On locally s-arc transitive graphs that are not of local characteristic p

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Abstract

Let Δ be a connected graph, without loops or multiple edges, such that each vertex has valency at least 3. Let $\{x, y\}$ be an edge. Let $G \leqslant \mathrm{Aut}(\Delta)$ acting locally s-arc transitively on Δ with $|G_z| < \infty$ for all $z \in \{x, y\}$. The amalgam $(G_x, G_y; G_{x,y})$ is determined in the case where $s \geqslant 4$ and there doesn't exists a prime p for which Δ is of local characteristic p with respect to G.

1 1. Introduction

In this paper we only consider graphs without loops and multiple edges. A G-graph Δ is a graph Δ together with a subgroup $G \leqslant \mathrm{Aut}(\Delta)$. Since we are interested in the action of groups on graphs our focus will be on such G-graphs.

From now on let Δ be a G-graph. We use the following notation; for more notation see Section 3.

• Δ is locally finite if for each vertex z the stabilizer G_z is finite.

• Δ is vertex (edge) transitive if G acts transitively on the vertices (edges) of Δ.

• Δ *is locally s-arc transitive* if for each vertex z the stabilizer G_z acts transitively on the set of arcs of length s originating at z.

• Let Δ be edge transitive and (x, y) be an 1-arc of Δ. Then $(G_x, G_y; G_{x,y})$ is the *vertex stabilizer amalgam* of Δ (with respect to (x, y)); see also Section 2 for notation about amalgams.

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Suppose that Δ is connected and locally s-arc transitive for some $s \geqslant 1$. Then G has at most 2 orbits on the vertex set $\mathrm{V}\Delta$ and is transitive on the edge set $\mathrm{E}\Delta$. For $\{x, y\} \in \mathrm{E}\Delta$, the vertex stabilizer amalgam $(G_x, G_y; G_{x,y})$ describes the structure of G and Δ locally. In the case of a finite vertex stabilizer amalgam - that is, G_x and G_y are finite - the structure of the amalgam seems to be very restricted. But there is no hope of classifying G and Δ, even for large s.

For this reason the main problem in the theory of locally s-arc transitive G-graphs is to bound s and to determine the structure of the corresponding vertex stabilizer amalgam.

The first results in this direction were the two seminal papers by Tutte [17, 18] where he showed that $s \leqslant 5$ for finite vertex transitive trivalent G-graphs. Later Weiss [22] gave the bound $s \leqslant 7$ for finite vertex transitive G-graphs of valency $\geqslant 3$.

Tutte's result was generalized in another direction by Goldschmidt in the groundbreaking paper [10] where he classified all finite vertex stabilizer amalgams of trivalent G-graphs. About at the same time Weiss [21] determined the finite vertex stabilizer amalgams if $s \geqslant 4$ under some additional assumptions on the induced groups $G_z^{\Delta(z)}$.

The direct study of finite vertex stabilizer amalgams using only group theoretic restrictions on the amalgams was made possible by Delgado and Stellmacher in [7]. In this paper they classified weak (B, N)-pairs of rank 2, and these weak (B, N)-pairs are very closely related to finite vertex stabilizers amalgams of locally s-transitive G-graphs. Indeed, in a recent paper by van Bon and Stellmacher [3] it was shown that if $s \geqslant 6$ and the valency of each vertex is at least 3, then any finite vertex stabilizer amalgam of a locally s-arc transitive G-graph is in fact a weak (B, N)-pair of rank 2. As a consequence one gets $s \leqslant 9$ for all such graphs.

The most general result about the structure of finite vertex stabilizer amalgams is due to Thompson [16] and Wielandt [20]. This result, the Thompson-Wielandt Theorem, states that – under some mild conditions on the amalgam – there always exists a non-trivial normal p-group in at least one of the vertex stabilizers (see [1] for an overview of various variations).

In this paper we investigate the structure of vertex stabilizer amalgams of a certain class of locally s-arc transitive graphs with $s \geqslant 4$. For $s = 4$ and 5 we give a detailed description of these amalgams since these are cases left out in [3].

For the formulation of our main result we need a notion that has been introduced in [3]. The G-graph Δ is called of *local characteristic* p if there exists a prime p such that

$$C_{G_z}(O_p(G_z^{[1]})) \leqslant O_p(G_z^{[1]}), \text{ for all } z \in \mathrm{V}\Delta.$$

One of the important steps in the proof of [3] is to show that if $s \geqslant 6$, then Δ is of local characteristic p.

The main result of this paper shows that, with the exception of three classes of vertex stabilizer amalgams that can be described precisely, see Section 2, this remains true under the weaker assumption $s \geqslant 4$. See Section 2 for the definition of amalgams of shape $\mathcal{A}_{q,d}$, $\mathcal{B}$ and $\mathcal{C}$.

Theorem 1. *Let Δ be a connected locally finite and locally s-transitive G-graph. Suppose that $s \geqslant 4$ and the valency of each vertex is at least 3. Then Δ is of local characteristic p, or there exists a 1-arc (x, y) such that the following hold:*

(1) *The vertex stabilizer amalgam $\mathfrak{A} = (G_x, G_y; G_{x,y})$ has shape $\mathcal{A}_{q,d}$, $\mathcal{B}$ or $\mathcal{C}$.*

(2) $s = 5$.

(3) *If $\mathfrak{A}$ has shape $\mathcal{A}_{q,d}$ then there exists $H \leqslant G_{x,y}$ such that $\mathfrak{A}$ has the splitting property with respect to H.*

The next theorem shows that the vertex stabilizer amalgams appearing in the conclusion of Theorem 1 are unique up to isomorphism.

Theorem 2. *Let Δ be a G-graph and $\widehat{\Delta}$ be a $\widehat{G}$-graph, both satisfying the hypothesis of Theorem 1. Suppose there exist 1-arcs $(x, y) \in E\Delta$ and $(\widehat{x}, \widehat{y}) \in E\widehat{\Delta}$ with $G_x \cong \widehat{G}_{\widehat{x}}$ and $G_y \cong \widehat{G}_{\widehat{y}}$. Then either Δ and $\widehat{\Delta}$ are both of local characteristic p, or the vertex stabilizer amalgams $(G_x, G_y; G_{x,y})$ and $(\widehat{G}_{\widehat{x}}, \widehat{G}_{\widehat{y}}; \widehat{G}_{\widehat{x},\widehat{y}})$ are isomorphic.*

The organization of the paper is as follows. Section 2 is devoted to the description of the three classes of amalgams which appear in the conclusion of Theorem 1 and to the proof of Theorem 2. In Section 3 (variations of) known results are given that are used in the proof of Theorem 1. In Section 4 properties of 2-transitive groups are collected which are relevant for this paper, and Sections 5 and 6 contain the proof of Theorem 1 and Theorem 2 split into a series of lemmas.

Throughout the paper we will assume the following hypothesis:

Main Hypothesis. *Let Δ be a connected locally finite and locally s-arc transitive G-graph for some integer $s \geqslant 1$. Suppose that each vertex of Δ has valency at least 3.*

The notation used in the paper is standard in the theory of (locally) s-arc transitive graphs, see also the notation given in Section 3. Basic facts about coprime action can be found in [11], [12] and [14]. We will use the classification of the finite 2-transitive groups, see [4] for a list.

2 2. Amalgams of Finite groups

The definitions and results presented in the beginning of this section follow [10]. As there an *amalgam* is a pair of group monomorphisms (ϕ_1, ϕ_2) with the same domain. That is, with each amalgam (ϕ_1, ϕ_2) comes a triple of groups (H_1, H_2, T) such that

$$H_1 \overset{\phi_1}{\leftarrow} T \overset{\phi_2}{\rightarrow} H_2;$$

and rather than by (ϕ_1, ϕ_2) we usually denote this amalgam by $H_1 \overset{\phi_1}{\leftarrow} T \overset{\phi_2}{\rightarrow} H_2$ and also call it an amalgam over T. The amalgam $H_1 \overset{\phi_1}{\leftarrow} T \overset{\phi_2}{\rightarrow} H_2$ is

- *finite* if H_1 and H_2 are finite groups;
- *faithful* if

$$X\phi_1 \ntrianglelefteq H_1 \quad \text{or} \quad X\phi_2 \ntrianglelefteq H_2 \quad \text{for all } 1 \neq X \trianglelefteq T.$$

Let $\mathfrak{A} : H_1 \overset{\phi_1}{\leftarrow} T \overset{\phi_2}{\rightarrow} H_2$ and $\widehat{\mathfrak{A}} : \widehat{H}_1 \overset{\widehat{\phi}_1}{\leftarrow} \widehat{T} \overset{\widehat{\phi}_2}{\rightarrow} \widehat{H}_2$ be two amalgams. Then

• $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ have the same *type* (via (ϕ_1, β, ϕ_2)) if there exists a triple (ψ_1, β, ψ_2) of group isomorphisms

$$(*) \qquad \psi_1 : H_1 \to \widehat{H}_1, \quad \beta : T \to \widehat{T}, \quad \psi_2 : H_2 \to \widehat{H}_2$$

such that

$$(1) \qquad T\phi_1\psi_1 = T\beta\widehat{\phi}_1 \quad \text{and} \quad T\phi_2\psi_2 = T\beta\widehat{\phi}_2.$$

• $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ are *isomorphic* (via (ϕ_1, β, ϕ_2)) if there exists a triple (ψ_1, β, ψ_2) of group isomorphisms satisfying $(*)$ such that

$$(2) \qquad \phi_1\psi_1 = \beta\widehat{\phi}_1 \quad \text{and} \quad \phi_2\psi_2 = \beta\widehat{\phi}_2.$$

Property (1) shows that $\phi_1\psi_1\widehat{\phi}_1^{-1}\beta^{-1}$ and $\phi_1\psi_1\widehat{\phi}_1^{-1}\beta^{-1}$ are two automorphisms of T, while property (2) shows that these two automorphisms both are the identity automorphism of T.

Now suppose that $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ have the same type via (ϕ_1, β, ϕ_2), so $(*)$ and (1) hold. Then $\widehat{\mathfrak{A}}$ is isomorphic to

$$H_1 \overset{\beta\widehat{\phi}_1\psi_1^{-1}}{\longleftarrow} T \overset{\beta\widehat{\phi}_2\psi_2^{-1}}{\longrightarrow} H_2.$$

Moreover, each amalgam over T of the same type as $\mathfrak{A}$ is isomorphic to

$$H_1 \overset{\phi_1}{\leftarrow} T \overset{\beta\phi_2}{\rightarrow} H_2 \text{ for some } \beta \in \mathrm{Aut}(T).$$

That is, amalgams of the form $H_1 \overset{\phi_1}{\leftarrow} T \overset{\beta\phi_2}{\rightarrow} H_2$ provide representatives for the isomorphism classes of amalgams which have the same type as $\mathfrak{A}$.

We now simplify the notation for $\mathfrak{A}$ slightly. We identify T, $T\phi_1$ and $T\phi_2$ by

$$t \equiv t\phi_1 \equiv t\phi_2, \quad t \in T.$$

Then ϕ_1 and ϕ_2 become the identity homomorphism, so T is contained in H_1 and H_2, and we write

$$H_1 \leftarrow T \rightarrow H_2.$$

(In the following $\mathfrak{A}$ is always such an amalgam if no homomorphisms show up in the notation.)

Let A_i be the subgroup of $\mathrm{Aut}(T)$ induced by $N_{\mathrm{Aut}(H_i)}(T)$. Then the double cosets $A_1 \cdot \beta \cdot A_2$ in $\mathrm{Aut}(T)$ are in one-to-one correspondence with the isomorphism classes of amalgams of type $\mathfrak{A}$, where $H_1 \leftarrow T \xrightarrow{\beta} H_2$ is a representative of the class corresponding to $A_1 \cdot \beta \cdot A_2$, see [10, p.381].

Now let Δ be an edge transitive G-graph and let $\{x, y\} \in \mathrm{E}\Delta$. Then *the vertex stabilizer amalgam* $(G_x, G_y; G_{x,y})$ of Δ is the amalgam $G_x \leftarrow G_{x,y} \rightarrow G_y$. The isomorphism type of $G_x \leftarrow G_{x,y} \rightarrow G_y$ does not depend on the choice of $\{x, y\}$ (possibly after interchanging the stabilizers G_x and G_y) since G acts edge-transitively on Δ – and this justifies to speak of "the" vertex stabilizer amalgam of Δ.

Also observe that vertex stabilizer amalgams of connected edge transitive G-graphs are faithful amalgams.

Notation. For a fixed amalgam $H_1 \leftarrow T \rightarrow H_2$ we define:

T_i is the largest subgroup of T such that $T_i \trianglelefteq H_i$ and

$$L_i := C_{H_i}(T_i).$$

Lemma 2.1. *Let $H_1 \leftarrow T \rightarrow H_2$ be an amalgam and suppose that T is finite. Then there exists a unique inclusion-minimal subgroup B of T satisfying:*

(i) $T_1T_2 \leqslant B$, *and*

(ii) $B = T \cap \langle B^{H_i} \rangle$ *for* $i = 1, 2$.

In particular, B is normal in T.

Proof. Let B_1 and B_2 be two subgroups of T satisfying (i) and (ii). Then $T_1T_2 \leqslant B_1 \cap B_2$ and

$$B_1 \cap B_2 \leqslant \langle (B_1 \cap B_2)^{H_i} \rangle \cap T \leqslant \langle B_k^{H_i} \rangle \cap T = B_k,\ k = 1, 2.$$

Hence $B_1 \cap B_2 = \langle (B_1 \cap B_2)^{H_i} \rangle \cap T$, and the minimality of B_1 and B_2 gives $B_1 = B_2$, so B is uniquely determined in T. In particular $B \trianglelefteq T$. □

Let $\mathfrak{A} :\ H_1 \leftarrow T \rightarrow H_2$ and B be as in 2.1. Define $H_i^* := \langle B^{H_i} \rangle$. We call the amalgam

$$\mathfrak{A}^* : \quad H_1^* \leftarrow B \rightarrow H_2^*$$

the *core* of $\mathfrak{A}$. We now use this core to define certain classes of amalgams which are of interest for this paper.

Let q be a prime power and d a divisor of $q-1$. Then each of the groups $\mathrm{GL}_1(q)$, $\mathrm{AGL}_1(q)$ and $\mathrm{AGL}_2(q)$ has a unique normal subgroup of index d which we denote by $\mathrm{GL}_1(q)^{(d)}$, $\mathrm{AGL}_1(q)^{(d)}$ and $\mathrm{AGL}_2(q)^{(d)}$, respectively, where in the last case d has to divide $\frac{q-1}{2}$ if q is odd.

Let q be even, $S \in Syl_2(\mathrm{AGL}_2(q))$, $V := O_2(\mathrm{AGL}_2(q))$ and $V_0 := C_V(S)$. Then $\mathrm{AGL}_2(q,S) := N_{\mathrm{AGL}_2(q)}(S)$ and $\mathrm{AGL}_2(q,S)^{(d)}$ is the (unique) subgroup X of index d in $\mathrm{AGL}_2(q,S)$ satisfying:

$$X = C_X(V_0)C_X(V/V_0),\ C_X(V/V_0)/S \cong \mathrm{GL}_1(q) \text{ and } C_X(V_0)/S \cong \mathrm{GL}_1(q)^{(d)}.$$

Finally, let q be odd. Then each of the groups $\mathrm{AGL}_1(q) \times \mathrm{AGL}_1(q)$ and $\mathrm{PGL}_2(q) \times \mathrm{AGL}_1(q)$ contains a unique normal subgroup of index 2 which has no direct factors. This subgroup we denote by

$$\mathrm{AGL}_1(q) \curlywedge \mathrm{AGL}_1(q) \text{ and } \mathrm{PGL}_2(q) \curlywedge \mathrm{AGL}_1(q), \text{ respectively.}$$

The following definitions only refer to amalgams of the form $H_1 \leftarrow T \rightarrow H_2$. That is, where the embedding homomorphisms are given "by inclusion".

Definition 2.2 (Amalgams of shape $\mathbf{L_2}(q,d)$ and $\mathcal{A}_{q,d}$). *Let $\mathfrak{A} : H_1 \leftarrow T \rightarrow H_2$ be a finite amalgam with core $\mathfrak{A}^* : H_1^* \leftarrow B \rightarrow H_2^*$. Then $\mathfrak{A}$ has shape $\mathrm{L}_2(q,d)$, where q is a prime power and d is a divisor of $q-1$, if the following hold:*

(i) *T is a maximal subgroup of H_i, $i = 1,2$.*

(ii) *$C_{H_1}(L_1) = T_1$.*

(iii) *$L_1 \cong \mathrm{PSL}_2(q)$ and $T_1 \cong \mathrm{AGL}_1(q)^{(d)}$.*

(iv) *$O_p(T) = C_{H_2}(O_p(T)) \trianglelefteq H_2$, and $O_p(T)$ is a 2-dimensional $\mathbb{F}_q H_2^*$-module.*

Let $\mathfrak{A}$ be an amalgam of shape $\mathrm{L}_2(q,d)$. We say that $\mathfrak{A}$ is of shape $\mathcal{A}_{q,d}$ if the following hold for the core $\mathfrak{A}^*$:

(1) *$q = 2^n > 2$, and $\frac{q-1}{d}$ does not divide $2^i - 1$ for $1 \leqslant i < n$.*

(2) *$H_1^* = L_1 \times T_1$, $B = (L_1 \cap B) \times T_1$ and $L_1 \cap B \cong \mathrm{AGL}_1(q)$.*

(3) *$L_1 \cap O_2(B)$ is the only 1-dimensional $\mathbb{F}_q H_2^*$- submodule.*

(4) *$H_2^* \cong \mathrm{AGL}_2(q,S)^{(d)}$ and $T_2/O_2(B) \cong \mathrm{GL}_1(q)^{(d)}$.*

Any amalgam of shape $\mathcal{A}_{q,d}$ has the splitting property *with respect to H if in addition there exists $t \in H_2^* \backslash B$ and $H \leqslant T \cap T^t$ such that:*

(5) *$H \cap T_1 = 1$ and $T = BH$,*

(6) H^{Σ} *is an affine* 2*-transitive group, where* $\Sigma := (B)^{H_1}\backslash\{B\}$.

