\{v_i\}$, for $i = 1, 2, 3$, is a maximal separator of $A$. Since each $\{v_i\}$ is a singleton it is a separator of $A$. Let us consider now $\{v_1\}$ and another vertex $v \neq v_1$. If $v \in \{v_2, v_3\}$, then $\{v_1, v\} \nsubseteq e_2$ and $\{v_1, v\} \cap e_2 \neq \emptyset$. If $v \notin \{v_2, v_3\}$, then $\{v_1, v\} \cap e_1 \neq \emptyset$ and $\{v_1, v\} \nsubseteq e_1$. It follows that $\{v_1\}$ is a maximal separator of $A$. The proof for $\{v_2\}$ and $\{v_3\}$ is similar.
($ii$) By hypothesis it follows that the subset $\{v, v'\}$ is a separator of $A$. Moreover, let $v'' \notin \{v, v'\}$ and $S = \{v, v', v''\}$. Then $S \cap e = \{v, v'\} \neq \emptyset$ and $S \nsubseteq e$ because $|S| = 3$, therefore $S$ is not a separator of $A$. This proves that $\{v, v'\}$ is a maximal separator of $A$, hence by Theorem 6.8 we obtain the thesis.
($iii$) We prove at first that $|B| \leq 2$. We assume by absurd that $|B| \geq 3$. By Theorem 6.8 we know that $B$ is a maximal separator of $A$, thus, if $e \in A$ then $B \cap e = \emptyset$ or $B \subseteq e$, that is impossible because $|B| \geq 3$. Hence we have that $B \cap e = \emptyset$ for all $e \in A$. Nevertheless, this contradicts the hypothesis that $B$ is non-empty and $B \subseteq T$. Therefore $|B| \leq 2$, and this implies that $B$ is a singleton or $|B| = 2$. Let us suppose that $B = \{v, v'\}$, with $v \neq v'$. We assume (by absurd) that $B$ is not an edge of $A$. We have then two possible cases: there exist $w, w' \in T$ such that $e := vw$ and $e' := v'w'$ are two disjoint edges in $A$, or there exists $u \in T$ such that $e := vu$ and $e' := v'u$ are two distinct edges in $A$ having $u$ as common vertex. Then, in both previous cases, it is immediate to note that $B \cap e \neq \emptyset$ and $B \nsubseteq e$. This shows that $B$ is not a separator of $A$, therefore, by Theorem 6.8, $B$ is not a block of $\pi^{\downarrow}_{\hat{G}}(A)$, a contradiction.
($iv$) It is a direct consequence of Corollary 6.12. □

**Example 6.15.** Let $G = K_8$ be the complete graph on 8 vertices and let $A = \{v_1v_2, v_2v_3, v_5v_6\}$. With the notations introduced in Proposition 6.14, we have that $T = \{v_1, v_2, v_3, v_5, v_6\}$, therefore by the same proposition we obtain $\pi^{\downarrow}_{\hat{G}}(A) = v_1|v_2|v_3|v_5v_6|v_4v_7v_8$.

6.2. **Discernibility Matrix in the Hypergraph Context.** The discernibility matrix of an information table $\mathcal{I}$ was introduced in [60] and it is an essential tool to study an information table $\mathcal{I}$ [75], in particular, to discover redundancies in the data. In this section we study the form of the discernibility matrix of $\hat{H}^{\uparrow}$.

**Proposition 6.16.** *Let $H = (V(H), E(H))$ be a hypergraph, where $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{Y_1, \ldots, Y_m\}$. Then:*
*(i) $\Delta^{\downarrow}_{\hat{H}}(v_i, v_j) = H(v_i) \triangle H(v_j)$;*
*(ii) $\Delta^{\uparrow}_{\hat{H}}(Y_i, Y_j) = Y_i \triangle Y_j$.*

*Proof.* ($i$) : By (35) we have that $\Delta^{\downarrow}_{\hat{H}}(v_i, v_j) = \{Y \in E(H) : F(v_i, Y) \neq F(v_j, Y)\}$. By definition of $F_H$, either $v_i \in Y$ and $v_j \notin Y$ or $v_i \notin Y$ and $v_j \in Y$. This means that $Y \in H(v_i) \triangle H(v_j)$, i.e.

| | $\mathbf{E_1}$ | $\mathbf{E_2}$ | $\mathbf{E_3}$ | $\mathbf{E_4}$ | $\mathbf{E_5}$ | $\mathbf{E_6}$ | $\mathbf{E_7}$ | $\mathbf{E_8}$ | $\mathbf{E_9}$ | $\mathbf{E_{10}}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| **2** | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| **3** | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| **4** | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| **5** | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |

TABLE 2. Knowledge table $T[\binom{\hat{5}}{3}]$.