Remark. In Condition (6) above the isomorphism $\mathrm{A\Gamma L}_1(q) \cong N_{\mathrm{P\Gamma L}_2(q)}(B)^{\Sigma}$ shows that H is isomorphic to a 2-transitive subgroup of $\mathrm{A\Gamma L}_1(q)$ (in its natural 2-transitive permutation representation). These subgroups have been classified by D. Foulser [8].

Definition 2.3 (Amalgams of shape $\mathcal{B}$). *Let $\mathfrak{A}$ be an amalgam of shape $L_2(7,2)$. We say that $\mathfrak{A}$ is of shape $\mathcal{B}$ if the following hold:*

(1) $H_1 = H_1^* \cong (\mathrm{PGL}_2(7) \curlywedge \mathrm{AGL}_1(7))$ *and* $B = T \cong (\mathrm{AGL}_1(7) \curlywedge \mathrm{AGL}_1(7))$.

(2) $H_2 = H_2^* \cong O_7(B) \rtimes (\mathrm{C}_6 \curlyvee \mathrm{SL}_2(3))$ *and* $T_2 \cong O_7(B) \rtimes \mathrm{C}_6$.

(3) $O_7(B)$ *is an irreducible* $\mathbb{F}_7 H_2$*-module.*

Definition 2.4 (Amalgams of shape $\mathcal{C}$). *Let $\mathfrak{A}$ be an amalgam of shape $L_2(5,2)$. We say that $\mathfrak{A}$ is of shape $\mathcal{C}$ if the following hold:*

(1) $H_1 = H_1^* \cong (\mathrm{PGL}_2(5) \curlywedge \mathrm{AGL}_1(5))$ *and* $B = T \cong (\mathrm{AGL}_1(5) \curlywedge \mathrm{AGL}_1(5))$.

(2) $H_2 = H_2^* \cong O_5(B) \rtimes (\mathrm{C}_4 \times \mathrm{Sym}(3))$ *and* $T_2 \cong O_5(B) \rtimes \mathrm{C}_4$.

(3) $O_5(B)$ *is an irreducible* $\mathbb{F}_5 H_2$*-module.*

In the remainder of this section we will show that faithful amalgams of shape $\mathcal{A}_{q,d}$, $\mathcal{B}$ or $\mathcal{C}$ are unique up to isomorphism provided that the corresponding co-domains are isomorphic. This will then later give Theorem 2.

In the following let $\mathfrak{A}$ be an amalgam of shape $\mathcal{A}_{q,d}$, $\mathcal{B}$ or $\mathcal{C}$. We first define some groups related to $\mathfrak{A}$.

$$\begin{array}{rlrl} V_0 & := O_p(L_1 \cap B), & Z_0 & := O_p(T_1), \\ A_1^* & := N_{\mathrm{Aut}(H_1^*)}(B), & A_1 & := N_{\mathrm{Aut}(H_1)}(T), \\ \mathrm{Aut}^{\circ}(B) & := N_{\mathrm{Aut}(B)}(Z_0), & A_2 & := N_{\mathrm{Aut}(H_2)}(T). \end{array}$$

Lemma 2.5. *Let $\mathfrak{A}$ be of shape $\mathcal{A}_{q,d}$.*

(a) $\mathrm{Aut}(T_1) \cong \mathrm{A\Gamma L}_1(q)$.

(b) $|V_0| = |Z_0| = q$, $O_2(B) = V_0 \times Z_0 = O_2(T)$, *and V_0 and Z_0 are the only non-trivial B-invariant proper subgroups of $O_2(B)$. In particular* $C_{O_2(B)}(B) = 1$.

(c) $C_T(V_0) = V_0 \times T_1$ *and* $C_T(Z_0) = Z_0 \times (L_1 \cap B)$.

(d) $C_{H_1}(B) = 1$ *and* $C_{H_2}(B) = 1$.

(e) *B is a characteristic subgroup of T.*

(f) *Let $\mathfrak{A}'$ be an amalgam over T of shape $\mathcal{A}_{q',d'}$. Then $q = q'$ and $d = d'$.*

(g) H_1^* *is a characteristic subgroup of* H_1 *and* $\mathrm{Aut}(H_1^*) \cong \mathrm{P\Gamma L}_2(q) \times \mathrm{A\Gamma L}_1(q)$.

(h) $O_2(H_2) = O_2(H_2^*)$, $H_2 = TO_2(H_2^*)$ *and* $C_T(O_2(H_2^*)/V) = T_2$.

(i) $\mathrm{Aut}^\circ(B) = A_1^* \cong \mathrm{A\Gamma L}_1(q) \times \mathrm{A\Gamma L}_1(q)$.

Proof. (a) and (b): By 2.2(iii) $T_1 \cong \mathrm{AGL}_1(q)^{(d)}$ and by 2.2(2) $T \cap L_1 = B \cap L_1 \cong \mathrm{AGL}_1(q)$. Hence $|Z_0| = |V_0| = q$ and by 2.2(2) $O_2(B) = V_0 \times Z_0$; in particular $|O_2(B)| = q^2$. Since $B \trianglelefteq T$, we get $O_2(B) \leqslant O_2(T)$, and since by 2.2(iv) $|O_2(T)| = q^2$, $O_2(B) = O_2(T)$ follows. Moreover, $(B \cap L_1)/V_0$ acts regularly on V_0 and T_1/Z_0 acts semi-regularly on Z_0. It follows that V_0 is an irreducible $B \cap L_1$-module.

Since by 2.2(1) $|T_1/Z_0|$ does not divide $2^i - 1$ for $1 \leqslant 2^i - 1 < q - 1$, also T_1 acts irreducibly on Z_0. In particular $\mathrm{Aut}(T_1) \cong \mathrm{A\Gamma L}_1(q)$, and (a) holds. The action of T_1 and $B \cap L_1$ on $O_2(B)$ also shows that V_0 and Z_0 are the only non-trivial B-invariant proper subgroups of $O_2(B)$. Since $q > 2$, this also shows that $C_{O_2(B)}(B) = 1$.

(c): By 2.2(ii), (2) $C_T(L_1) = T_1$ and $C_T(T_1) = B \cap L_1$. Now we apply 2.2(iii). Then $H_1/C_{H_1}(L_1)$ is a subgroup of $\mathrm{P\Gamma L}_2(q)$ and V_0 is isomorphic to a Sylow 2-subgroup of $\mathrm{PGL}_2(q)$. Since these Sylow subgroups are self-centralizing in $\mathrm{P\Gamma L}_2(q)$, we get that

$$V_0 \times T_1 \leqslant C_T(V_0) \leqslant V_0 C_T(L_1) = V_0 \times T_1.$$

Similarly, by (a) Z_0 is self-centralizing in $\mathrm{Aut}(T_1)$. This gives

$$(B \cap L_1)Z_0 \leqslant C_T(Z_0) \leqslant Z_0 C_T(T_1) = Z_0 \times (B \cap L_1).$$

(d): By (b) $O_2(B) = V_0 Z_0$ and $C_{O_2(B)}(B) = 1$, and by 2.2(iv) $Z(B) \leqslant C_{H_2}(B) \leqslant C_{H_2}(O_2(B)) = O_2(B)$. It follows that $Z(B) = 1$. Thus, it suffices to show that $C_{H_1}(B) \leqslant B$ and $C_{H_2}(B) \leqslant B$.

As seen above $C_{H_2}(B) \leqslant B$, so it remains to show that $C_{H_1}(B) \leqslant B$. We apply (c). Then

$$C_{H_1}(O_2(B)) = C_{H_1}(V_0) \cap C_{H_1}(Z_0) = (V_0 \times T_1) \cap (L_1 \times Z_0) = O_2(B)(L_1 \cap T_1).$$

As by 2.2(2) $L_1 \cap T_1 = 1$, we get $C_{H_1}(O_2(B)) = O_2(B) \leqslant B$. Hence also $C_{H_1}(B) \leqslant B$.

(e): By (b) V_0 and Z_0 are the only normal subgroups of order q in T. Thus, any automorphism of T has to act on $\{V_0, Z_0\}$ and so normalizes $C_T(V_0)C_T(Z_0)$, and by (c) this group is equal to $T_1 \times (B \cap L_1) = B$. This shows that B is characteristic in T.

(f): Let $\mathfrak{A}' : \widehat{H}_1 \leftarrow T \rightarrow \widehat{H}_2$, and let $\widehat{B}$, $\widehat{T}_1$, $\widehat{Z}_0$, $\widehat{V}_0$ be the subgroups of T which have for $\mathfrak{A}'$ the meaning of B, T_1, Z_0 and V_0. By (b) $|V_0|^2 = |O_2(T)| = |\widehat{V}_0|^2$, and so $q = |V_0| = |\widehat{V}_0| = q'$. Moreover, again by (b) $\{Z_0, V_0\} = \{\widehat{Z}_0, \widehat{V}_0\}$. That is, $C_T(\widehat{V}_0) = C_T(V_0)$ or $C_T(\widehat{V}_0) = C_T(Z_0)$. In the first case (c) shows that $V_0 \times T_1 = \widehat{V}_0 \times \widehat{T}_1$ and then 2.2(iii) that $d = d'$. In the second case $C_T(\widehat{Z}_0) = C_T(V_0)$ and $C_T(Z_0) = C_T(\widehat{V}_0)$. Now 2.2(iii),(2) show that $d = d' = 1$.

(g): By 2.2(ii),(iii) L_1 is the only component of H_1 and $L_1 \cong \mathrm{PSL}_2(q)$. Thus, L_1 and so also $L_1C_{H_1}(L_1)$ is characteristic in H_1. Since by 2.2(ii) $C_{H_1}(L_1) = T_1$, 2.2(2) and (a) imply (g).

(h): By 2.2(i) T is a maximal subgroup of H_2 and by 2.2(4) $H_2^* \not\leqslant T$, so $H_2 = TH_2^*$; and as seen in the proof of (a), $O_2(T) = O_2(B)$. It follows that $H_2 = O_2(H_2^*)T$ and $O_2(T) \leqslant O_2(H_2^*)$. Hence $O_2(H_2) = O_2(H_2^*)O_2(T) = O_2(H_2^*)$.

To prove that last part of (h) put $T_2^* := C_T(O_2(H_2)/O_2(T))$. Then T_2^* is normalized by $O_2(H_2)T = H_2$, so $T_2^* \leqslant T_2$. Conversely, $[T_2, O_2(H_2)] \leqslant T_2 \cap O_2(H_2) = O_2(T)$, so $T_2 \leqslant T_2^*$. Hence $T_2 = T_2^*$.

(i): By (g) $\mathrm{Aut}(H_1^*) \cong \mathrm{P\Gamma L}_2(q) \times \mathrm{A\Gamma L}_1(q)$. Since B embeds in H_1^* as $\mathrm{AGL}_1(q) \times \mathrm{AGL}_1(q)^{(d)}$, we get $A_1^* \cong \mathrm{A\Gamma L}_1(q) \times \mathrm{A\Gamma L}_1(q)$.

Suppose that $d = 1$. Then $B \cong \mathrm{AGL}_1(q) \times \mathrm{AGL}_1(q)$ and so $\mathrm{Aut}(B) \cong \mathrm{Aut}(\mathrm{A\Gamma L}_1(q)) \wr \mathrm{C}_2$. Since A_1^* normalizes Z_0, we get $A_1^* = \mathrm{Aut}^\circ(B)$.

Suppose that $d \neq 1$. Then $\mathrm{Aut}(B) \cong \mathrm{A\Gamma L}_1(q) \times \mathrm{A\Gamma L}_1(q)$. This time $A_1^* = \mathrm{Aut}(B) = \mathrm{Aut}^\circ(B)$. □

By 2.5(d) $C_{H_i}(B) = 1$, $i = 1, 2$. Thus, we can identify B, T, H_1, H_2 with the corresponding subgroups in their automorphism group. That is, $H_i \leqslant \mathrm{Aut}(H_i)$, $H_i^* \leqslant \mathrm{Aut}(H_i^*)$, $A_i \leqslant \mathrm{Aut}(T)$ and $A_i^* \leqslant \mathrm{Aut}(B)$.

Lemma 2.6. *Let $\mathfrak{A} : H_1 \leftarrow T \rightarrow H_2$ and $\widehat{\mathfrak{A}} : \widehat{H}_1 \leftarrow \widehat{T} \rightarrow \widehat{H}_2$ be two amalgams of shape $\mathcal{A}_{q,d}$. Suppose that $H_i \cong \widehat{H}_i$, $i = 1, 2$. Then $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ are isomorphic.*

Proof. In order to use the same notation for $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ we use the $\widehat{\ }$-convention to distinguish corresponding subgroups used in definition 2.2. We also put $V := V_0 \times Z_0$ and $\widehat{V} := \widehat{V}_0 \times \widehat{Z}_0$.

Let $\psi_i : H_i \rightarrow \widehat{H}_i$, $i = 1, 2$ be isomorphisms. We first show that $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ have the same type. For doing this it suffices to show that $\widehat{T}$ and $T\psi_i$ are conjugate in $\mathrm{Aut}(\widehat{H}_i)$, $i = 1, 2$.

The case $i = 1$. By 2.2(ii),(iii) L_1 is the only component of H_1 and $T_1 = C_{H_1}(L_1)$, so $L_1\psi_1 = \widehat{L}_1$ and $C_{H_1}(L_1)\psi_1 = C_{\widehat{H}_1}(\widehat{L}_1)$. Thus $T_1\psi_1 = \widehat{T}_1$. Since by 2.2(2) $H_1^* = L_1 \times T_1$, also $H_1^*\psi_1 = \widehat{H}_1^*$.

By 2.2(2) $O_2(T) \in Syl_2(H_1^*)$ and by 2.2(1) $T = N_{H_1}(O_2(T))$. It follows that $T\psi_1$ and $\widehat{T}$ are normalizers of Sylow 2-subgroups of $\widehat{H}_1^*$ in $\widehat{H}_1$, so they are conjugate in $\widehat{H}_1$.

The case $i = 2$. Put

$$T_0 := L_1 \cap B,\ \widetilde{V} := V\psi_2,\ \widetilde{T}_k := T_k\psi_2,\ k = 0, 1, 2.$$

By 2.5(h) $H_2 = TO_2(H_2^*)$ and $O_2(H_2) = O_2(H_2^*)$. Note that $V_0 = Z(O_2(H_2))$ and $\widehat{V}_0 = Z(O_2(\widehat{H}_2))$. Hence $V_0\psi_2 = \widehat{V}_0$ and

$$\widetilde{T}_1 \leqslant C_{H_2}(V_0)\psi_2 = C_{\widehat{H}_2}(\widehat{V}_0) = O_2(\widehat{H}_2)C_{\widehat{T}}(\widehat{V}_0) \overset{2.5(c)}{=} O_2(\widehat{H}_2)\widehat{T}_1 \leqslant \widehat{H}_2^*.$$

We now use that by 2.2(iv) $\widehat{V}$ is a 2-dimensional $\mathbb{F}_q\widehat{H}_2^*$-module. Note that then $\widehat{H}_2$ induces $\mathbb{F}_q$-semilinear transformations on $\widehat{V}$. Since $\widehat{V}$ is 2-dimensional and $\widetilde{V} \trianglelefteq \widehat{H}_2$, either $\widehat{V} = \widetilde{V}$ or $\widehat{V} \cap \widetilde{V} = \widehat{V}_0$ and $O_2(\widehat{H}_2) = \widehat{V}\widetilde{V}$.

In the first case we use that $\widetilde{T}_1 \leqslant \widehat{H}_2^*$ and $Z_0 = [V, T_1]$. Hence $\widetilde{T}_1$ acts $\mathbb{F}_q$-linearly on $\widehat{V}$ and $Z_0\psi_2 = [\widetilde{V}, \widetilde{T}_1]$ is a 1-dimensional subspace of $\widehat{V}$. Since $O_2(\widehat{H}_2)$ is transitive on the 1-dimensional subspaces different from $\widehat{V}_0$, $Z_0\psi_2$ and $\widehat{Z}_0$ are conjugate in $\widehat{H}_2^*$, so we may assume that $Z_0\psi_2 = \widehat{Z}_0$. Hence $T\psi_2 \leqslant N_{\widehat{H}_2}(\widehat{Z}_0)$. Then by 2.2(i) $T\psi_2 = T$, and we are done.