$\Delta^{\downarrow}_{\hat{H}}(v_i, v_j) = H(v_i) \,\triangle\, H(v_j)$.

$(ii)$ : By (36) we have that $\Delta^{\uparrow}_{\hat{H}}(Y_i, Y_j) = \{v \in V(H) : F(v, Y_i) \neq F(v, Y_j)\}$. By definition of $F$, either $v \in Y_i$ and $v \notin Y_j$ or $v \notin Y_i$ and $v \in Y_j$, and this proves that $\Delta^{\uparrow}_{\hat{H}}(Y_i, Y_j)$ is exactly the symmetric difference of $Y_i$ and $Y_j$. □

In [15], the classical complete $(n,k)$-uniform hypergraph $\binom{\hat{n}}{k}$ has been extensively studied from the up-indiscernibility perspective. In our paper, we provide a general form of the up-discernibility hypergraph.

**Example 6.17.** Let $\hat{n} = \{1, \dots, 5\}$, $k = 3$ and consider $(\binom{\hat{5}}{3})^{\uparrow}$. The hypergraph $\binom{\hat{5}}{3}$ has vertices $V(\binom{\hat{5}}{3}) = \{1,2,3,4,5\}$ and the following hyperedges: $E_1 = \{1,2,3\}, E_2 = \{1,2,4\}, E_3 = \{1,2,5\}, E_4 = \{1,3,4\}, E_5 = \{1,3,5\}, E_6 = \{1,4,5\}$, $E_7 = \{2,3,4\}, E_8 = \{2,3,5\}, E_9 = \{2,4,5\}, E_{10} = \{3,4,5\}$. In Table 2 below, we represent the incidence matrix of $\binom{\hat{5}}{3}$.

By Proposition 6.16, we can express the corresponding discernibility matrices $\Delta^{\downarrow}[\binom{\hat{5}}{3}]$ and $\Delta^{\uparrow}[\binom{\hat{5}}{3}]$ using the symmetric differences, as the following two tables.

| $\Delta^{\downarrow}[\binom{\hat{5}}{3}]$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | $\emptyset$ | $E_4, E_5, E_6$ $E_7, E_8, E_9$ | $E_2, E_3, E_6$ $E_7, E_8, E_{10}$ | $E_1, E_3, E_5, E_7, E_9, E_{10}$ | $E_1, E_2, E_4, E_8, E_9, E_{10}$ |
| 2 | * | $\emptyset$ | $E_2, E_3, E_4$ $E_5, E_9, E_{10}$ | $E_1, E_3, E_4, E_6, E_8, E_{10}$ | $E_1, E_2, E_5, E_6, E_7, E_{10}$ |
| 3 | * | * | $\emptyset$ | $E_1, E_2, E_5, E_6, E_8, E_9$ | $E_1, E_3, E_4, E_6, E_7, E_9$ |
| 4 | * | * | * | $\emptyset$ | $E_2, E_3, E_4, E_5, E_7, E_8$ |
| 5 | * | * | * | * | $\emptyset$ |

| $\Delta^{\uparrow}[\binom{\hat{5}}{3}]$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ | $E_8$ | $E_9$ | $E_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $E_1$ | $\emptyset$ | 3, 4 | 3, 5 | 2, 4 | 2, 5 | 2, 3, 4, 5 | 1, 4 | 1, 5 | 1, 3, 4, 5 | 1, 2, 4, 5 |
| $E_2$ | * | $\emptyset$ | 4, 5 | 2, 3 | 2, 3, 4, 5 | 2, 5 | 1, 3 | 1, 3, 4, 5 | 1, 5 | 1, 2, 3, 5 |
| $E_3$ | * | * | $\emptyset$ | 2, 3, 4, 5 | 2, 3 | 2, 4 | 1, 3, 4, 5 | 1, 3 | 1, 5 | 1, 2, 3, 5 |
| $E_4$ | * | * | * | $\emptyset$ | 4, 5 | 3, 5 | 1, 2 | 1, 2, 4, 5 | 1, 2, 3, 5 | 1, 5 |
| $E_5$ | * | * | * | * | $\emptyset$ | 3, 4 | 1, 2, 4, 5 | 1, 2 | 1, 2, 3, 4 | 1, 4 |
| $E_6$ | * | * | * | * | * | $\emptyset$ | 1, 2, 3, 5 | 1, 2, 3, 4 | 1, 2 | 1, 3 |
| $E_7$ | * | * | * | * | * | * | $\emptyset$ | 4, 5 | 3, 5 | 2, 5 |
| $E_8$ | * | * | * | * | * | * | * | $\emptyset$ | 3, 4 | 2, 4 |
| $E_9$ | * | * | * | * | * | * | * | * | $\emptyset$ | 2, 3 |
| $E_{10}$ | * | * | * | * | * | * | * | * | * | $\emptyset$ |

In the next proposition we provide the general form of the up-discernibility hypergraph of $\hat{H} = \binom{\hat{n}}{k}$.

**Proposition 6.18.** *Let $\binom{\hat{n}}{k}$ where $k < n$. Then:*
*(i) If $k \leq \frac{n}{2}$ we have*

$$DISC^{\uparrow}(\binom{\hat{n}}{k}) = \binom{\hat{n}}{2k} \cup \binom{\hat{n}}{2(k-1)} \cup \cdots \cup \binom{\hat{n}}{2}$$

*(ii) If $k > \frac{n}{2}$ we have*

$$DISC^{\uparrow}(\binom{\hat{n}}{k}) = \binom{\hat{n}}{2(n-k)} \cup \binom{\hat{n}}{2(n-k-1)} \cup \cdots \cup \binom{\hat{n}}{2}$$

*Proof.* $(i)$ : Let $Y \in \binom{\hat{n}}{2k} \cup \binom{\hat{n}}{2(k-1)} \cup \cdots \cup \binom{\hat{n}}{2}$, so that $|Y| = 2(k-l)$, for some integer $l$ such that $0 \leq l \leq k-1$. Let $Y = \{y_1, \ldots, y_{k-l}, z_1 \ldots, z_{k-l}\}$. In order to show that $Y \in DISC^{\uparrow}(\binom{\hat{n}}{k})$, by Proposition 6.16 we must find two hyperedges $Y_i, Y_j \in \binom{\hat{n}}{k}$ such that $Y_i \Delta Y_j = Y$. Since $k \leq \frac{n}{2}$, we have that $|\hat{n} \setminus Y| = n - 2(k-l) \geq 2l$, therefore we can choose at least $l$ distinct elements $t_1, \ldots, t_l \in \hat{n} \setminus Y$. We take then $Y_i := \{y_1, \ldots, y_{k-l}, t_1, \ldots, t_l\}$ and $Y_j := \{z_1, \ldots, z_{k-l}, t_1, \ldots, t_l\}$. It is clear then that $Y_i, Y_j \in \binom{\hat{n}}{k}$ and $Y_i \Delta Y_j = Y$. This proves that $\binom{\hat{n}}{2(n-k)} \cup \binom{\hat{n}}{2(n-k-1)} \cup \cdots \cup \binom{\hat{n}}{2} \subseteq DISC^{\uparrow}(\binom{\hat{n}}{k})$. On the other hand, if we take $Y \in DISC^{\uparrow}(\binom{\hat{n}}{k})$, by Proposition 6.16 we have $Y = Y_i \Delta Y_j$, for some $Y_i, Y_j \in \binom{n}{k}$. It is immediate then to note that $|Y| = 2(k-l)$, for some integer $l$ such that $0 \leq l \leq k-1$. This proves that $DISC^{\uparrow}(\binom{\hat{n}}{k}) \subseteq \binom{\hat{n}}{2(n-k)} \cup \binom{\hat{n}}{2(n-k-1)} \cup \cdots \cup \binom{\hat{n}}{2}$.
$(ii)$ : Let $Y \in \binom{\hat{n}}{2(n-k)} \cup \binom{\hat{n}}{2(n-k-1)} \cup \cdots \cup \binom{\hat{n}}{2}$. Thus there exists some integer $l$ such that $0 \leq l \leq n-k-1$ and that $|Y| = 2(n-k-l)$; therefore $Y = \{y_1, \ldots, y_{n-k-l}, z_1, \ldots, z_{n-k-l}\}$. Since $k > \frac{n}{2}$, we have $|\hat{n} \setminus Y| = n - 2(n-k-l) > 2l$ and $l < 2k+l-n < 2l$. Hence, there exist at least $2k+l-n$ elements $t_1, \ldots, t_{2k+l-n} \in \hat{n} \setminus Y$. If we set $Y_i = \{y_1, \ldots, y_{n-k-l}, t_1, \ldots, t_{2k+l-n}\}$ and $Y_j = \{z_1, \ldots, z_{n-k-l}, t_1, \ldots, t_{2k+l-n}\}\}$, we have $|Y_i| = |Y_j| = k$ and $Y = Y_i \triangle Y_j$. Thus $Y \in DISC^{\uparrow}(\binom{\hat{n}}{k})$. Vice versa, let $Y \in DISC^{\uparrow}(\binom{\hat{n}}{k})$. By Proposition 6.16 we have $Y = Y_i \Delta Y_j$, for some $Y_i, Y_j \in \binom{n}{k}$. It is immediate then to note that $|Y| = 2(k-l)$, for some integer $l$ such that $0 \leq l \leq k-1$. But we have $k - l = n - k - i$ for some integer $i$. We have to show that $0 \leq i \leq n-k-1$. By the previous equality, we obtain $l = 2k - n + i$. Substituting in the relation $0 \leq l \leq k-1$, we have $0 \leq 2k - n + i \leq k-1$, from which we deduce $0 \leq i \leq n-k-1$. This means that $Y \in \binom{\hat{n}}{2(n-k)} \cup \binom{\hat{n}}{2(n-k-1)} \cup \cdots \cup \binom{\hat{n}}{2}$. □