In the second case $\widetilde{V} \cap \widehat{V} = \widehat{V}_0$ and $O_2(\widehat{H}_2) = \widehat{V}\widetilde{V} = \widehat{V}\widetilde{Z}_0$; in particular $\widetilde{T}_0 \leqslant C_{\widehat{H}_2}(O_2(\widehat{H}_2)/\widehat{V}))$. Now 2.5(h) gives

$$\widetilde{T}_0 \leqslant C_{\widehat{H}_2}(O_2(\widehat{H}_2/\widehat{V})) = O_2(\widehat{H}_2)C_{\widehat{T}}(O_2(\widehat{H}_2/\widehat{V})) = O_2(\widehat{H}_2)\widehat{T}_2 \leqslant \widehat{H}_2.$$

As we have seen already that $\widetilde{T}_1 \leqslant \widehat{H}_2^*$, we get from 2.2(2), $B\psi_2 = \widetilde{T}_0\widetilde{T}_2 \leqslant \widehat{H}_2$. It follows that $H_2^*\psi_2 = \widehat{H}_2^*$; in particular $\widetilde{V} \trianglelefteq \widehat{H}_2$.

Note that $O_2(\widehat{H}_2)/\widetilde{V} \cong \widehat{V}/\widehat{V}_0$ as $\widehat{H}_2^*$-modules. Hence

$$C_{H_2^*}(O_2(H_2)/V) \cong C_{\widehat{H}_2^*}(O_2(\widehat{H}_2)/\widetilde{V}) = C_{\widehat{H}_2^*}(\widehat{V}/\widehat{V}_0).$$

The left hand side is equal to $T_2O_2(H_2)$, so $C_{H_2^*}(O_2(H_2)/V)/O_2(H_2) \cong GL_1(q)^{(d)}$, while the right hand side is equal to $C_{\widehat{T}}(\widehat{Z}_0)O_2(\widehat{H}_2)$ and so $C_{\widehat{H}_2^*}(\widehat{V}/\widehat{V}_0)/O_2(\widehat{H}_2) \cong \mathrm{GL}_1(q)$. This shows that $d = 1$, so $H_2^* \cong \mathrm{AGL}_2(q, S) \cong \widehat{H}_2^*$.

Since $p = 2$ the $\mathbb{F}_q$-action of $\widetilde{V}$ on $\widehat{V}$ shows that each element in $O_2(\widehat{H}_2)$ which is not in $\widehat{V} \cup \widetilde{V}$ has order 4. Hence $\widehat{V}$ and $\widetilde{V}$ are the only elementary abelian subgroups of order q^2 in $O_2(\widehat{H}_2)$. We now use the embedding

$$\widehat{H}_2 \leqslant \mathrm{A\Gamma L}_2(S, q) \leqslant \mathrm{A\Gamma L}_2(q) \leqslant \mathrm{Aut}(\mathrm{PGL}_3(q)).$$

Also observe that $O_2(\mathrm{A\Gamma L}_2(q))$ is a Sylow 2-subgroup of $\mathrm{PSL}_3(q)$ and $\widehat{H}_2 \cap \mathrm{PGL}_3(q) = \mathrm{AGL}_2(q, S) = \widehat{H}_2^*$. Thus, there exists a duality automorphism $\tau \in \mathrm{Aut}(\mathrm{PGL}_3(q))$ with $\widehat{V}^\tau = \widetilde{V}$ and $\widehat{H}_2^{*\tau} = \widehat{H}_2^*$. As seen above, also $H_2^*\psi_2 = \widehat{H}_2^*$, and so $\widehat{Z}_0\tau$ is a 1-dimensional subspace of the $\mathbb{F}_qH_2^*\psi_2$-module $\widetilde{V}$ different from $\widetilde{V}_0\,(= \widehat{V}_0)$. As $O_2(H_2^*)\psi_2$ is transitive on the 1-dimensional subspaces of $\widetilde{V}$ different from $\widetilde{V}_0$ and $O_2(H_2^*)\psi_2 = O_2(\widehat{H}_2)$, we may assume that $\widehat{Z}_0^\tau = \widetilde{Z}_0$.

Since $\mathrm{Aut}(\mathrm{PGL}_3(q))/\mathrm{PGL}_3(q)$ is abelian we get $[\widehat{T}, \tau] \leqslant \widehat{H}_2 \cap \mathrm{PGL}_3(q) = \widehat{H}_2^*$ and so τ normalizes $\widehat{T}\widehat{H}_2^* = \widehat{H}_2$. Thus

$$T\psi_2 = N_{H_2\psi_2}(\widetilde{Z}_0) = N_{\widehat{H}_2}(\widehat{Z}_0^\tau) = N_{\widehat{H}_2}(\widehat{Z}_0)^\tau = \widehat{T}^\tau,$$

and $T\psi_2$ and $\widehat{T}$ are conjugate in $\mathrm{Aut}(\widehat{H}_2)$.

We have show that $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ have the same type. As we have seen at the beginning of this section the isomorphism classes of amalgams with the same type as $\mathfrak{A}$ correspond to the double cosets $A_2 \cdot \beta \cdot A_1$ in $\mathrm{Aut}(T)$, where $H_1 \leftarrow T \xrightarrow{\beta} H_2$ is a representative of the class corresponding to $A_2 \cdot \beta \cdot A_1$. We will show that there is only one double coset that leads to faithful amalgams. Since $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ both are faithful, they then have to be isomorphic.

First observe that by 2.5(e),(g) $\mathrm{Aut}(T) \leqslant Aut(B)$ and $\mathrm{Aut}(H_1) \leqslant \mathrm{Aut}(H_1^*)$. Moreover, by 2.5(i) $A_1^* = \mathrm{Aut}^\circ(B)$. Since $\mathrm{Aut}(H_1^*) = H_1^* A_1^*$ we get that

$$\mathrm{Aut}(H_1) = H_1^*(\mathrm{Aut}(H_1) \cap A_1^*).$$

Since $A_1 \leqslant A_1^*$ and A_1 is a maximal subgroup of $\mathrm{Aut}(H_1)$, we get $A_1 = \mathrm{Aut}(H_1) \cap A_1^*$.

Now let $\beta \in \mathrm{Aut}(T)$, so $\beta \in \mathrm{Aut}(B)$. If $\beta \in \mathrm{Aut}^\circ(B)$ then $\beta \in A_1^*$. Hence, β normalizes H_1^* and T and so $H_1^* T = H_1$. It follows that $\beta \in A_1$ and $A_2 \cdot a \cdot A_1 = A_2 \cdot 1 \cdot A_1$.

If $\beta \notin \mathrm{Aut}^\circ(B)$ then $Z_0^\beta = V_0$. Thus, $Z_0 = O_2(T_1) \trianglelefteq H_1$ and by 2.2(3) $Z_0^\beta = V_0 \trianglelefteq H_2$. This means that the amalgams in the isomorphism class corresponding to $A_2 \cdot \beta \cdot A_1$ are not faithful. □

Lemma 2.7. *Any two amalgams of shape $\mathcal{B}$ or $\mathcal{C}$, respectively, are isomorphic.*

Proof. For the amalgams of shape $\mathcal{B}$ and $\mathcal{C}$ we have $T = N_{H_1}(O_p(B)) = N_{H_2}(O_p(B))$, and $O_p(B)$ is a Sylow p-group of both H_1 and H_2, where $p = 7$ or 5, respectively. Hence, there is only one conjugacy class of subgroups isomorphic to T in H_i, $i \in \{1, 2\}$. This shows that any two amalgams of shape $\mathcal{B}$ or $\mathcal{C}$, respectively, have the same type. We will show that $A_1 \cdot 1 \cdot A_2$ is the only double coset in $\mathrm{Aut}(T)$.

Let $A := \mathrm{Aut}(B)$. Then $A \cong \mathrm{AGL}_1(p) \wr \mathrm{C}_2$, with $p = 7$ or 5 respectively. Since $\mathrm{Aut}(H_1) \cong \mathrm{PGL}_2(p) \times \mathrm{AGL}_1(p)$, we have $A_1 = \mathrm{Aut}^\circ(B) \cong \mathrm{AGL}_1(p) \times \mathrm{AGL}_1(p)$. In particular $|A/A_1| = 2$. Then either $A_2 \leqslant A_1$ or $A_1 \cdot 1 \cdot A_2$ is the only double coset. Thus, it suffices to show that $A_2 \nleqslant A_1$.

Since by 2.3(3) and 2.4(3), respectively, $O_p(B)$ is a 2-dimensional $\mathbb{F}_p H_2$-module and by 2.2(iv) $O_p(B) = O_p(H_2) = C_{H_2}(O_p(B))$ we can consider A_2 and $\mathrm{Aut}(H_2)$ as subgroups of $\mathrm{AGL}_2(p)$, where $O_p(B) = O_p(\mathrm{AGL}_2(p))$.

Table 1:

p	$\mathrm{AGL}_2(p)/T_2$	H_2/T_2	B/T_2	$\mathrm{Aut}(H_2)/T_2$	A_2/T_2
7	$\mathrm{PGL}_2(7)$	$\mathrm{Alt}(4)$	C_3	$\mathrm{Sym}(4)$	$\mathrm{Sym}(3)$
5	$\mathrm{PGL}_2(5)$	$\mathrm{Sym}(3)$	C_2	$\mathrm{C}_2 \times \mathrm{Sym}(3)$	$\mathrm{C}_2 \times \mathrm{C}_2$

We now apply the properties given in 2.3 and 2.4. Then $T_2/O_p(B) = Z(\mathrm{AGL}_2(p)/O_p(B))$, so $T_2 \trianglelefteq \mathrm{AGL}_2(p)$ and $\mathrm{AGL}_2(p)/T_2 \cong \mathrm{PGL}_2(p)$, $p = 7$ or 5. Thus, we get the two cases described in Table 1. In both cases $|A_2/B| = 2$ and so $A_2 \nleqslant B$. Note that the normalizer in $\mathrm{PGL}_2(p)$ of both Z_0 and V_0 is a cyclic group. Since A_2 normalizes B and A_2/T_2 is not cyclic it follows from 2.5(b) that A_2 interchanges Z_0 and V_0. It follows that $A_2 \nleqslant \mathrm{Aut}^\circ(B) = A_1$. □

Remark. The faithful amalgam of shape $\mathcal{A}_{q,1}$ with $T/B \cong \mathrm{C_n}$ has a completion inside $(\mathrm{PGL}_2(q) \wr \mathrm{AGL}_1(q)) \rtimes \mathrm{C_n}$, see [9].

The amalgams of shape $\mathcal{B}$ and $\mathcal{C}$ have a completion inside the G, where $G \cong \mathrm{Aut}({}^3D_4(2))$ and $G \cong \mathrm{Aut}(J_2)$, respectively. This can be seen by observing that $O_p(B)$ is a Sylow p-subgroup of G ($p = 7$ or 5, respectively), $T = N_{H_1}(O_p(B))$ and $H_2 = N_G(O_p(B))$. See [5].

3 3. Variations of known results

In this section we fix some of the notation which we will use in the remainder of the paper, and prove some properties of locally s-transitive graphs which we need. Recall that we assume the Main Hypothesis given in the introduction.

Notation. *Let $G \leqslant \mathrm{Aut}(\Delta)$ and $d(w,u)$ be the usual distance function on Δ. For a vertex $w \in \mathrm{V}\Delta$ we define*

$$\Delta(w) = \{u \in \mathrm{V}\Delta \mid d(w,u) = 1\},\ q_w = |\Delta(w)| - 1,$$

$$G_w = \langle g \in G \mid w^g = w\rangle, \quad G_w^{[k]} = \bigcap\nolimits_{d(w,u)\leqslant k} G_w \quad \textit{and} \quad G_w^{\Delta(w)} = G_w/G_w^{[1]}.$$

Thus, $\Delta(w)$ is the set of neighbours of w, $q_w + 1$ is the valency of w, G_w is stabilizer of x in G, $G_w^{\Delta(w)}$ is the group induced by G_w on $\Delta(w)$, and $G_w^{[1]}$ is the kernel of this action.

We first give a general version of the Thompson-Wielandt Theorem for locally 1-transitive G-graphs using [1].

Theorem 3.1. *Suppose that for every vertex z the induced action of G_z on $\Delta(z)$ is quasi-primitive. Then either Δ is of local characteristic p or for any edge $\{x,y\}$ (possibly after interchanging x and y), $G_{x,y}^{[1]} = G_x^{[2]}$ and $G_y^{[2]} = 1$.*

Proof. Let $\{x,y\}$ be any edge of Δ. According to [1, 1.2], possibly after interchanging x and y, either $G_{x,y}^{[1]} = G_x^{[2]}$ and $G_y^{[2]} = 1$, or there exists a prime p such that $F^*(G_x)$, $F^*(G_y)$ and $F^*(G_{x,y}^{[1]})$ are p-groups. In order to prove our theorem we may assume that the first case does not hold. Since $G_x^{[2]}G_y^{[2]} \leqslant G_{x,y}^{[1]}$, we get $G_{x,y}^{[1]} \neq 1$. In particular, $F^*(G_x)$, $F^*(G_y)$ and $F^*(G_{x,y}^{[1]})$ are non-trivial p-groups.

Let $u \in \{x,y\}$ and $v \in \{x,y\}$ with $v \neq u$. Suppose first that $O_p(G_u) \nleqslant G_u^{[1]}$. Since $1 \neq O_p(G_{x,y}^{[1]}) \leqslant O_p(G_u^{[1]})$ it follows from [1, 2.8] that $G_{u,v}^{[1]}$ is a p-group.

Hence $G_{u,v}^{[1]}$ and $G_v^{[2]}$ are subgroups of $O_p(G_u^{[1]})$. If $C_{G_u}(O_p(G_u^{[1]})) \nleqslant G_u^{[1]}$, then by the quasi-primitivity of G_u on $\Delta(u)$, $C_{G_u}(O_p(G_u^{[1]}))$ is transitive on $\Delta(u)$. Thus, also $C_{G_u}(G_{u,v}^{[1]})$ and $C_{G_u}(G_v^{[2]})$ are transitive on $\Delta(u)$. It follows that $G_{u,v}^{[1]} = G_u^{[2]}$ and $G_v^{[2]} = 1$, which is not the case since $\{u,v\} = \{x,y\}$. Hence $C_{G_u}(O_p(G_u^{[1]})) \leqslant G_u^{[1]}$ and thus $C_{G_u}(O_p(G_u^{[1]})) = C_{G_u^{[1]}}(O_p(G_u^{[1]})) \leqslant O_p(G_u^{[1]})$.

Suppose now that $O_p(G_u) \leqslant G_u^{[1]}$. Then $C_{G_u}(O_p(G_u^{[1]})) = C_{G_u}(O_p(G_u)) \leqslant O_p(G_u) = O_p(G_u^{[1]})$.

We have show that $C_{G_u}(O_p(G_u^{[1]})) \leqslant O_p(G_u^{[1]})$ for any $u \in \{x,y\}$. Since G is transitive on $\mathrm{E}\Delta$ we conclude that Δ is of local characteristic p. □

For locally s-arc transitive graphs with $s \geqslant 4$ a stronger version of 3.1 holds. Its proof uses Section 6 of [3], where it was also shown that Δ has local characteristic p if $s \geqslant 6$.

Proposition 3.2. *Suppose that $s \geqslant 4$ and let $\{x,y\} \in \mathrm{E}\Delta$. Then either Δ is of local characteristic p, or (possibly after interchanging x and y) the following hold:*

(a) $1 \neq G_{x,y}^{[1]} = G_x^{[2]}$ *and* $G_y^{[2]} = 1$.

(b) $G_x^{[1]} \cap G_z^{[2]} = 1$ *for any* $z \in \Delta(y)\backslash\{x\}$.

(c) $G_x^{[2]}$ *is abelian or* $G_x^{[1]} = G_x^{[2]}$.

Proof. It follows immediately from the main theorem of [2] together with [3, 6.6, 6.4, and 6.2] that either Δ is of local characteristic p, or (possibly after interchanging x and y) $1 \neq G_{x,y}^{[1]} = G_x^{[2]}$ and $G_x^{[1]} \cap G_z^{[2]} = 1$ for any $z \in \Delta(y)\backslash\{x\}$.

Suppose that we are in the latter case. Then by the transitivity of G_y on $\Delta(y)$ also $G_{y,z}^{[1]} = G_z^{[2]}$, and

$$G_y^{[2]} \leqslant G_{x,y,z}^{[1]} = G_x^{[1]} \cap G_{y,z}^{[1]} = G_x^{[1]} \cap G_z^{[2]} = 1.$$

Thus, (a) and (b) hold. To prove (c) assume that $G_x^{[1]} \neq G_x^{[2]}$. Then there exists an $h \in G_x^{[1]} \backslash G_x^{[2]}$ with $z^h \neq z$ since $G_{x,y}^{[1]} = G_x^{[2]}$. Now $G_z^{[2]} \neq G_{z^h}^{[2]}$ and $(G_z^{[2]})^{\Delta(x)} = (G_{z^h}^{[2]})^{\Delta(x)} \cong G_z^{[2]}$. Since $[G_z^{[2]}, G_{z^h}^{[2]}] \leqslant G_{z,y,z^h}^{[1]} = 1$ it follows that $G_z^{[2]}$ is abelian. Again by the transitivity of G_y on $\Delta(y)$, also $G_x^{[2]}$ is abelian. Thus, (c) holds. □

The following lemma is a special case of [3, 4.8].

Lemma 3.3. *Suppose that $s \geqslant 4$. Let (x_0, x_1, x_2, x_3) be a 3-arc and p be a prime with $p|q_{x_0}$. Then for any $P \in Syl_p(G_{x_0,x_1,x_2,x_3})$ we have $N_{G_{x_1}}(P) \leqslant G_{x_2}$.*

The next lemma is a variation of [2, 2.3].