## 7. Knowledge pairing systems in digraph and graph context

In this section we will introduce two new knowledge pairing systems induced respectively by adjacency in digraphs and graphs context.

7.1. **The digraph case.** We start our discussion by introducing the knowledge pairing system associated to a digraph $\vec{D}$.

**Definition 7.1.** *Let $\vec{D} = (V(\vec{D}), Arc(\vec{D}))$ be a digraph. We associate to $\vec{D}$ the following knowledge pairing system:*

$$\hat{D} := \langle V(\vec{D}), V(\vec{D}), \{0, 1\}, F \rangle,$$

*where $F(u, v) := 1$ if $u \to v$ and $F(u, v) := 0$ otherwise.*

For a digraph $\vec{D}$ the description of the indiscernibility relation in $\hat{D}$ has been given in [27]. We recall this characterization (clearly, the same results can be rewritten for the equivalence relation $\equiv^{\uparrow}$ with the appropriate changes):

**Theorem 7.2.** *Let $A \subseteq V(D)$ and $v, v' \in V(D)$. The following conditions are equivalent:*
*(i) $v \equiv_A^{\downarrow} v'$.*
*(ii) For all $z \in A$ it results that $v \to z$ if and only if $v' \to z$.*
*(iii) $N_{\vec{D}}^{+}(v) \cap A = N_{\vec{D}}^{+}(v') \cap A$.*

*Proof.* See [27] □

As a direct consequence of previous theorem we obtain the following results (the same results can be rewritten for the equivalence relation $\equiv^{\uparrow}$ with the appropriate changes):

**Corollary 7.3.** *(i) If $v \in V(\vec{D})$ and $A \subseteq V(\vec{D})$, then $[v]_A^{+} = \{v' : N_{\vec{D}}^{+}(v) \cap A = N_{\vec{D}}^{+}(v') \cap A\}$.*
*(ii) If $v \equiv_A^{\downarrow} v'$ and $v' \in A$, then $v \not\to v'$.*

*Proof.* See [27] □

Thus, if $v \in V(\vec{D})$, it results by Theorem 7.7 that $v' \in [v]_A^{\downarrow}$ (resp. $v' \in [v]_U^{\uparrow}$) if and only if $v$ and $v'$ *"out-see"* (resp. *"in-see"*) all the vertices $z \in A$ in the same way, that is, $v \to z$ if and only if $v' \to z$ (resp. $z \to v$ if and only if $z \to v'$) for all $z \in A$. Hence any down-indiscernibility block with respect to a vertex subset $A$ consists of all vertices having all the same out-adjacency relation with respect to all vertices in $A$ while the up-indiscernibility blocks consist of all vertices with the same in-adjacency relation with respect to all vertices in $A$.