Lemma 3.4. *Let (x_0, x_1, x_2) be a 2-arc. Suppose that $s \geqslant 4$ and $C_{G_{x_0}}(G_{x_0}^{[1]}) \nleqslant G_{x_0}^{[1]}$. Then $G_{x_2} \cap G_{x_0}^{[1]} = G_{x_0}^{[2]}$. In particular $G_{x_0}^{[1]\Delta(x_1)}$ acts semi-regularly on $\Delta(x_1)\backslash\{x_0\}$ and $|G_{x_0,x_1,x_2}| = |G_{x_0,x_1,x_2}^{\Delta(x_0)}||G_{x_0}^{[2]}|$.*

Proof. Since $C_{G_{x_0}}(G_{x_0}^{[1]}) \nleqslant G_{x_0}^{[1]}$, there exists $g \in C_{G_{x_0}}(G_{x_0}^{[1]})$ with $x_1^g \neq x_1$. Let $x_1' = x_1^g$ and $x_2' = x_2^g$. Then g centralizes $G_{x_2} \cap G_{x_0}^{[1]}$, so

$$G_{x_2} \cap G_{x_0}^{[1]} = (G_{x_2} \cap G_{x_0}^{[1]})^g = G_{x_2'} \cap G_{x_0}^{[1]}.$$

Since $s \geqslant 4$, G_{x_2',x_1',x_0,x_1} is transitive on $\Delta(x_1)\backslash\{x_0\}$ and normalizes $G_{x_2} \cap G_{x_0}^{[1]}$. Hence $G_{x_2} \cap G_{x_0}^{[1]} \leqslant G_{x_1}^{[1]}$. Since $C_{G_{x_0}}(G_{x_0}^{[1]})$ is transitive on $\Delta(x_0)$, it follows that $G_{x_2} \cap G_{x_0}^{[1]} = G_{x_0}^{[2]}$. In particular $(G_{x_2} \cap G_{x_0}^{[1]})^{\Delta(x_1)} = 1$. That is, $G_{x_0}^{[1]}$ induces a semi-regular group on $\Delta(x_1)\backslash\{x_0\}$. $\square$

A group G acting on a set Ω is called *generously transitive* if for any two $\omega_1, \omega_2 \in \Omega$, with $\omega_1 \neq \omega_2$, there is a $g \in G$ such that $\omega_1^g = \omega_2$ and $\omega_2^g = \omega_1$.

Lemma 3.5. *Let (x_0, x_1, x_2) be a 2-arc. Suppose that $s \geqslant 4$ and that there exists $N \trianglelefteq G_{x_1}$ such that $N^{\Delta(x_1)}$ is generously transitive. Then $N \cap G_{x_0,x_1,x_2}$ is transitive on $\Delta(x_2)\backslash\{x_1\}$.*

Proof. Since $N^{\Delta(x_1)}$ is generously transitive there exists an $a \in N$ with $x_0^a = x_2$ and $x_2^a = x_0$.

Let $x_3 \in \Delta(x_2)\backslash\{x_1\}$ and consider the 4-arc $(x_3^a, x_0, x_1, x_2, x_3)$. For any $g \in G_{x_0,x_1,x_2,x_3}$ we have:

$$(x_3^a)^g = x_3^{ag} = x_3^{ga[a,g]} = x_3^{a[a,g]} = (x_3^a)^{[a,g]}.$$

Since a normalizes $\{x_0, x_1, x_2\}$, we also have $[a, g] \in N \cap G_{x_0,x_1,x_2}$. Thus, the displayed equation shows that the G_{x_0,x_1,x_2,x_3}-orbit of x_3^a is contained in the $N \cap G_{x_0,x_1,x_2}$-orbit of x_3^a. Since $s \geqslant 4$, G_{x_0,x_1,x_2,x_3} is transitive on $\Delta(x_0)\backslash\{x_1\}$. Hence $N \cap G_{x_0,x_1,x_2}$ is transitive on $\Delta(x_0)\backslash\{x_1\}$, too. Since a normalizes $N \cap G_{x_0,x_1,x_2}$ the lemma follows. $\square$

Lemma 3.6. *Let (x_0, x_1, x_2, x_3) be a 3-arc. Suppose that $s \geqslant 4$ and that $G_{x_1}^{[2]}$ transitive on $\Delta(x_3)\backslash\{x_2\}$. Then $s \geqslant 5$.*

Proof. Let $H := G_{x_0,x_1,x_2,x_3}$ and $x_4 \in \Delta(x_3)\backslash\{x_2\}$. The transitivity of $G_{x_1}^{[2]}$ on $\Delta(x_3)\backslash\{x_2\}$ gives $H = G_{x_1}^{[2]}H_{x_4}$. Since $s \geqslant 4$, H is transitive on $\Delta(x_0)\backslash\{x_1\}$. Then $G_{x_1}^{[2]} \leqslant G_{x_0}^{[1]}$ shows that also H_{x_4} is transitive on $\Delta(x_0)\backslash\{x_1\}$.

Let $x_{-1} \in \Delta(x_0)\backslash\{x_1\}$. Since $G_{x_1}^{[2]} \leqslant H_{x_{-1}}$, we also have that $H_{x_{-1}}$ is transitive on $\Delta(x_3)\backslash\{x_2\}$. As each 4-arc is either conjugated to $(x_{-1}, x_0, x_1, x_2, x_3)$ or to $(x_4, x_3, x_2, x_1, x_0)$ it follows that $s \geqslant 5$. $\square$

Lemma 3.7. *Let (x_0, x_1, x_2, x_3) be a 3-arc. Suppose that there exists $t \in G_{x_2}$ with $x_1^t = x_3$ and $[t, G_{x_0,x_1,x_2,x_3,x_0^t}] \leqslant G_{x_2}^{[3]}$. Then $s \leqslant 5$.*

Proof. Let $x_4 := x_0^t$, so $x_4 \in \Delta(x_3)\backslash\{x_2\}$. Pick $x_5 \in \Delta(x_4)\backslash\{x_3\}$ and put $K := G_{x_0,x_1,x_2,x_3,x_4,x_5}$ and $x_{-1} := x_5^{t^{-1}}$, so $x_{-1} \in \Delta(x_0)\backslash\{x_1\}$. Then

$$[t, K] \leqslant [t, G_{x_0,x_1,x_2,x_3,x_4}] \leqslant G_{x_2}^{[3]} \leqslant K.$$

Hence $t \in N_G(K)$ and so $K = K^{t^{-1}} \leqslant G_{x_{-1}}$. But then

$$G_{x_{-1},x_0,x_1,x_2,x_3,x_4,x_5} = G_{x_{-1}} \cap K = G_{x_0,x_1,x_2,x_3,x_4,x_5}.$$

In particular $G_{x_0,x_1,x_2,x_3,x_4,x_5}$ is not transitive on $\Delta(x_0)\backslash\{x_1\}$. Hence $s \leqslant 5$. □

Let p be a prime and $r \geqslant 2$ a natural number. A prime u is called a Zsigmondy prime for (r,p) if u divides $p^r - 1$, but u does not divide $p^i - 1$, for $1 \leqslant i < r$. The first part of the following lemma is due to Zsigmondy [23].

Lemma 3.8. *Zsigmondy primes for (r,p) exist except for $(6,2)$ and $(2,p)$ with p a Mersenne prime. Moreover, if u is a Zsigmondy prime for (r,p), then u does not divide r.*

Proof. See [23] for the proof of the existence of Zsigmondy primes when (r,p) is not equal to $(6,2)$ or $(2,p)$ with p a Mersenne prime.

Suppose now that u is a Zsigmondy prime which divides r, so $r = uu_0$ for some $1 \leqslant u_0 < r$. Since $u \neq p$ we have $p^{u-1} \equiv 1 \pmod u$. Hence

$$0 \equiv p^r - 1 \equiv p^{uu_0} - 1 \equiv p^{(u-1)u_0+u_0} - 1 \equiv p^{u_0} - 1 \pmod u.$$

Thus u divides $p^{u_0} - 1$, a contradiction. □

Lemma 3.9. *Let $q = p^r$ and $a = gcd(r, q-1)$. Let $\mathbb{F}$ be a finite field of characteristic p with multiplicative group $\mathbb{F}^*$ such that $a^{-1}(q-1)$ divides $2|\mathbb{F}^*|$. Then $|\mathbb{F}| \geqslant q$ or $q = 9$ and $|\mathbb{F}| = 3$.*

Proof. If $r = 1$, then $q = p$ and $|\mathbb{F}| \geqslant q$. So we may assume $r \geqslant 2$.

Suppose there exists a Zsigmondy prime u for (r,p). Note that $u \neq 2$. By Lemma 3.8 u does not divide r whence also not a, so u divides $a^{-1}(q-1)$. Since $u > 2$ it also divides $|\mathbb{F}^*|$. Hence $|\mathbb{F}| \geqslant q$.

Suppose that there is no Zsigmondy prime for (r,p). Then either $q = 2^6$ or $q = p^2$ and $p = 2^k - 1$, for some $k \in \mathbb{Z}$ with $k > 1$. In the first case $a = 3$ and $a^{-1}(q-1) = 21$. Hence 21 divides $|\mathbb{F}^*|$ from which it easily follows that $|\mathbb{F}| \geqslant q$. In the second case we have $a = 2$. If $|\mathbb{F}| < q = p^2$, then $|\mathbb{F}| = p$ and so $\frac{1}{2}(p^2 - 1) \leqslant 2(p-1)$ which gives $p = 3$ and $q = 9$. □

4 4. Preliminary results on 2-transitive groups

In this section we discuss properties of 2-transitive permutation groups that are either elementary or consequences of the classification of the the finite 2-transitive groups of simple type. All of them will be needed in the proof of the main theorem. So throughout this section G is a finite 2-transitive permutation group.

We call G of *regular type* if $F^*(G)$ is an abelian regular subgroup and of *simple type* if $F^*(G)$ is a non-abelian simple group. Any finite 2-transitive group is either of regular or of simple type, see [3, 2.2].

We start with the case when G is of simple type and $F^*(G)$ is a rank 1 group of Lie type in characteristic p (including $Ree(3)'$). We call the action of G ($F^*(G)$) the *natural permutation representation* of G ($F^*(G)$) in characteristic p if $F^*(G_x) = O_p(G_x)$ where $P = char\,\mathbb{F}$, $\mathbb{F}$ is a field of definition for $F^*(G)$ as a rank 1 group of Lie type.

Remarks. 1. The 2-transitive permutation representations of the finite Chevalley groups were obtained in [6] and shows that G and $F^*(G)$ have a unique such natural permutation representation in characteristic p.

2. If $F^*(G) \cong \mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5)$ or $F^*(G) \cong Ree(3)' \cong \mathrm{PSL}_2(8)$ then G and $F^*(G)$ have two natural permutation representations but in different characteristics. In the first case the characteristics are 2 and 5, in the second case 3 and 2.

3. If $F^*(G) \cong \mathrm{PSL}_2(9)$ or $\mathrm{PSL}_2(11)$ then there exists also a 2-transitive permutation representation on 6 and 11 points, respectively, which has no characteristic since in both cases $F^*(G_x) \cong \mathrm{A}lt(5)$.

4. If $F^*(G) \cong \mathrm{PSL}_2(7)$ then G has two 2-transitive permutation representation which satisfy $F^*(G_x) = O_p(G_x)$ for $p = 2$ and 7, respectively. But only 7 is the characteristic of a field of definition for $F^*(G)$ as a rank 1 group of Lie type.

5. Only for $F^*(G) \cong Ree(3)'$ the natural permutation representation in characteristic 3 for$F^*(G)$ is not 2-transitive.

Notation 4.1. *Suppose that $F^*(G)$ is a non-abelian simple rank* 1 *group of Lie type in its natural permutation representation in characteristic p.*

(1) $q := |\mathbb{F}|$, *where $\mathbb{F}$ is the field of definition in characteristic p for $F^*(G)$. That is,*

$$F^*(G) \cong \mathrm{PSL}_2(q),\ \mathrm{PSU}_3(q),\ \mathrm{Ree}(q)' \text{ (and } p = 3\text{), or } \mathrm{Sz}(q) \text{ (and } p = 2\text{).}$$

(2) *Let X be a 2-point stabilizer of G. Then $T := X \cap F^*(G)$ is a* torus *of $F^*(G)$ and $T_0 := C_X(T)$ is a* torus *of G.*

REMARK. *Since all 2-point stabilizers are conjugate in G, also all tori of $F^*(G)$ and G, respectively, are conjugate in G. Moreover, for each torus T_0 of G, $T_0 \cap F^*(G)$ is a torus of $F^*(G)$.*

(3) *Let T_0 be a torus of G. An element $u \in T_0$ is called a Zsigmondy element of T_0 if its order $o(u)$ is a Zsigmondy prime for (r, p), where $p^r = q$ if $F^*(G) \not\cong \mathrm{PSU}_3(q)$ and $p^r = q^2$ if $F^*(G) \cong \mathrm{PSU}_3(q)$.*

In the next lemma we use that for a simple rank 1 group of Lie type the natural permutation permutation representation in characteristic p extents to a permutation representation of $\mathrm{Aut}(K)$.

Lemma 4.2. *Let K be a simple rank 1 group of Lie type in its natural permutation representation in characteristic p on n points. Let X be a 2-point stabilizer of $\mathrm{Aut}(K)$, T_0 a torus of $\mathrm{Aut}(K)$ in X and $T := T_0 \cap K$. Then there exists a power $q := p^r$ such that Table 2 holds. Moreover,*

Table 2: 2-Point Stabilizers in Rank 1 Groups of Lie type.

K	n	$\lvert X/T_0\rvert$	$\lvert T_0/T\rvert$	$\lvert T_0\rvert$	Remarks
$\mathrm{PSL}_2(q)$	$q+1$	r	a	$q-1$	$q \geqslant 4$, $a = (q-1, 2)$
$\mathrm{PSU}_3(q)$	q^3+1	$2r$	b	q^2-1	$q \geqslant 3$, $b = (q+1, 3)$
$\mathrm{Sz}(q)$	q^2+1	r	1	$q-1$	$p = 2$, $q \geqslant 8$, r odd
$\mathrm{Ree}(3)'$	28	1	1	2	$\mathrm{Ree}(3)' \cong \mathrm{PSL}_2(8)$
$\mathrm{Ree}(q)$	q^3+1	r	1	$q-1$	$p = 3$, $q \geqslant 27$, r odd

(a) *T_0 and X/T_0 are cyclic groups.*

(b) *Suppose that no Zsigmondy element of T_0 is contained in T. Then one of the following holds:*

  (i) *$K \cong \mathrm{PSL}_2(q)$, where $q = p$, 2^6, or $q = p^2$ and p a Mersenne prime.*

  (ii) *$K \cong \mathrm{PSU}_3(q)$, where $q = p$ and $p + 1 = 2^t$ or $3 \cdot 2^t$, or $q = 2^3$.*

  (iii) *$K \cong \mathrm{Ree}(3)'$.*

Lemma 4.3. *Let $K := F^*(G)$ be a simple rank 1 group of Lie type in its natural permutation representation in characteristic p, and let P a point stabilizer of G and $T_0 \leqslant P$ a torus of G.*

(a) *Suppose that $K \not\cong \mathrm{Ree}(3)'$. Then for all Zsigmondy elements $u \in T_0$, $\langle u\rangle$ acts irreducibly on $O_p(P)/\Phi(O_p(P))$.*

(b) *Either $O_p(P) \leqslant K$ or $G \cong \mathrm{Ree}(3)$. Moreover, either $O_p(P)$ is regular on the points not fixed by P or $G \cong \mathrm{Ree}(3)'$ and $O_p(P)$ is fixed-point-free but not regular on the points not fixed by P.*

(c) *$\Omega_1(Z(O_p(P))) = \Omega_1(Z(O_p(P\cap K)))$ and $|\Omega_1(Z(O_p(P)))| = q$, and either $Z(O_p(P)) = Z(O_p(P\cap K)) = \Omega_1(Z(O_p(P\cap K)))$ or $G \cong \mathrm{Ree}(3)'$.*

(d) *Let A be an abelian normal subgroup of P. Then either $|A| \leqslant q$ or $K \cong \mathrm{Ree}(q)'$ and $|A| \leqslant q^2$.*

(e) *$[O_p(P), \langle t\rangle] = O_p(P)$ for all $1 \neq t \in T_0$ or $G \cong \mathrm{Ree}(3)$.*

(f) *$[Z(O_p(P)), \langle t\rangle] = Z(O_p(P))$ for all $1 \neq t \in T_0$, or $K \cong \mathrm{PSU}_3(q)$, and $[Z(O_p(P)), \langle t\rangle] = 1$ for all $t \in T_0$ with $t^{q+1} = 1$.*

(g) *$T_0 \cap K$ is irreducible on $\Omega_1(Z(O_p(P)))$.*

Proof. For $K \cong \mathrm{Ree}(3)'$ the lemma follows from the isomorphism $\mathrm{Ree}(3)' \cong \Gamma\mathrm{L}_2(8)$. Thus from now on we assume $K \neq \mathrm{Ree}(3)'$.

We start by listing some properties which can easily be deduced from the description of the groups given in [15] in case $K \cong \mathrm{PSL}_2(q)$, $q \neq 2, 3$, or $K \cong \mathrm{PSU}_3(q)$, $q \neq 2$, and which are proven in [19] and [13], for $K \cong \mathrm{Ree}(q)$, $q \neq 3$, and $K \cong \mathrm{Sz}(q)$, $q \neq 2$, respectively.

$$|O_p(P)/\Phi(O_p(P))| = q, \text{ if } K \ncong \mathrm{PSU}_3(q), \text{ and } q^2 \text{ otherwise.} \tag{1}$$

For $1 \neq t \in T_0$

$$C_{O_p(P)}(t) \leqslant \Phi(O_p(P)). \tag{2}$$

and

$$C_{Z(O_p(P))}(t) = 1, \text{ or } K \cong \mathrm{PSU}_3(q),\ t^{q+1} = 1, \text{ and } [Z(O_p(P)), \langle t\rangle] = 1. \tag{3}$$

(a): By (2) and coprime action $C_{O_p(P)/\Phi(O_p(P))}(t) = 1$. Hence T_0 acts semiregularly on $O_p(P)/\Phi(O_p(P))$ and (a) follows from the definition of a Zsigmondy element and (1).

(b) and (c): This follows immediately from the structure of the Sylow p-groups given in the above mentioned references.

(d): Let $A \trianglelefteq P$ be abelian. For any prime r dividing $|A|$ we have $1 \neq O_r(N) \leqslant F^*(P)$. Hence A is a p-group and $A \leqslant O_p(P)$. As seen in the proof of (a) T_0 acts semiregular on $O_p(P)/\Phi(O_p(P))$, whence by (1) and Table 2 of 4.2 T_0 acts irreducibly on $O_p(P)/\Phi(O_p(P))$. Hence if $A \nleqslant \Phi(O_p(P))$ then $O_p(P) \leqslant A$ and thus $K \cong PSL_2(q)$ and $|A| = q$. If $A \leqslant \Phi(O_p(P))$ then by (1), $|O_p(P)/A| \geqslant q$ and q^2, respectively, and so either $|A| \leqslant q$, or $K \cong Ree(q)$ and $|A| \leqslant q^2$.

(e): By coprime action and (2) $O_p(P) = [O_p(P), \langle t\rangle]\Phi(O_p(P)) = [O_p(P), \langle t\rangle]$ and (e) follows.

(f): This follows from (3) and coprime action.

(g): This follows from (c), (3) and coprime action. □

Lemma 4.4. *Suppose that G is of simple type. Let P be a point stabilizer of G and $X \leqslant P$ be a 2-point stabilizer.*

(a) *One of the following holds:*

(1) *$F^*(P)$ is non-abelian simple.*

(2) *$F^*(P) = O_p(P)$ and either $F^*(G) \cong \mathrm{PSL}_n(p^e)$, $n \geqslant 3$, in its permutation representation on the points (or hyperplanes) of a corresponding projective space, or $F^*(G)$ is a non-abelian simple rank 1 group of Lie type in its natural permutation representation in characteristic p.*

(b) *Suppose that N is a subnormal subgroup of P such that*

$$F^*(P) \nleqslant N, \quad N \nleqslant F^*(P) \quad \text{and} \quad [N, N^x] = 1 \text{ for all } x \in P \backslash N_P(N).$$

Then $F^*(P) = O_p(P)$, X is a p'-group, and $F^*(G) \cong Ree(3)'$, $\mathrm{PSU}_3(p)$, or $\mathrm{PSL}_2(p^2)$ in its natural permutation representation in characteristic p.

(c) *Suppose that there exists a cyclic normal subgroup $N \trianglelefteq P$ which is regular on the points not fixed by P. Then $N \leqslant F^*(G) \cong \mathrm{PSL}_2(p)$, $p \geqslant 5$ a prime.*

Proof. (a): This is Lemma 2.5 of [3].

(b): Put $\overline{P} := P/O_p(P)$. Then $F^*(N) \trianglelefteq\trianglelefteq F^*(P)$ since N is subnormal in P, and $N \neq 1$ and so $F^*(N) \neq 1$ since $N \nleqslant F^*(P)$. If $F^*(P)$ is simple, then $F^*(N) = F^*(P) \leqslant N$, a contradiction. Thus, (a) shows that (a:2) holds; in particular $F^*(P) = O_p(P)$.

Suppose that the first case of (a:2) holds. That is, $F^*(G) \cong \mathrm{PSL}_n(p^e)$, $n \geqslant 3$, and P fixes a point or a hyperplane on the projective space for $\mathrm{PSL}_n(p^e)$. Then there exists $O_p(P) \leqslant A \trianglelefteq P$ such that

$$\overline{A} \cong \mathrm{SL}_{n-1}(p^e), \text{ and } O_p(P) \text{ is a natural } \mathrm{SL}_{n-1}(p^e)\text{-module for } \overline{A}.$$

In particular, $\overline{A}$ acts irreducibly on $O_p(P)$.

Put $A_0 := A \cap NO_p(P)$. Then A_0 is subnormal in A, and the subnormal subgroup structure of $\mathrm{SL}_{n-1}(q)$ yields $A_0 \trianglelefteq A$. The irreducible action of $\overline{A}$ on $O_p(P)$ gives $O_p(P) \leqslant O^p(A_0)$ or $O_p(P) \cap O^p(A_0) = 1$.

In the first case $O_p(P) \leqslant N$ since $O^p(A_0) \leqslant O^p(NO_p(P)) = O^p(N)$, which contradicts the hypothesis. In the second case $O^p(A_0) = 1$ since $O_p(P) = F^*(P)$. Hence $A_0 = O_p(P)$ and $A \cap N \leqslant O_p(P)$. Now the subnormality of $\overline{N}$ shows that $[\overline{N}, \ \overline{A}] = 1$. But then again since $O^p(N) = O^p(NO_p(P))$, A normalizes $O^p(N)$ and so $[A, O^p(N)] \leqslant O_p(P) \cap O^p(N)$. Thus, by the irreducible action of $\overline{A}$ on $O_p(P)$ either $O_p(P) \leqslant O^p(N)$ or $O^p(N) = 1$ and $N \leqslant O_p(N)$, both cases contradicting the hypothesis.

Suppose now that the second case of (a:2) holds. That is, $F^*(G)$ is a non-abelian simple rank 1 group of Lie type in its natural permutation representation in characteristic p. We use Notation 4.1. Let X be a 2-point stabilizer in P and T a torus of $F^*(G)$ in X. As $N \trianglelefteq\trianglelefteq P$, we get $O^p(N) = O^p(NO_p(P))$. Hence $O^p(N) \trianglelefteq NO_p(P)$. If $O^p(N) \leqslant C_P(O_p(P))$ then $O^p(N) \leqslant O_p(P)$ since $O_p(P) = F^*(P)$. But then $O^p(N) = 1$ and $N \leqslant O_p(P)$, which contradicts the hypothesis on N. We have shown:

$$(*) \qquad 1 \neq [O^p(N), O_p(P)] \leqslant O^p(N) \cap O_p(P) \leqslant O_p(N).$$

Assume first that T does not contain Zsigmondy elements. We apply 4.2(b) and discuss the cases given there. Suppose first that $F^*(G) \cong \mathrm{PSL}_2(p), \mathrm{PSL}_2(p^2)$, $\mathrm{PSU}_3(p)$ or $Ree(3)'$. In all cases Table 2 shows that X is a p'-group. Moreover, in the $\mathrm{PSL}_2(p)$-case $|O_p(P)| = p$, and so by $(*)$ $F^*(P) = O_p(P) \leqslant N$, which is not the case. Hence (b) holds in these case.

Suppose now that $K \cong \mathrm{PSL}_2(2^6)$ or $\mathrm{PSU}_3(2^3)$. Let T_0 be a torus of P with $T_0 \cap F^*(G) = T$. Then either $T = T_0$ and $|T| = 3^2 \cdot 7$, or $|T| = 3 \cdot 7$ and $|T_0/T| = 3$. Let $S \in Syl_7(P)$. Then we may assume that $S \leqslant T$, and from the above order formula we get that $O^2(C_X(S)) \leqslant C_X(T)$.

Assume that 7 divides the order of $\overline{O^2(N)}$, so $\overline{S} \leqslant \overline{O^2(N)}$. Now $(*)$ and 4.3(e) yield $O_2(P) \leqslant O_2(N)$ a contradiction.

Assume that $\overline{O^2(N)}$ is a $7'$-group. Then $|\overline{O^2(N)}| = 3^2$ or 3. Since $\overline{S}$ is normal and $\overline{O^2(N)}$ subnormal in $\overline{P}$, we get $[\overline{S}, \overline{O^2(N)}] = 1$. Coprime action gives $O^2(N) \leqslant C_X(T)O_2(P)$, and so $O^2(N) \leqslant T_0O_2(P)$. Now (*) and 4.3(e) gives $O_2(P) \leqslant O^2(N)$. Hence (b) holds.

Assume now that T contains a Zsigmondy element u. Suppose first that u normalizes $O^p(N)$. By 4.3(a) $\langle u \rangle$ acts irreducibly on $O_p(P)/\Phi(O_p(P))$. Since $O_p(N) \neq O_p(P)$, we get $O_p(N) \leqslant \Phi(O_p(P))$. But then $(*)$ yields

$$1 \neq [O^p(N), O_p(P)] \leqslant \Phi(O_p(P)),$$

which is impossible by coprime action.

Now suppose that u does not normalize $O^p(N)$. Put $M = O^p(N)O^p(N)^u$. Then $M \cap TO_p(P)$ contains an element $h \notin O_p(P)$. Observe that M is normalized by $O_p(P)$, so $[O_p(P), h] \leqslant O_p(M)$. Since by 4.3(e) $[O_p(P), h] = O_p(P)$, we get $O_p(P) = O_p(M) = O_p(N)O_p(N)^u$. As no conjugate of N centralizes $O_p(P)$, we get $N^{\langle u \rangle} = \{N, N^u\}$. But then $u^2 \in N_T(O^p(N))$. Since u has prime order, we get $u^2 = 1$. But then 2 is a Zsigmondy prime, which is impossible.

(c): For any prime p dividing $|N|$ we have $1 \neq O_p(N) \leqslant O_p(P)$. It follows from (a) that N is an p-group and $N \leqslant F^*(P) = O_p(P)$ and that $F^*(G) \cong \mathrm{PSL}_n(p^e)$, $n \geqslant 3$, or $F^*(G)$ is a rank 1 group of Lie type in characteristic p. By 4.3(b) $G \not\cong Ree(3)'$.

In the first case $O_p(P)$ is an irreducible P-module, so $N = O_p(P)$. But this is impossible since is the latter group is not cyclic for $n \geqslant 3$.

In the second case $O_p(P)$ is regular on the points not fixed by P, whence $N = O_p(P) = Z(O_p(P))$. By 4.3(d),(c) $F^*(G) \cong \mathrm{PSL}_2(q)$ and N is elementary abelian of order q. Since N is cyclic we conclude that q is a prime and $N \leqslant F^*(G)$. □

5 5. The Structure of $F^*(G_x^{\Delta(x)})$

Throughout this section we assume the following hypothesis:

Hypothesis 5.1. *The Main Hypothesis holds with $s \geqslant 4$. In in addition, there exists a 2-arc (x, y, z) such that*

$$G_{x,y}^{[1]} = G_x^{[2]} \neq 1 \quad \text{and} \quad G_y^{[2]} = G_{x,y,z}^{[1]} = 1.$$

Moreover, either $G_x^{[2]}$ is abelian or $G_x^{[1]} = G_x^{[2]}$.

A comparison with 3.2 shows that Hypothesis 5.1 corresponds to the case in Theorem 1, where Δ is not of local characteristic p. Moreover, since $s \geqslant 2$ all arcs of length 2 with initial vertex conjugate to x satisfy the hypothesis in place of (x, y, z). We will use this trivial fact without reference.

Also recall that $G_u^{\Delta(u)}$ is a 2-transitive permutation group for all $u \in \mathrm{V}\Delta$ since $s \geqslant 4$.

Notation 5.2. *We fix (x, y, z) with the properties given in Hypothesis 5.1. In addition, we define $L_x := \langle G_z^{[2]G_x} \rangle$. Also recall that $q_w = |\Delta(w)| - 1$ for $w \in \mathrm{V}\Delta$.*

Lemma 5.3. (a) *$G_x^{[1]} \cap G_z^{[2]} = 1$ and $[G_x^{[2]}, G_z^{[2]}] = 1$.*

(b) *$[L_x, G_x^{[1]}] \leqslant G_x^{[2]}$ and $[L_x, G_x^{[2]}] = 1$.*

(c) *$[L_x, L_x] \nleqslant G_x^{[1]}$ and $[L_x, L_x] \leqslant C_{G_x}(G_x^{[1]})$.*

Proof. (a): Each of the subgroups $G_x^{[1]} \cap G_z^{[2]}$ and $[G_x^{[2]}, G_z^{[2]}]$ is contained in $G_{x,y,z}^{[1]}$, and by Hypothesis 5.1 $G_{x,y,z}^{[1]} = 1$.

(b): As $G_x^{[2]} \trianglelefteq G_x$ and $G_{x,y}^{[1]} = G_x^{[2]}$, $[G_x^{[1]}, G_z^{[2]}] \leqslant G_x^{[2]} \trianglelefteq G_x$. Now (b) follows from (a) and the definition of L_x.

(c): From (b) we get $[L_x, G_x^{[1]}, L_x] = [G_x^{[1]}, L_x, L_x] = 1$. Hence, the Three Subgroups Lemma gives $[L_x, L_x, G_x^{[1]}] = 1$.

Assume that $[L_x, L_x] \leqslant G_x^{[1]}$. Then $L_x^{\Delta(x)}$ is abelian and by (a) non-trivial. Hence $G_x^{\Delta(x)}$ is of regular type and $L_x^{\Delta(x)}$ the regular normal subgroup. In particular, $(L_x \cap G_{x,y})^{\Delta(x)} = 1$. But then $G_z^{[2]} \leqslant G_x^{[1]}$, which contradicts (a). □

Lemma 5.4. *$G_x^{\Delta(x)}$ is of simple type.*

Proof. Put $\overline{G_x} := G_x^{\Delta(x)}$ and assume that $\overline{G_x}$ is of regular type. Then $\overline{A} := F^*(\overline{G_x})$ is a minimal normal and regular p-subgroup of $\overline{G_x}$.

1°. $O_p(G_y^{[1]}) = 1$.

Since $\overline{G_x} = \overline{AG_{x,y}}$ and $\overline{A} \cap \overline{G_{x,y}} = 1$, we get first $O_p(\overline{G_{x,y}}) \leqslant \overline{A}$ and then $O_p(\overline{G_{x,y}}) = 1$. We conclude that $O_p(G_{x,y}) \leqslant G_x^{[1]}$ and so by Hypothesis 5.1, $O_p(G_y^{[1]}) \leqslant G_{x,y}^{[1]} = G_x^{[2]}$.

By 5.3(c) $[L_x, L_x]$ centralizes $O_p(G_y^{[1]})$ and $\overline{[L_x, L_x]} \neq 1$. It follows that $N_{G_x}(O_p(G_y^{[1]}))^{\Delta(x)}$ is transitive. Since also $N_{G_y}(O_p(G_y^{[1]}))^{\Delta(y)}$ is transitive, [12, 10.3.3] gives $O_p(G_y^{[1]}) = 1$.

Now let N be the inverse image of $\overline{A}$ in $[L_x, L_x]$. As $\overline{A}$ is abelian, $[N, N] \leqslant G_x^{[1]}$, and by 5.3(c) $[N \cap G_x^{[1]}, N] = 1$. Hence N is nilpotent. Since $\overline{A}$ is a p-group, we get $\overline{O_p(N)} = \overline{A}$.

2°. $\quad N_0 := O_p(N) \cap G_x^{[1]} \nleqslant G_y^{[1]}$.

Assume that $N_0 \leqslant G_y^{[1]}$. Then $N_0 \leqslant O_p(G_y^{[1]})$ and so by (1°) $N_0 = 1$. Let $u \in \Delta(x)\backslash\{y\}$. By [3, 2.4] there exists $g \in O_p(N)$ with $u^g = y$ and $[g, G_{u,x,y}] \leqslant G_x^{[1]}$. Hence

$$[g, G_{u,x,y}] \leqslant O_p(N) \cap G_x^{[1]} = 1.$$

Now let q be a divisor of q_u. Then g centralizes any q-Sylow subgroup of $G_{u,x,y,z}$. Since $s \geqslant 4$, 3.3 shows that $g \in N \cap G_y$, which contradicts $u^g = y$.

From (2°) we get that $N_0 \nleqslant G_y^{[1]}$ and

$$[N_0, G_y^{[1]}] \leqslant N_0 \cap G_y^{[1]} \leqslant O_p(G_y^{[1]}) \overset{(1^\circ)}{=} 1.$$

That is, $N_0 \leqslant C_{G_y}(G_y^{[1]})$ and $C_{G_y}(G_y^{[1]})^{\Delta(y)}$ is transitive. As $G_x^{[2]} \leqslant G_y^{[1]}$, it follows that also $C_{G_y}(G_x^{[2]})^{\Delta(y)}$ is transitive. But then again [12, 10.3.3] shows that $G_x^{[2]} = 1$, which contradicts Hypothesis 5.1. □

Lemma 5.5. *$F^*(G_{x,y}^{\Delta(x)})$ and $G_x^{[2]}$ are p-groups. In particular $O^p(L_x) \leqslant C_{G_x}(G_x^{[1]})$ and $O^p(L_x) \nleqslant G_x^{[1]}$.*

Proof. Put $\overline{G_x} = G_x^{\Delta(x)}$, $N = \overline{G_z^{[2]}}$ and $P = \overline{G_{x,y}}$. Note that $N \trianglelefteq \overline{G_y^{[1]}} \trianglelefteq P$ and by 5.3(a) $N \cong G_z^{[2]}$. Moreover, by 5.4 $\overline{G_x}$ is of simple type.