**Example 7.4.** Let us consider the digraph $\vec{D}$ represented in Figure 7 of Example 2.2.
We have the following down-indistinguishability classes of the knowledge pairing system $\hat{D}$:

$$[\emptyset]^{\downarrow}_{\approx} = \{\emptyset\},\ [a]^{\downarrow}_{\approx} = \{a\},\ [b]^{\downarrow}_{\approx} = \{b\},\ [c]^{\downarrow}_{\approx} = \{c\},\ [d]^{\downarrow}_{\approx} = \{d\},\ [e]^{\downarrow}_{\approx} = \{e\},\ [f]^{\downarrow}_{\approx} = \{f\},$$
$$[ac]^{\downarrow}_{\approx} = \{ac\},\ [ad]^{\downarrow}_{\approx} = \{ad\},\ [ae]^{\downarrow}_{\approx} = \{ae\},\ [af]^{\downarrow}_{\approx} = \{af\},$$
$$[bc]^{\downarrow}_{\approx} = \{bc\},\ [bd]^{\downarrow}_{\approx} = \{bd\},\ [be]^{\downarrow}_{\approx} = \{be\},\ [bf]^{\downarrow}_{\approx} = \{bf\},$$
$$[de]^{\downarrow}_{\approx} = \{de\},\ [ef]^{\downarrow}_{\approx} = \{ef\},\ [abf]^{\downarrow}_{\approx} = \{abf, ab\},$$
$$[ade]^{\downarrow}_{\approx} = \{ade\},\ [aef]^{\downarrow}_{\approx} = \{aef\},\ [bde]^{\downarrow}_{\approx} = \{bde\},\ [bef]^{\downarrow}_{\approx} = \{bef\},$$
$$[cdf]^{\downarrow}_{\approx} = \{cdf, cd, cf, df\},\ [cdef]^{\downarrow}_{\approx} = \{cdef, cde, cef, def, ce\},$$
$$[bcdf]^{\downarrow}_{\approx} = \{bcdf, bcd, bcf, bdf\},\ [acdf]^{\downarrow}_{\approx} = \{acdf, acd, acf, adf\},$$
$$[abef]^{\downarrow}_{\approx} = \{abef, abe\},\ [bcdef]^{\downarrow}_{\approx} = \{bcdef, bcde, bcef, bdef, bce\},$$
$$[acdef]^{\downarrow}_{\approx} = \{acdef, acde, acef, adef, ace\},$$
$$[abcdf]^{\downarrow}_{\approx} = \{abcdf, abcd, abcf, abdf, abc, abd\}$$
$$[abcdef]^{\downarrow}_{\approx} = \{abcdef, abcde, abcef, abdef, abce, abde\}.$$

Therefore the down-maximum partitioners of $\hat{D}$ are

$$\emptyset, a, b, c, d, e, f, ac, ad, ae, af, bc, bd, be, bf, de, ef, abf, ade, aef, bde, bef,$$
$$cdf, abef, acdf, bcdf, cdef, bcdef, acdef, abcdf, abcdef,$$

whereas the down-minimal partitioners of $\hat{D}$ are the following:

$$\emptyset, a, b, c, d, e, f, ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ef, abc, abd,$$
$$abe, acd, ace, acf, ade, adf, aef, bcd, bce, bcf, bde, bdf, def,$$
$$abce, abde, adef, bdef.$$

Furthermore, we have the following up-indistinguishability classes of the knowledge pairing system $\hat{D}$:

$$[\emptyset]^{\uparrow}_{\approx} = \{\emptyset\},\ [a]^{\uparrow}_{\approx} = \{a\},\ [b]^{\uparrow}_{\approx} = \{b\},\ [c]^{\uparrow}_{\approx} = \{c\},\ [d]^{\uparrow}_{\approx} = \{d\},\ [e]^{\uparrow}_{\approx} = \{e\},\ [f]^{\uparrow}_{\approx} = \{f\},$$
$$[ac]^{\uparrow}_{\approx} = \{ac\},\ [ad]^{\uparrow}_{\approx} = \{ad\},\ [af]^{\uparrow}_{\approx} = \{af\},$$
$$[bc]^{\uparrow}_{\approx} = \{bc\},\ [bd]^{\uparrow}_{\approx} = \{bd\},\ [be]^{\uparrow}_{\approx} = \{be\},\ [bf]^{\uparrow}_{\approx} = \{bf\},$$
$$[ce]^{\uparrow}_{\approx} = \{ce\},\ [de]^{\uparrow}_{\approx} = \{de\},\ [ef]^{\uparrow}_{\approx} = \{ef\},\ [abd]^{\uparrow}_{\approx} = \{abd, ab\},$$
$$[ade]^{\uparrow}_{\approx} = \{ade, ae\},\ [bce]^{\uparrow}_{\approx} = \{bce\},\ [bde]^{\uparrow}_{\approx} = \{bde\},\ [bef]^{\uparrow}_{\approx} = \{bef\},$$
$$[cdf]^{\uparrow}_{\approx} = \{cdf, cd, cf, df\},\ [cdef]^{\uparrow}_{\approx} = \{cdef, cde, cef, def\},$$
$$[bcdf]^{\uparrow}_{\approx} = \{bcdf, bcd, bcf, bdf\},\ [acdf]^{\uparrow}_{\approx} = \{acdf, acd, acf, adf\},$$
$$[abde]^{\uparrow}_{\approx} = \{abde, abe\},\ [bcdef]^{\uparrow}_{\approx} = \{bcdef, bcde, bcef, bdef\},$$
$$[acdef]^{\uparrow}_{\approx} = \{acdef, acde, acef, adef, ace, aef\},$$
$$[abcdf]^{\uparrow}_{\approx} = \{abcdf, abcd, abcf, abdf, abc, abf\}$$
$$[abcdef]^{\uparrow}_{\approx} = \{abcdef, abcde, abcef, abdef, abce, abef\}.$$