Assume first that $F^*(P) \leqslant N$. Since $|\Delta(y)| \geqslant 3$ there exists $v \in \Delta(y)\backslash\{x, z\}$. By 5.3(a) $[G_z^{[2]}, G_v^{[2]}] = 1$. Hence $[\overline{G_v^{[2]}}, N] = 1$ and so $[\overline{G_v^{[2]}}, F^*(P)] = 1$, too. It follows that $\overline{G_v^{[2]}} \leqslant Z(F^*(P))$. This shows that $F^*(P) = N = Z(F^*(P))$, and by 4.4(a) both N and $F^*(P)$ are p-groups.

Next assume that $N \leqslant F^*(P)$ and $F^*(P) \neq N$. Since N is subnormal in P, 4.4(a) shows that $F^*(P)$ is a p-group, and we are done.

Thus we may assume that $F^*(P) \nleqslant N$ and $N \nleqslant F^*(P)$. Then by 4.4(b) $F^*(\overline{G_x})$ is a rank 1 group of Lie type in its natural permutation representation in characteristic p on $\Delta(x)$, and $P/O_p(P)$ is a p'-group. In particular, $F^*(P) =$

$O_p(P)$, $q_x = |O_p(P)|$ and $\overline{G_{u,x,y}} \cap O_p(P) = 1$, so $\overline{G_{u,x,y}}$ is a p'-group. On the other hand, 3.4 implies that q_x divides $|\overline{G_{u,x,y,z}}||N|$. It follows that q_x divides $|N|$ and so $F^*(P) = O_p(P) \leqslant N$, a contradiction.

We have shown that $F^*(G_{x,y}^{\Delta(x)})$ and $G_x^{[2]}$ are p-groups. Now the definition of L_x shows that $L_x/[L_x, L_x]$ is a p-group. Hence $O^p(L_x) \leqslant [L_x, L_x]$ and so by 5.3(c) $O^p(L_x) \leqslant C_{G_x}(G_x^{[1]})$. Moreover, by 5.4 $O^p(L_x) \nleqslant G_x^{[1]}$. □

Lemma 5.6. *$F^*(G_x^{\Delta(x)})$ is a non-abelian simple rank 1 group of Lie type in its natural permutation representation in characteristic p.*

Proof. By 5.4 $G_x^{\Delta(x)}$ is of simple type, and by 5.5 $F^*(G_{x,y}^{\Delta(x)})$ is a p-group. Thus, according to 4.4(a) we may assume that $F^*(G_x^{\Delta(x)}) \cong \mathrm{PSL}_n(q)$, $n \geqslant 3$ and $q = p^k$, in its natural permutation representation on the 1-dimensional subspaces of an n-dimensional $\mathbb{F}_q$-vector space. In particular $G_x^{\Delta(x)}$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}_n(q)$.

Put $\overline{G}_x := G_x/G_x^{[1]}$ and let $u \in \Delta(x)\backslash\{y\}$, so (u, x, y, z) is a 3-arc. By 5.5 $G_z^{[2]}$ is a p-group. Since $G_z^{[2]}$ is subnormal in $G_{x,y}$, 5.3(a) shows that $\overline{G_z^{[2]}}$ is a non-trivial subgroup of $O_p(\overline{G_{x,y}})$.

We use the following facts about the action of $\overline{G}_x$ on $\Delta(x)$:

1°. $q_x = q\frac{q^{n-1}-1}{q-1}$.

2°. *There exists $E_{x,y} \trianglelefteq G_{x,y}$ such that $\overline{E}_{x,y}/O_p(\overline{E}_{x,y}) \cong \mathrm{SL}_{n-1}(q)$ and $|\overline{G}_{x,y}/\overline{E}_{x,y}|$ divides $(q-1)k$. Moreover, $O_p(\overline{E}_{x,y})$ is a natural module for $\overline{E}_{x,y}/O_p(\overline{E}_{x,y})$.*

3°. *Either $n = 3$ and $|\overline{G}_{u,x,y}/O_p(\overline{G}_{u,x,y})|$ divides $(q-1)^2k$ or there exists $E_{u,x,y} \trianglelefteq G_{u,x,y}$ such that $\overline{E}_{u,x,y}/O_p(\overline{E}_{u,x,y}) \cong \mathrm{SL}_{n-2}(q)$ and $|\overline{G}_{u,x,y}/\overline{E}_{u,x,y}|$ divides $(q-1)^2k$.*

Let t be a prime dividing q_x with $t \neq p$.

4°. *$G_x^{[1]}$ is a t'-group.*

Pick $w \in \Delta(z)\backslash\{y\}$ and $R \in Syl_t(G_{x,y,z,w})$. There exists an R-invariant Sylow t-subgroup $X \in Syl_t(G_x^{[1]})$. By 3.3 $N_X(R) \leqslant G_x^{[1]} \cap G_z$. On the other hand, 5.3(c) shows that the hypothesis of 3.4 is fulfilled for (x, y, z), so $G_x^{[1]} \cap G_z = G_x^{[2]}$. Since $G_x^{[2]}$ is a p-group and $t \neq p$, we get $N_X(R) = 1$ and so $X = 1$.

5°. *$q^{n-1} = 2^6$ or $q^{n-1} - 1 = p^2 - 1$, p Mersenne prime.*

Suppose that t is a Zsigmondy prime for $((n-1)k, p)$. By 3.8 t does not divide k, and since $n - 1 \geqslant 2$, t does not divide $(q-1)^2$. On the other hand, since $s \geqslant 4$, t divides $|G_{u,x,y}|$ and so by (4°) t divides $|\overline{G_{u,x,y}}|$. Thus, by (3°) $n \geqslant 4$ and t divides $|\mathrm{PSL}_{n-2}(q)|$. The group order of $\mathrm{PSL}_{n-2}(q)$ shows that this is impossible since t is Zsigmondy prime.

We have shown that there are no Zsigmondy primes for $((n-1)k, p)$, and so 3.8 implies (5°).

We now discuss the two cases given in (5°), separately. Suppose first that $q^{n-1} = 2^6$. Then $(n, k) \in \{(3, 3), (4, 2), (7, 1)\}$.

Assume that $(n, k) = (3, 3)$. Then 9 divides q_x and we can choose $t = 3$. By (2°) and (4°) 9^2 does not divide $|G_{x,y}|$ which contradicts $s \geqslant 4$.

Next assume that $(n, k) = (4, 2)$. Then 7 divides q_x and we can choose $t = 7$. By (2°) and (4°) 7^2 does not divide $|G_{x,y}|$ which contradicts $s \geqslant 4$.

Finally assume that $(n, k) = (7, 1)$. Then 7 divides q_x and we can choose $t = 7$. Let $R \in Syl_7(G_{x,y})$. Then by (2°) and (4°)

$$G_x = L_x G_x^{[1]},\ \overline{E}_{x,y}/O_2(\overline{E}_{x,y}) \cong \mathrm{SL}_6(2), |R| = 7^2.$$

In particular $1 \neq \overline{G_z^{[2]}} \leqslant O_2(\overline{G_{x,y}}) = O_2(\overline{E_{x,y}})$.

By 5.3(b) L_x centralizes $G_x^{[2]}$ and by (4°) $G_x^{[1]}$ is a $7'$-group. Hence $O^{7'}(G_x) \leqslant C_{G_x}(G_x^{[2]})$. Since $s \geqslant 4$, $q_x^2 q_y$ divides $|G_{x,y}|$. As $|R|$ divides q_x^2, q_y is not divisible by 7. Thus we may assume that $R \leqslant G_z$. Then $R \leqslant O^{7'}(G_z) \leqslant C_{G_z}(G_z^{[2]})$. Since $1 \neq \overline{G_z^{[2]}} \leqslant O_2(\overline{E}_{x,y})$, we conclude that $\overline{R}$ has a fixed point in $O_2(\overline{E}_{x,y})$. But $O_2(\overline{E}_{x,y})$ is a natural $\mathrm{SL}_6(2)$-module, and no Sylow 7-subgroup has a fixed point in this module. This contradiction shows that the first case in (5°) does not occur.

Assume now that $q^{n-1} - 1 = p^2 - 1$, p Mersenne prime. Then $q = p = 2^r - 1$, for some $r \geqslant 2$, $q_x = p+1 = 2^r$ and $p-1 = 2(2^{r-1}-1)$. By (2°) $G_x^{[1]}$ is a $2'$ group and by (3°) $|\overline{G}_{u,x,y}/O_p(\overline{G}_{u,x,y})|$ divides $(p-1)^2$. Since q_x divides $|G_{u,x,y}|$ we have that $p+1$ divides $(p-1)^2$. Thus we have $r = 2$ and so $p = 3$ and $q_x = 12$. In particular $\overline{G}_x \cong \mathrm{PSL}_3(3)$ and $t = 2$. Hence by 5.5 $G_x = C_{G_x}(G_x^{[1]})G_x^{[1]}$ and so again by (4°), $O^{2'}(G_x) \leqslant C_{G_x}(G_x^{[1]})$. Let $R \in Syl_2(G_{x,y})$. Note that $|\overline{G_{x,y}}| = 3^2|\mathrm{GL}_3(3)| = 3^3 \cdot 2^4$ and that $q_x^2 q_y$ divides $|G_{x,y}|$ since $s \geqslant 4$. It follows that q_y divides $3|G_x^{[1]}|$ and so by (4°) q_y is odd. Thus R fixes a vertex in $\Delta(y)\backslash\{x\}$, and we may assume that $R \leqslant G_z$. Then $R \leqslant O^{2'}(G_z) \leqslant C_{G_z}(G_z^{[1]})$. In particular $1 \neq \overline{G_z^{[2]}} \leqslant O_2(\overline{E_{x,y}})$ is centralized by $\overline{R}$. But $|R| = 4^2$ and by (3°) $O_2(\overline{E_{x,y}})$ is a natural module for $\overline{E_{x,y}}/O_2(\overline{E}_{x,y})$, and no Sylow 2-subgroup of $\mathrm{GL}_2(3)$ has a fixed point on its natural module. This contradiction shows that also the second case in (5°) does not occur. □

6 6. The Type of the Vertex Stabilizer Amalgam.

In this section we assume Hypothesis 5.1. In addition we use 5.6 without reference. That is, $F^*(G_x^{\Delta(x)})$ is a non-abelian simple rank 1 group of Lie type in its natural permutation representation in characteristic p.

The basic facts about these permutation groups we have already collected in 4.2, more information relevant for us can be found in 4.3. We will use Notation 4.1 and the following additional notation.

Notation 6.1 (Notation for rank 1 Groups). *We fix $u \in \Delta(x)\backslash\{y\}$, so (u,x,y,z) is a 3-arc. We also fix the prime p given by 5.6, and q is the order of the field of definition for $F^*(G_x^{\Delta(x)})$. That is, $F^*(G_x^{\Delta(x)}) \cong \mathrm{PSL}_2(q)$, $\mathrm{Sz}(q)$, $\mathrm{PSU}_3(q)$ or $\mathrm{Ree}(q)'$. Let*

$$L := O^p(L_x),\, T := O^p(L \cap G_{u,x,y}),\ H := G_{u,x,y,z},\ D := C_H(T),$$

$$L_y := \langle T^{G_y}\rangle,\, Q := O_p(G_y^{[1]}),\ V := \Omega_1(Z(Q)),\ V_0 := [V,T],\, Z_0 := C_V(T).$$

As before we begin with some elementary consequences.

Lemma 6.2. (a) *q_x is a power of p and $G_z^{[2]} \leqslant Q \leqslant O_p(G_{x,y})$.*

(b) *$L \leqslant C_{G_x}(G_x^{[1]})$, $L \nleqslant G_x^{[1]}$ and $G_x^{[1]} \cap G_z = G_x^{[2]}$.*

(c) *$L^{\Delta(x)} = F^*(G_x^{\Delta(x)})$ is a non-abelian simple group, and $T^{\Delta(x)}$ is a torus of $F^*(G_x^{\Delta(x)})$.*

(d) *$L \cap G_{u,x,y} = TZ(L)$ and $Z(L) = L \cap G_x^{[1]}$.*

(e) *$TZ(L)$ is transitive on $\Delta(y)\backslash\{x\}$.*

(f) *$Z(L)$ is a p-group, T is a cyclic p'-group, and $T = O^p(TZ(L))$. In particular $TZ(L)$ is abelian and $T \cap Z(L) = 1$.*

(g) *$T \trianglelefteq G_{u,x,y}$ and $T \cap G_x^{[1]} = T \cap G_z = 1$.*

Proof. (a): The first part follows from 4.2, the second part from 5.5.

(b): The first two statements follow from the definition of L and 5.5, the third one then from 3.4.

(c): Put $\overline{G}_x := G_x^{\Delta(x)}$. Since $F^*(\overline{G}_x)$ is a non-abelian simple group and since $L \nleqslant G_x^{[1]}$, we have $F^*(\overline{G}_x) \leqslant \overline{L}$. Now 4.3(b) shows that $p = 3$ and $\overline{G}_x \cong \mathrm{Ree}(3)$, or $O_p(\overline{G_{x,y}}) \leqslant F^*(\overline{G}_x) \leqslant \overline{L}$. In the first case $\overline{L} = F^*(\overline{G}_x)$. In the second case (a) shows that $\overline{G_z^{[2]}} \leqslant F^*(\overline{G}_x)$, and so by the definition of L_x, $\overline{L}_x = F^*(\overline{G}_x) = \overline{L}$.

By the definition of T and 4.2 also the second part of (c) follows.

(d): By (c) $T^{\Delta(x)}$ is a torus of $F^*(G_x^{\Delta(x)}) = L^{\Delta(x)}$. Hence $L \cap G_{u,x,y} = T(L \cap G_x^{[1]})$, and (b) implies (d).

(e): By (c) $L^{\Delta(x)} = F^*(G_x^{\Delta(x)})$ is a non-abelian simple group, and so by [3, 2.4] $L^{\Delta(x)}$ is generously transitive. Hence, by 3.5 $L \cap G_{u,x,y}$ is transitive on $\Delta(y)\backslash\{x\}$. Now (d) implies (e).

(f): By (d) $Z(L) = L \cap G_x^{[1]}$. In particular $[Z(L),Q] \leqslant Z(L) \cap Q \leqslant O_p(Z(L))$. Hence $O^p(Z(L))$ centralizes Q and so by (a) also $G_z^{[2]}$. Thus $O^p(Z(L)) \leqslant G_z$.

The transitivity of $G_{x,y}$ on $\Delta(y)\backslash\{x\}$ gives $O^p(Z(L)) \leqslant G_y^{[1]}$. Now Hypothesis 5.1 implies $O^p(Z(L)) \leqslant G_x^{[1]} \cap G_y^{[1]} = G_x^{[2]}$. As $G_x^{[2]}$ is a p-group, $O^p(Z(L)) = 1$.

This shows that $Z(L)$ is a p-group. By (c) $T^{\Delta(x)}$ is a cyclic p'-group. In particular by (d), $TZ(L)$ is abelian and $O^p(TZ(L)) = T$ is also a cyclic p'-group.

(g): By (d) $TZ(L) \trianglelefteq G_{u,x,y}$, so by (f) $T \trianglelefteq G_{u,x,y}$. Moreover, by (d) $T \cap G_x^{[1]} \leqslant L \cap G_x^{[1]} = Z(L)$, and by (f) $T \cap Z(L) = 1$. Hence $T \cap G_x^{[1]} = 1$.

Put $R := T \cap G_z$. By (f) R is a p'-group and by (e) $R = T \cap G_y^{[1]}$. On the other hand, by (a) q_z is a power of p. Thus, R has a fixed point in $\Delta(z)\backslash\{y\}$, and since $R \trianglelefteq H$ and H is transitive on $\Delta(z)\backslash\{y\}$, we get $R \leqslant G_z^{[1]}$. But then by Hypothesis 5.1 $R \leqslant G_y^{[1]} \cap G_z^{[1]} = G_z^{[2]}$. Now again (a) shows that R is a p-group and so $R = 1$. $\square$

Lemma 6.3. (a) *$Z(L) = L \cap G_x^{[2]} = TZ(L) \cap G_z$.*

(b) *T is a cyclic p'-group regular on $\Delta(y)\backslash\{x\}$.*

(c) *$|T| = q_y$ and $C_{G_{x,y}}(T) \cap G_z = C_{G_y^{[1]}}(T)$.*

(d) *$Q = O_p(G_{x,y})$ and $Q^{\Delta(x)} = O_p(G_{x,y}^{\Delta(x)})$. In particular Q is transitive on $\Delta(x)\backslash\{y\}$ and $T^{\Delta(y)} \trianglelefteq G_{x,y}^{\Delta(y)}$.*

(e) *$G_y^{\Delta(y)}$ is of regular type or $L_y^{\Delta(y)} \cong \mathrm{PSL}_2(q_y)$, q_y an odd prime and $q_y \geqslant 5$.*

(f) *$C_{G_{u,x,y}}(T^{\Delta(x)}) = C_{G_{u,x,y}}(T^{\Delta(y)}) = C_{G_{u,x,y}}(T)$ and $C_{G_{u,x,y}}(T) = TD$.*

Proof. (a): By 6.2(e) $TZ(L)$ is transitive on $\Delta(y)\backslash\{x\}$, and by 6.2(f) $TZ(L)$ is abelian. Hence

$$(*) \qquad R := TZ(L) \cap G_z = TZ(L) \cap G_y^{[1]} \text{ and } Z(L) \cap G_z \leqslant R.$$

By Hypothesis 5.1 $G_x^{[1]} \cap G_y^{[1]} = G_x^{[2]}$ and by 6.2(d) $Z(L) = L \cap G_x^{[1]}$. Hence (a) follows from $(*)$ if we can show that $R \leqslant Z(L)$ and $Z(L) \leqslant G_z$.

By 6.2(f) T is the unique maximal p'-subgroup of $TZ(L)$, and $O_p(TZ(L)) = Z(L)$. Since by 6.2(g) $T \cap G_z = 1$, this shows R is a p-group and so $R \leqslant O_p(TZ(L)) = Z(L)$. Assume that $Z(L) \nleqslant G_z$. Since two distinct G_x-conjugates of $G_z^{[2]}$ have trivial intersection, we get $G_z^{[2]} \cap LG_x^{[1]} = 1$. On the other hand, by 6.2(a) $G_z^{[2]} \leqslant O_p(G_{x,y})$ and so

$$O_p(G_{x,y}^{\Delta(x)}) \nleqslant (L \cap G_{x,y})^{\Delta(x)}.$$

Since by 6.2(c) also $(L \cap G_{x,y})^{\Delta(x)} \nleqslant O_p(G_{x,y}^{\Delta(x)})$, we get from 4.3(b) that $G_x^{\Delta(x)} \cong \mathrm{Ree}(3)$. In particular $|G_z^{[2]}| = 3$ and $q_x = 27$. Moreover, 6.2(b) shows that we can apply 3.4. Thus

$$|G_{x,y,z}| = |G_{x,y,z}^{\Delta(x)}||G_x^{[2]}| \leqslant 3^3 \cdot 2 \cdot 3.$$

But $s \geqslant 4$ and so $q_x^2 = 3^6$ divides $|G_{x,y,z}|$, a contradiction. Hence (a) is proved.

(b): As already used in (a), $TZ(L)$ is transitive on $\Delta(y)\backslash\{x\}$. Now (b) follows from 6.2(f) and (a).

(c): This is a direct consequence of (b).

(d): We first show that $Q^{\Delta(x)} = O_p(G_{x,y}^{\Delta(x)})$.

By (b) $O_p(G_{x,y})$ fixes a vertex in $\Delta(y)\backslash\{x\}$. Since $G_{x,y}$ normalizes $O_p(G_{x,y})$, we get $O_p(G_{x,y}) \leqslant G_y^{[1]}$ and so $O_p(G_{x,y}) = Q$. In particular $Q^{\Delta(x)} \leqslant O_p(G_{x,y}^{\Delta(x)})$.

Let $Q \leqslant Q_0 \trianglelefteq G_{x,y}$ be minimal with $Q_0^{\Delta(x)} = O_p(G_{x,y}^{\Delta(x)})$. Then $Q_0 = O^{p'}(Q_0)$. It suffices to show that $Q_0 \leqslant O_p(G_{x,y})$.

Clearly, $Q_0 = Q_1(Q_0 \cap G_x^{[1]})$, where $Q_1 \in Syl_p(Q_0)$. Thus, by 6.2(b) $Q_0 = Q_1 C_{Q_0}(T)$ and

$$[Q_0, T] = [Q_1, T] \leqslant Q_0 \cap L.$$

Moreover, since by 6.2(d) $L \cap G_x^{[1]} = Z(L)$ and by 6.2(f) $Z(L)$ is a p-group, we get that

$$[Q_1, T] \leqslant Q_0 \cap L \leqslant O_p(L \cap G_y) \leqslant Q \leqslant Q_1.$$

Hence Q_1 is T-invariant, and coprime action gives $Q_1 = C_{Q_1}(T)[Q_1, T] \leqslant C_{Q_1}(T)Q$. It follows that $Q_0 = C_{Q_0}(T)Q$.

Since $Q_0 = O^{p'}(Q_0)$ also $O^{p'}(C_{Q_0}(T)) = C_{Q_0}(T)$. On the other hand, by (b) and (c) $C_{Q_0}(T) \leqslant TG_y^{[1]}$ and so since T is a p'-group, $C_{Q_0}(T) = O^{p'}(C_{Q_0}(T)) \leqslant G_y^{[1]}$. In particular $C_{Q_0}(T) \cap G_x^{[1]} = G_{x,y}^{[1]} = G_x^{[2]} \leqslant Q$. It follows that $C_{Q_0}(T)$ is a p-group. Hence also Q_0 is a p-group and $Q_0 \leqslant O_p(G_{x,y})$.

We have shown that $Q^{\Delta(x)} = O_p(G_{x,y}^{\Delta(x)})$. As $O_p(G_{x,y}^{\Delta(x)})$ is transitive on $\Delta(x)\backslash\{y\}$ also Q is. Then $G_{x,y} = QG_{u,x,y}$ and by 6.2(g) $TQ \trianglelefteq G_{x,y}$. Thus, since $Q \leqslant G_y^{[1]}$, we have $T^{\Delta(y)} \trianglelefteq G_{x,y}^{\Delta(y)}$.

(e): Suppose that $G_y^{\Delta(x)}$ is of simple type. According to (b) and (d), $|T^{\Delta(x)}| = q_y$ and $T^{\Delta(y)}$ is a normal cyclic subgroup of $G_{x,y}^{\Delta(x)}$ regular on $\Delta(y)\backslash\{x\}$. Hence 4.4(c) gives $L^{\Delta(y)} = F^*(G_y^{\Delta(y)}) \cong \mathrm{PSL}_2(q_y)$, where $q_y \geqslant 5$ is an odd prime.

(f): By 6.2(g) $T \trianglelefteq G_{u,x,y}$ and $T \cap G_x^{[1]} = 1$, and by 6.3(b) $T \cap G_y^{[1]} = 1$. This shows that

$$[C_{G_{u,x,y}}(T^{\Delta(y)}), T] = [C_{G_{u,x,y}}(T^{\Delta(x)}), T] = 1,$$

and the first part of (f) follows.

Since by (b) $G_{u,x,y} = TH$, we get $C_{G_{u,x,y}}(T) = TC_H(T) = TD$, and this is the second part of (f). □

Lemma 6.4. *Let $q = p^r$. Then $|H/D|$ divides r and $D/G_x^{[2]}$ is a p'-group. In particular, q_x divides $r|G_x^{[2]}|$ and q_y divides $r|D/G_x^{[2]}|$.*

Proof. Observe that $C_{H^{\Delta(x)}}(T^{\Delta(x)}) = C_H(T^{\Delta(x)})/G_x^{[1]}$. Thus, using the isomorphism theorems

$$(*) \qquad H^{\Delta(x)}/C_{H^{\Delta(x)}}(T^{\Delta(x)}) \cong H/C_H(T^{\Delta(x)}).$$

By 6.3(f) $C_H(T^{\Delta(x)}) = D$, so the right hand side of $(*)$ is isomorphic to H/D. By Table 2 of 4.2 the order of the left hand side divides r and $D^{\Delta(x)}$ is a p'-group. The latter fact also shows that $O^{p'}(D) = G_x^{[1]} \cap G_y^{[1]} = G_x^{[2]}$, so $D/G_x^{[2]}$ is a p'-group.

Now observe that $q_x q_y$ divides $|H|$ and that by 6.2(a) q_x is a power of p while by 6.3(b) q_y is coprime to p. Hence also the divisor conditions hold since $G_x^{[2]}$ is a p-group. □

Lemma 6.5. (a) *$|V_0| = q$ and T is irreducible on V_0.*

(b) *$V = V_0 \times Z_0$, $Z_0 = V \cap G_x^{[1]} = V \cap G_x^{[2]} \trianglelefteq G_{x,y}$ and $|Z_0| \leqslant |V_0| = q$.*

(c) *$C_T(V) = 1$.*

(d) *Either $q_y = \frac{q-1}{2}$ and p is odd, or $q_y = q - 1$.*

Proof. (a) and (b): By 6.3(d) and 4.3(c),(g) $|\Omega_1(Z(Q^{\Delta(x)}))| = q$ and T acts irreducibly on $\Omega_1(Z(Q^{\Delta(x)}))$. This shows that either $V^{\Delta(x)} = 1$ or $V^{\Delta(x)} = V_0^{\Delta(x)} = \Omega_1(Z(Q)^{\Delta(x)})$.

In the first case $V \leqslant G_x^{[1]}$, so by 6.2(a) V is centralized by L. Since V is normal in G_y and $L^{\Delta(x)}$ is transitive, [3, 4.4] shows that $V = 1$. But then $Q = 1$ which contradicts $G_x^{[2]} \leqslant Q$.

Thus, we have $V_0^{\Delta(x)} = \Omega_1(Z(Q)^{\Delta(x)})$ and $|V_0^{\Delta(x)}| = q$. Since T is a p'-group which centralizes $G_x^{[1]}$, we get $V = V_0 \times Z_0$, $|V_0| = q$ and $Z_0 = V \cap G_x^{[1]} \trianglelefteq G_{x,y}$, and T is irreducible on V_0. Moreover, by Hypothesis 5.1 $G_{x,y}^{[1]} = G_x^{[2]}$ and so $Z_0 = V \cap G_x^{[2]}$. Since $G_x^{[2]G_y}$ is a conjugacy class of TI-subgroups, $|Z_0| \leqslant |V_0| = q$.

(c): By 6.2(a) $G_z^{[2]} \leqslant Q$. Since $G_z^{[2]}$ is a non-trivial normal subgroup of $G_y^{[1]}$, we get $V \cap G_z^{[2]} \neq 1$. Hence $C_T(V) \leqslant G_z$ and by 6.3(b) $C_T(V) = 1$.

(d): As seen above $V_0^{\Delta(x)} = \Omega_1(Z(Q^{\Delta(x)}))$ and $|V_0| = q$. Moreover by (b) and (c) $C_T(V) = C_T(V_0) = 1$. Hence, 4.3(f) excludes the case $L^{\Delta(x)} \cong \mathrm{PSU}_3(q)$, and Table 2 of 4.2 gives (d). □

Lemma 6.6. (a) *$D \cap LG_x^{[1]} \nleqslant G_x^{[2]}$.*

(b) *$G_x^{[1]} \nleqslant G_y^{[1]}$. In particular $G_x^{[1]} \neq G_x^{[2]}$.*

(c) *$G_x^{[2]}$ is abelian.*

(d) *$V_0 \cap G_e^{[2]} = 1$ for all $e \in \Delta(y) \backslash \{x\}$.*

Proof. (a): First observe that by 6.2(b),(d) $TZ(L)G_x^{[1]} \leqslant C_{G_{u.x,y}}(T)$. This gives

$$G_{u,x,y} \cap LG_x^{[1]} = (G_{u,x,y} \cap L)G_x^{[1]} = TZ(L)G_x^{[1]} \leqslant C_{G_{u.x,y}}(T).$$

Hence $H \cap LG_x^{[1]} = D \cap LG_x^{[1]}$.

Assume that $H \cap LG_x^{[1]} \leqslant G_x^{[2]}$. Since $G_x^{[2]}$ centralizes T, $H \cap LG_x^{[1]} = G_x^{[2]}$. Then $H/H \cap LG_x^{[1]} = H/G_x^{[2]}$, and the isomorphism theorem shows that $|H/G_x^{[2]}|$ divides $|G_x/LG_x^{[1]}|$. By 6.3(c), 6.5(d) and Table 2 of 4.2 $|G_x/LG_x^{[1]}|$ divides $2r$, where $q = p^r$, and so $|H/G_x^{[2]}|$ divides $2r$.

As $s \geqslant 4$, q_y divides $|H|$ and by 6.3(b) q_y is prime to p. Since $G_x^{[2]}$ is a p-group, q_y divides $|H/G_x^{[2]}|$ and so q_y divides $2r$. But then again by 6.5(d)

$$\frac{p^r - 1}{2} \leqslant q_y \leqslant 2r.$$

It follows that $p = 2$ and $r = 1$, $p = 3$ and $r = 1$ or 2, or $p = 5$ and $r = 1$. If $r = 1$ and $p \neq 5$ then $G_x^{\Delta(x)}$ is of regular type, which contradicts 5.4.

Assume that $p = 5$. Then $|H^{\Delta(x)}| = |T^{\Delta(x)}| = 2$ and $G_x^{\Delta(x)} \cong \mathrm{PGL}_2(5)$. Since H normalizes T and $T \cap H = 1$, $G_{u,x,y}^{\Delta(x)}$ contains an elementary abelian group of order 4. But 2-point stabilizers in $\mathrm{PGL}_2(5)$ are cyclic, a contradiction.

Thus we have $p = 3$ and $r = 2$. In particular $q_y = 4 = |T|$ and $|H/D| = 2 = |D/G_x^{[2]}|$. Moreover, by 6.3(e) $G_y^{\Delta(y)}$ is of regular type and permutation degree $q_y + 1 = 5$, so $G_y^{\Delta(y)} \cong Fb(20)$. Hence $H^{\Delta(y)} = 1$ and so $[T, H] \leqslant G_y^{[1]}$. But then by 6.3(f) $H = C_H(T^{\Delta(y)}) = C_H(T) = D$, which contradicts $|H/D| = 2$. This contradiction shows that $D \cap LG_x^{[1]} \neq G_x^{[2]}$.

(b): Assume that $G_x^{[1]} \leqslant G_y^{[1]}$. Then by Hypothesis 5.1 $G_x^{[1]} = G_x^{[2]}$. Hence $H \cap LG_x^{[1]} = H \cap LG_x^{[2]} = (H \cap L)G_x^{[2]}$, and by 6.2(d) $H \cap L = TZ(L) \cap G_z$. Now 6.3(a) gives $H \cap L \leqslant G_x^{[2]}$ and so $H \cap LG_x^{[1]} = G_x^{[2]}$, which contradicts (a).

(c): This follows from (b) and Hypothesis 5.1.

(d): Assume that $V_0 \cap G_e^{[2]} \neq 1$. By 6.2(b) $G_x^{[1]}$ centralizes $V_0 \cap G_e^{[2]}$ and so $G_x^{[1]} \leqslant G_e$. The transitive action of $G_{x,y}$ on $\Delta(y) \backslash \{x\}$ gives $G_x^{[1]} \leqslant G_y^{[1]}$. This contradicts (b). □

Lemma 6.7. $L^{\Delta(x)} \cong \mathrm{PSL}_2(q)$.

Proof. Since $s \geqslant 4$, q_x divides $|H|$ and so by 6.4 q_x divides $r|G_x^{[2]}|$, where $q = p^r$. Since $r < q$ we get $q^{-1}q_x < |G_x^{[2]}|$.

By 6.3(c) $|T| = q_y$ and by 6.5(d) either $q_y = q - 1$, or p is odd and $q_y = \frac{q-1}{2}$. Suppose $L^{\Delta(x)} \ncong \mathrm{PSL}_2(q)$. Then by Table 2 of 4.2 $q_y = q - 1$ and either

$$L^{\Delta(x)} \cong \mathrm{Sz}(q),\ p = 2 \text{ and } q_x = q^2, \text{ or } L^{\Delta(x)} \cong \mathrm{Ree}(q)',\ p = 3 \text{ and } q_x = q^3.$$

On the other hand, by 6.6(c) $G_x^{[2]}$ is abelian. Let $W := \langle (G_x^{[2]})^{G_y}\rangle$. Then also W is abelian and $W \trianglelefteq G_{x,y}$. Now 4.3(d) shows that either $|W^{\Delta(x)}| \leqslant q$ and $q_x = q^2$ or $|W^{\Delta(x)}| \leqslant q^2$ and $q_x = q^3$. In both cases $|G_x^{[2]}| \leqslant |W^{\Delta(x)}| \leqslant q^{-1}q_x$, a contradiction to the above inequality. Thus $L^{\Delta(x)} \cong \mathrm{PSL}_2(q)$. □

Lemma 6.8. (a) *$Q = V$, $G_x^{[2]} \leqslant V$, and V is transitive on $\Delta(x)\backslash\{y\}$.*

(b) *$H \cap G_y^{[1]} = D \nleqslant G_x^{[2]}$, $G_y^{[1]} = DV$ and $[G_y^{[1]}, L_y] \leqslant V$.*

(c) *$D/G_x^{[2]}$ is a cyclic p'-group, and $C_V(d) = 1$ for all $d \in D\backslash G_x^{[2]}$.*

(d) *$C_{G_y}(V) = V$ and $Z(L) = 1$.*

Proof. (a): By 6.7 $L^{\Delta(x)} \cong \mathrm{PSL}_2(q)$ and by 4.3(b) $Q^{\Delta(x)} \leqslant L^{\Delta(x)}$, so $Q^{\Delta(x)}$ is elementary abelian. Hence $\Phi(Q)$ is normalized by L and G_y, and so by [3, 4.4] $\Phi(Q) = 1$. Thus $Q = V$, and by 6.2(a) $G_x^{[2]} \leqslant V$. The last part of (a) follows from 6.3(d).