Therefore the up-maximum partitioners of $\hat{D}$ are

$$\emptyset, a, b, c, d, e, f, ac, ad, af, bc, bd, be, bf, ce, de, ef, abd, ade,$$
$$bce, bde, bef, cdf, cdef, bcdf, acdf, abde, bcdef, acdef, abcdf, abcdef,$$

whereas the up-minimal partitioners of $\hat{D}$ are the following:

$$\emptyset, a, b, c, d, e, f, ab, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ef,$$
$$abc, abe, abf, acd, ace, acf, adf, adf, aef, bcd, bce, bcf, bde, bdf, bef,$$
$$cde, cef, def, abce, abef, bcde, bdef.$$

We now interpret the role assumed by the maps $\Gamma^{\uparrow}$ and $\Gamma^{\downarrow}$. Let us fix a set $Z$ of airports, for example $Z = \{c, d\}$. Then, by (28), it is immediate to see that $\Gamma^{\downarrow}(Z) = \{c, d, f\}$. By taking account of the adjacency matrix of the digraph $\vec{D}$ represented in Table 8 of Example 2.2, we deduce that $\Gamma^{\downarrow}(Z)$ is the biggest set of airports from which there are direct flights towards both Copenaghen and Dublin or there are no flights to both the previous destinations. In our case, there are no flights starting from Copenaghen, Dublin and Florence and arriving in Copenaghen and Dublin.
On the other hand, if we fix a vertex subset $A = \{c, d, f\}$. Then, by (29), by referring to the rows of the adjacency table of $\vec{D}$, we see that $\Gamma^{\uparrow}(A) = \{c, d, e, f\}$ and argue that the set $\Gamma^{\uparrow}(A)$ is the biggest set of airports in which flights from Copenaghen, Dublin and Florence arrive or not. In our particular

situation, there are no direct flights from Copenaghen, Dublin and Florence to Copenaghen, Dublin, Edinburgh and Florence.
We now compute and interpret $\gamma^{\uparrow\downarrow}(Z)$ and $\gamma^{\downarrow\uparrow}(A)$. As before, fix $Z = \{c, d\}$. By the previous two computations, it's clear that

$$\gamma^{\uparrow\downarrow}(Z) = \{c, d, e, f\}.$$

We deduce that if we fix a subset $Z$ of airports, then $\gamma^{\uparrow\downarrow}(Z) = \{c, d, e, f\}$. We deduce that $\gamma^{\uparrow\downarrow}(Z)$ is the biggest set of airports in which arrive or not flights from all the airports starting from which there are (or not) flights directed to each airport of $Z$.
Finally, let us fix $A = \{b, c\}$. By taking account of the rows of the Table 8 of Example 2.2, we have $\Gamma^{\downarrow}(A) = \{a, d\}$ and by referring to its columns, we have $\Gamma^{\uparrow}(\{a, d\}) = \{b, c, e\}$, it follows immediately that

$$\gamma^{\downarrow\uparrow}(A) = \{b, c, e\}.$$

In other terms, $\gamma^{\downarrow\uparrow}(A)$ is the biggest set of airports in which arrive or not flights from all the airports starting from which there are (or not) flights going towards each airport of $A$.

7.2. **Adjacency in graphs.** We now see what happens when we consider a simple graph $G$ as knowledge pairing system. In this case, it is natural to give the following definition.

**Definition 7.5.** *Let $G = (V(G), E(G))$ be a simple graph. We associate to $G$ the following knowledge pairing system:*

$$\hat{G} := \langle V(G), V(G), \{0, 1\}, F\rangle,$$

*where $F(u, v) := 1$ if $u \sim v$ and $F(u, v) := 0$ otherwise.*

**Remark 7.6.** *Let us note that the adjacency matrix of a graph is symmetric, hence the down-indiscernibility and the up-indiscernibility coincide. In the sequel we always write $\equiv_A$ instead of $\equiv_A^{\downarrow}$ or $\equiv_A^{\uparrow}$.*

For a simple graph $G$ the description of the indiscernibility relation in $\hat{G}$ has been given in [20]. We recall this characterization:

**Theorem 7.7.** *Let $G$ be a simple graph, $A \subseteq V(G)$ and $v, v' \in V(G)$. The following conditions are equivalent:*
*(i) $v \equiv_A v'$.*
*(ii) For all $z \in A$ it results that $v \sim z$ if and only if $v' \sim z$.*
*(iii) $N_G(v) \cap A = N_G(v') \cap A$.*

*Proof.* See [20]. □

As an example, we give the general form of any partition of the complete graph $K_n$, for all $n \geq 1$ and for all $A \subseteq V(K_n)$.

**Proposition 7.8.** *Let $n \geq 1$ and let $A = \{w_1, \dots, w_k\}$ be a subset of $V(K_n) = \{v_1, \dots, v_n\}$. Then*

$$\pi_A(K_n) = w_1|w_2|\dots|w_k|A^c, \tag{38}$$

*where $A^c$ is the complementary subset of $A$ in $V(K_n)$.*

*Proof.* See [20]. □

One could think that adjacency induces the same knowledge pairing system of incidence in graph context, since any graph can be seen as an hypergraph. It fails to be true, as we can see by comparing Example 6.15 and Proposition 7.8. In fact, in the case of down-incidence we consider as attributes some edges of the graph and we are interested in finding which vertices have the same behavior with respect to the edges of $A$, i.e. two vertices are down-indiscernible if they belong to the same edges of $A$. On the other hand, in the case of adjacency, two vertices are indiscernible whenever they are linked to the same vertices of $A$. We then argue that in graph context there exists more than one canonical knowledge pairing system.

**Remark 7.9.** *We have shown how to associate a knowledge pairing system to a hypergraph, a digraph and a simple undirected graph. These knowledge pairing systems have been studied with respect to two fundamental aspects of rough set theory: the indiscernibility relation and the discernibility matrix.*
*We can summarize and highlight the differences among the three systems discussed in this section according to the following table.*

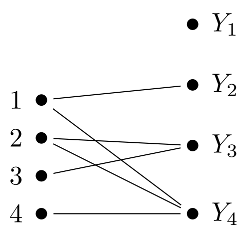
| | $\hat{H}^{\downarrow}$ | $\hat{H}^{\uparrow}$ | $\hat{D}$ | $\hat{G}$ |
|---|---|---|---|---|
| *attributes* | $E(H)$ | $V(H)$ | $V(\vec{D})$ | $V(G)$ |
| *objects* | $V(H)$ | $E(H)$ | $V(\vec{D})$ | $V(G)$ |
| *indiscernibility* | $\{v_i, v_j\}$ *is a separator* | $Y \cap A = Z \cap A$ | $N^+_{\vec{D}}(v_i) \cap A = N^+_{\vec{D}}(v_j) \cap A$ | $N_G(v_i) \cap A = N_G(v_j) \cap A$ |
| *discernibility matrix* | $Y_i \triangle Y_j$ | $H(v_i) \triangle H(v_j)$ | $N^+_G(v_i) \triangle N^+_G(v_j)$ | $N_G(v_i) \triangle N_G(v_j)$ |

7.3. **Knowledge pairing systems of Bipartite Graphs and Hypergraphs.** In this sub-section we want to analyze in a deeper way the relation between the knowledge pairing systems induced by graphs and by hypergraphs. To be more specific, it is well known that hypergraphs and bipartite graphs are equivalent structures. Indeed, to any bipartite graph $B = (B_1|B_2)$, with $B_1 = \{x_1, \dots, x_p\}$ and $B_2 = \{y_1, \dots, y_q\}$, we can associate the hypergraph $H_B = (X_B, \mathcal{F}_B)$, where $X_B := \{x_1, \dots, x_p\}$ and $\mathcal{F}_B = \{Y_1, \dots, Y_q\}$, with $Y_j$ the subset of $X_B$ defined as $Y_j = \{x_i \in X_B : \{x_i, y_j\} \in E(B)\}$, for $j = 1, \dots, q$. On the other hand, if we have a hypergraph $H = (X, \mathcal{F})$ with $X = \{x_1, \dots, x_p\}$ and $\mathcal{F} = \{Y_1, \dots, Y_q\}$, then we can define the bipartite graph $B_H := (X|\{Y_1, \dots, Y_q\})$, where $\{x_i, y_j\} \in E(B)$ if and only if (by definition) $x_i \in Y_j$. Then, it results that $H_{B_H} = H$ and $B_{H_B} = B$.

**Example 7.10.** Let $H$ be the hypergraph with vertex set $X = \{1, 2, 3, 4\}$ and hyperedges $Y_1 = \emptyset, Y_2 = \{1\}, Y_3 = \{2, 3\}, Y_4 = \{1, 2, 4\}$. Then the knowledge table $T[\hat{H}]$ is the following:

| | $\emptyset$ | $\{\mathbf{1}\}$ | $\{\mathbf{2}, \mathbf{3}\}$ | $\{\mathbf{1}, \mathbf{2}, \mathbf{4}\}$ |
|---|---|---|---|---|
| **1** | 0 | 1 | 0 | 0 |
| **2** | 0 | 0 | 0 | 0 |
| **3** | 0 | 0 | 1 | 0 |
| **4** | 0 | 0 | 1 | 1 |

It is clear that we can associate to the hypergraph $H$ the following bipartite graph $B_H$:




Now, given a fixed bipartite graph $B$, we study the links among the knowledge pairing system $\hat{B}$ introduced in Definition 7.5 and the knowledge pairing system $\hat{H_B}$ given in Definition 6.1. It is also clear that $T[\hat{G}]$ is a $(p+q) \times (p+q)$ matrix having four blocks: one block $T_1(B)$ of order $p \times p$ with all zeroes in its entries, corresponding to the object row $x_1, \dots, x_p$ and to the attribute row $x_1, \dots, x_p$; one block $T_2(B)$ of order $p \times q$ corresponding to the object row $x_1, \dots, x_p$ and to the attribute row $y_1, \dots, y_q$; one block $T_3(B)$ of order $q \times p$ corresponding to the object-row $y_1, \dots, y_q$ and to the attribute row $x_1, \dots, x_p$; finally, a last block $T_4(B)$ of order $q \times q$ with all zeroes in its entries, corresponding to the object-row $y_1, \dots, y_q$ and to the attribute-row $y_1, \dots, y_q$. With these notations, we have then the following result.

**Proposition 7.11.** *The matrix $T[\hat{H_B}]$ coincides with the block $T_2(B)$ while its transposed matrix coincides with the block $T_3(B)$.*

**Example 7.12.** Let us consider the bipartite graph $B_{\mathcal{J}}$ drawn in Figure 2. Then

$$H_{B_{\mathcal{J}}} = (\{O, P, Q\}, \{\{O, Q\}; \{P, Q\}\}).$$

Therefore the knowledge table of $\hat{B_{\mathcal{J}}}$ is the one given in Figure 3.
Let us note that the knowledge table of the knowledge pairing system $\hat{H_{B_{\mathcal{J}}}}$ is the block $2 \times 3$ in the southwest corner of the table in the previous figure. Analogously, the knowledge table of the knowledge pairing system $\hat{H_{B_{\mathcal{J}}}}$ is the block $3 \times 2$ in the northeast corner of the table in the same figure.

## 8. Conclusions

In our paper we discussed a new interpretation of an information table, that we called *knowledge pairing system.* We have a knowledge pairing system whenever an a priori distinction between potential objects of a set $U$ and potential attributes of a set $\Omega$ is not evident or, also, when both the interpretations, of the elements in $U$ as objects and of the elements of $\Omega$ as attributes (and vice versa), are equally admissible. In these cases we treat both the elements of $U$ and $\Omega$ as attributes. In this way, the interrelations between the indiscernibility induced on $U$ by $\Omega$ and the indiscernibility induced by $U$ on $\Omega$ provide the possibility to define a new type of set operator that can be studied with techniques similar to that of FCA. We show as three types of different real models induce a data interpretation by means via knowledge pairing systems: graph, digraph and hypergraph models. For each one of these three different cases, we widely discussed the convenience of a study approach based on appropriate knowledge pairing systems and related up-down order-structures and up-down operators. For the hypergraph case, we continued a precedent study started in [15]. In this context, we analyzed a hypergraph as a knowledge pairing system and used both the incidence matrix and its transpose in order to obtain new theoretical and interpretative results (see Theorem 6.8, Proposition 6.14 and their consequences) concerning the indiscernibility relations induced by hyperedge subsets.

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