(b): By 6.3(c) $D = C_H(T) \leqslant C_{G_{x,y}}(T) \cap G_z \leqslant G_y^{[1]}$. Moreover, by 6.6(a) $D \nleqslant G_x^{[2]}$. Conversely, by (a) V is transitive on $\Delta(x)\backslash\{y\}$, and so $G_y^{[1]} = V(G_y^{[1]} \cap G_u) = V(G_y^{[1]} \cap H)$. By 6.2(g) H normalizes T and $T \cap G_y^{[1]} \leqslant T \cap G_z = 1$. Hence $[T, G_y^{[1]} \cap H] \leqslant T \cap G_y^{[1]} = 1$ and $G_y^{[1]} \cap H \leqslant D$. This gives $G_y^{[1]} \cap H = D$ and $G_y^{[1]} = DV$. Since $[G_y^{[1]}, T] \leqslant V$, also $[G_y^{[1]}, L_y] \leqslant V$.

(c): By 6.2(c) $D^{\Delta(x)}$ is contained in a torus of $G_x^{\Delta(x)}$. Hence by 4.2 $D^{\Delta(x)}$ is a cyclic p'-group. Since by (b) $D \cap G_x^{[1]} = G_x^{[1]} \cap G_y^{[1]} = G_x^{[2]}$, also $D/G_x^{[2]}$ is a cyclic p'-group.

Let $d \in D\backslash G_x^{[2]}$. As seen above $1 \neq d^{\Delta(x)}$ is contained in a torus of $G_x^{\Delta(x)}$. Hence 6.7 and 4.3(f) show that $[V^{\Delta(x)}, d^{\Delta(x)}] = V^{\Delta(x)}$. It follows that $V = [V, d](V \cap G_x^{[1]})$ and then by 6.5(b) $V = [V, d]Z_0$. As T normalizes $[V, d]$ and $C_V(d)$ we get $V_0 \leqslant [V, d]$ and $C_V(d) \leqslant Z_0$. Since V is elementary abelian, (b) implies that L_y normalizes $C_V(d)$. Hence $C_V(d) \leqslant G_y^{[2]} = 1$.

(d): By (b) $G_y^{[1]} = DV$ and so by (c) $C_{G_y}(V) \cap G_y^{[1]} = V$. Moreover, $C_{G_y}(V)^{\Delta(y)} = 1$ since $G_x^{[2]} \leqslant V$. Hence $C_{G_y}(V) = V$.

By 6.3(a) $Z(L) \leqslant V$, by 6.2(b) $Z(L)$ centralizes $D \cap LG_x^{[1]}$, and by 6.6(a) $D \cap LG_x^{[1]} \nleqslant G_x^{[2]}$. Hence (c) gives $Z(L) = 1$. □

Lemma 6.9. $L \cong \mathrm{PSL}_2(q)$ *and either*

(i) *$p = 2$, $q_y = q - 1$ and $V_0 \trianglelefteq G_y$, or*

(ii) *p is odd and $q_y = \frac{q-1}{2}$.*

Proof. By 6.7 $L^{\Delta(x)} \cong \mathrm{PSL}_2(q)$ and by 6.8(d) $Z(L) = 1$. Hence 6.2(d) gives $L \cong \mathrm{PSL}_2(q)$. Then 6.5(d) and Table 2 of 4.2 imply that either $p = 2$ and $q_y = q - 1$ or p is odd and $q_y = \frac{q-1}{2}$. In the latter case (ii) holds.

Assume that $p = 2$ and $q_y = q - 1$. As we have already seen in 6.5(b), $V = V_0 \times Z_0$, $Z_0 = V \cap G_x^{[2]} \trianglelefteq G_{x,y}$ and $|Z_0| \leqslant |V_0| = q$. Put $\Lambda := Z_0^{G_y}$ and $\Lambda^* := \bigcup_{X \in \Lambda} X$.

Since $Z_0 \trianglelefteq G_{x,y}$, we get that $|\Lambda| = q_y + 1$ and $\Lambda = \{Z_0\} \cup A^{G_{x,y}}$, where $A \in \Lambda \backslash \{Z_0\}$. Moreover, since $G_x^{[2]G_y}$ is a conjugacy class of TI-subgroups, the same holds for Λ. Thus, we can count the elements in Λ^* easily:

$$|\Lambda^*| = (q_y + 1)(|Z_0| - 1) + 1 = q|Z_0| - (q - 1),$$

and so

$$|V \backslash \Lambda^*| = |V_0||Z_0| - q|Z_0| + (q - 1) = q - 1.$$

By 6.3(a) and 6.8(d) $V_0 \cap G_x^{[2]} = 1$. This together with 6.6(d) gives $V_0 \cap \Lambda^* = \{1\}$ and so $V_0 \backslash \{1\} = V \backslash \Lambda^*$. Since G_y normalizes $V \backslash \Lambda^*$, we get $V_0 \trianglelefteq G_y$, and (i) holds. □

Let V be a $\mathbb{F}_p$-vector space and X a group acting on V. By $\mathrm{End}_X(V)$ we denote the ring of endomorphisms of V which commute with the action of X.

Lemma 6.10. *Let $\mathbb{F} := \mathrm{End}_{L_yD}(V)$. Then $\mathbb{F} \cong \mathbb{F}_q$, and V is a 2-dimensional $\mathbb{F}_q L_y D$-module. Moreover, the elements of $G_{x,y}$ act $\mathbb{F}$-semi-linearly on V, and each non-trivial D-orbit of elements of V generates a 1-dimensional $\mathbb{F}_q$ space.*

Proof. Let $\overline{G}_y := G_y/V$. By 6.8(d) $\overline{G}_y$ acts faithfully on V, by 6.8(a) $G_x^{[2]} \leqslant V$, and by 6.8(c) $\overline{D}$ is cyclic and acts elementwise fixed-point-freely on V. Also observe that each element of $\mathbb{F}$ normalizes $C_V(T) = Z_0$. Hence we get embeddings

$$\overline{D} \xrightarrow{\tau} \mathbb{F} \xrightarrow{\sigma} \mathrm{End}_D(Z_0),$$

where τ is injective and σ is a ring homomorphism. It suffices to show that σ is bijective and that $\mathrm{End}_D(Z_0) \cong \mathbb{F}_q$. Then $G_{x,y}$ induces field automorphisms in $\mathbb{F}$ and acts $\mathbb{F}$-semilinearly on V.

Let $x \in \mathbb{F}$ such that $x\sigma = 0$. Then $Z_0 \leqslant \mathit{kern}\ x$ since $x\sigma$ is the restriction of x to Z_0. Since T acts on $\mathit{kern}\ x$ and is irreducible on V/Z_0 either $\mathit{kern}\ x = Z_0$ or $\mathit{kern}\ x = V$. In the first case Z_0 is L_yD-invariant. But then $Z_0 \leqslant G_x^{[2]} \cap G_z^{[2]} = 1$, a contradiction since $Z_0 \neq 1$. Thus, $\mathit{kern}\ x = V$ and so $x = 0$. This shows that σ is injective.

Let W be an irreducible D-submodule of Z_0 and $\mathbb{K} := \mathrm{End}_D(W)$. Then $\overline{D}$ embeds into $\mathbb{K}$, and we can identify $\overline{D}$ with its image, so $\overline{D} \leqslant \mathbb{K}$. Schur's lemma shows that $\mathbb{K}$ is a finite division ring. Since $\overline{D}$ is abelian it generates a finite subfield $\mathbb{K}_0$ of $\mathbb{K}$.

By 6.4 q_y divides $r|\overline{D}|$, where $q = p^r$. Let $r_y = (r, q-1)$. Hence $r_y^{-1} q_y$ divides $|\overline{D}|$ and so by 6.5(d) $r_y^{-1}(q-1)$ divides $2|\mathbb{K}_0^*|$. Now 3.9 gives $|\mathbb{K}_0| \geqslant q$ or $q = 9$ and $|\mathbb{K}_0| = 3$.

Assume first that $q = 9$ and $|\mathbb{K}_0| = 3$. Then $|\overline{D}| = 2$ and $T \cong C_4$. Let $w \in N_L(T)$ with $y^w = u$. Since $(TD)^{\Delta(x)}$ is contained in a torus of $G_x^{\Delta(x)}$, 4.2 shows that $(TD)^{\Delta(x)}$ is cyclic. Hence $D \leqslant \Omega_1(T)G_x^{[1]} \leqslant C_{G_x}(w)$. But then $D \leqslant G_u^{[1]}$ and so H/D induces a transitive group on $\Delta(u)\backslash\{x\}$. But $|H/D| = 2$ while $q_y = q_u = 4$, a contradiction.

Thus we have $|\mathbb{K}_0| \geqslant q$, so

$$q \leqslant |\mathbb{K}_0| \leqslant |\mathbb{K}| \leqslant |W| \leqslant |Z_0|.$$

Now 6.5(b) gives $|Z_0| = |W| = q$ and $\mathbb{K}_0 = \mathbb{K} \cong \mathbb{F}_q$. Since $\mathbb{K}_0$ is generated by $\overline{D}$, we also have that σ is surjective and so $\mathbb{F} \cong \mathbb{K}$. Since $\mathbb{K}_0 = \mathbb{K}$ the D-orbit of each non-trivial vector generates a 1-dimensional $\mathbb{F}_q$-subspace. □

Notation 6.11.

$$B := (L_y \cap G_{x,y})G_y^{[1]},\ G_x^* := LB,\ G_y^* := L_yB,\ d := \frac{q(q-1)}{|G_x^{[1]}|}.$$

Moreover, $\mathfrak{A} := (G_x, G_y; G_{x,y})$ is the vertex stabilizer amalgam of the G-graph Δ, and by $\mathfrak{A}^*$ we denote the amalgam $G_x^* \leftarrow B \rightarrow G_y^*$.

Lemma 6.12. *$\mathfrak{A}^*$ is the core of $\mathfrak{A}$.*

Proof. Recall that $G_x^{[1]}$ and $G_y^{[1]}$ are the largest subgroups of G_x and G_y, respectively, which are contained in $G_{x,y}$.

1°. $G_x^{[1]}G_y^{[1]} \leqslant B$.

By the definition of B, $TG_y^{[1]} \leqslant B$, and by 6.2(b) $G_x^{[1]} \leqslant C_{G_{x,y}}(T)$. Then by 6.3(b),(c) $G_x^{[1]} \leqslant TC_{G_y^{[1]}}(T) \leqslant B$, and (1°) follows.

2°. $G_x^* \cap G_{x,y} = B = G_y^* \cap G_{x,y}$.

We have

$$G_y^* \cap G_{x,y} = L_yB \cap G_{x,y} = (L_y \cap G_{x,y})B = B,$$

and

$$G_x^* \cap G_{x,y} = LB \cap G_{x,y} = (L \cap G_{x,y})B.$$

In the second equation we use 6.9. Then $L \cap G_{x,y} = TV_0 \leqslant TG_y^{[1]} \leqslant B$, and $G_x^* \cap G_{x,y} = B$ follows.

3°. $G_x^* = \langle B^{G_x} \rangle$ *and* $G_y^* = \langle B^{G_y} \rangle$.

Put $E_x := \langle B^{G_x}\rangle$ and $E_y := \langle B^{G_y}\rangle$. Since $B \trianglelefteq G_{x,y}$, also $G_x^* \trianglelefteq G_x$ and $G_y^* \trianglelefteq G_y$. Hence $E_x \leqslant G_x^*$ and $E_y \leqslant G_y^*$. Since by (1°) $G_x^{[1]} \leqslant E_x$ and $G_y^{[1]} \leqslant E_y$, it suffices to show that $E_x^{\Delta(x)} = G_x^{*\Delta(x)}$ and $E_y^{\Delta(y)} = G_y^{*\Delta(y)}$. But this follows from (2°) and the fact that for $a \in \mathrm{V}\Delta$, $F^*(G_a^{\Delta(a)})$ is of simple or regular type.

What we have proved so far shows that the amalgam $\mathfrak{A}^*$ satisfies conditions (i) and (ii) of 2.1. It remains to show that B is inclusion-minimal with respect to these properties.

Let $B_0 \leqslant B$ such that also B_0 satisfies the conditions (i) and (ii). As by 6.2(c) $L^{\Delta(x)} = F^*(G_x^{\Delta(x)})$ and $G_x^{[1]} \leqslant B_0$, we get $L \leqslant \langle B_0^{G_x}\rangle$. Thus, by property (ii) $T \leqslant B_0$. Hence also $L_y \leqslant \langle B_0^{G_y}\rangle$. But then again by property (ii) and since $G_y^{[1]} \leqslant B_0$, also $B = (L_y \cap G_{x,y})G_y^{[1]} \leqslant B_0$.

This shows that B is inclusion-minimal and that $\mathfrak{A}^*$ is the core of the vertex stabilizer amalgam $(G_x, G_y; G_{x,y})$. □

In the following lemmas we will have to verify the conditions given in 2.2, 2.3 and 2.4 for the vertex stabilizer amalgam $\mathfrak{A}$. That is, we verify these conditions with $(G_x, G_y, G_x^{[1]}, G_y^{[1]}, G_{x,y})$ in pace of (H_1, H_2, T_1, T_2, T) and also all the other notation has to be changed accordingly.

Lemma 6.13. $\mathfrak{A}$ *has shape* $\mathrm{L}_2(\mathrm{q}, \mathrm{d})$.

Proof. We have to verify the conditions 2.2(i) – (iv).

Since $G_x^{\Delta(x)}$ and $G_y^{\Delta(y)}$ are 2-transitive, $G_{x,y}$ is a maximal subgroup of G_x and G_y. This is 2.2(i).

Now let $L_1 := C_{G_x}(G_x^{[1]})$ and $T_1 := L_1 \cap G_{x,y}$. By 6.2(b) $L \leqslant L_1$ and so $L_1 = LT_1$. Also note that by 6.8(b),(c) $T_1 \cap V = V_0$.

In order to show that $L_1 = L$ we need to show that $T_1 \leqslant L$. By 6.10 T_1 induces $\mathbb{F}_q$-semilinear transformations on V. Since T_1 centralizes the $\mathbb{F}_q$-space Z_0, T_1 induces $\mathbb{F}_q$-linear transformations. As V is 2-dimenional and T_1 also normalizes the 1-subspace V_0 we conclude that Z_0 and V_0 are the only 1-spaces normalized by T_1. This shows that $T_1 \cap G_z = C_{T_1}(V) = T_1 \cap V = V_0$. On the other hand, by the transitive action of T on $\Delta(y)\backslash\{x\}$, $T_1 = T(T_1 \cap G_z)$. Hence $T_1 = TV_0 \leqslant L$.

By 6.2(b) and 6.9 $C_{G_x}(L_1) = G_x^{[1]}$ and 2.2(ii) follows.

By 6.2(b),(g) $G_x^{[1]} \leqslant C_{G_{u,x,y}}(T) = TD$ and $T \cap G_x^{[1]} = 1$. Hence $G_x^{[1]}$ is isomorphic to a subgroup of D. On the other hand, by 6.10 $D \cap V = Z_0$ is a 1-dimensional $\mathbb{F}_q$-space of V and D embeds into $\mathrm{AGL}_1(q)$. Hence $G_x^{[1]} \cong \mathrm{GL}_1(q)^{(d)}$. Since $L \cong \mathrm{PSL}_2(q)$, 2.2(iii) follows.

By 6.3(d) and 6.8(a) $O_p(G_{x,y}) = Q = V \trianglelefteq G_y$, and by 6.8(d) $C_{G_y}(V) = V$. By 6.12 $\mathfrak{A}^*$ is the core of $\mathfrak{A}$. Since $G_y^* = L_yB = L_yG_y^{[1]}$ and by 6.8(b) $G_y^{[1]} = DV$, we get $G_y^* = L_yDV$. Hence, 6.10 shows that V is 2-dimesional $\mathbb{F}_qG_y^*$-module. Thus, also 2.2(iv) holds. □

Lemma 6.14. *One of the following holds:*

(a) $p = 2$ *and* $\mathfrak{A}$ *has shape* $\mathcal{A}_{q,d}$ *and* $D \cong G_x^{[1]}$.

(b) $p = 7$ *and* $\mathfrak{A}$ *has shape* $\mathcal{B}$.

(c) $p = 5$ *and* $\mathfrak{A}$ *has shape* $\mathcal{C}$.

In all cases $G_y^{\Delta(y)}$ *is of regular type.*

Proof. According to 6.12 and 6.13 $\mathfrak{A}^*$ is the core of $\mathfrak{A}$ and $\mathfrak{A}$ has shape $\mathrm{L_2(q,d)}$. We now treat the cases $p = 2$ and p odd separately.

Case 1. *The case* $p = 2$.

In this case we want to show (a). Recall that $L \cong \mathrm{PSL}_2(q)$, $q > 2$, and $TV_0 \cong \mathrm{AGL}_1(q)$.

1°. $G_y^{\Delta(x)}$ *is of regular type.*

By 6.8(d) $C_{G_y}(V) = V$ and by 6.9 $V_0 \trianglelefteqslant G_y$. Moreover, by 6.10 V is a 2-dimensional $\mathbb{F}_q L_y$-module and V_0 is a 1-dimesional subspace. Hence $L_y^{\Delta(y)}$ is solvable. Now (1°) follows.

2°. $G_x^{[1]} \cong D \cong \mathrm{AGL}_1(q)^{(d)}$, $B \cap L \cong \mathrm{AGL}_1(q)$ *and* $B = (L \cap B) \times G_x^{[1]}$.

According to (1°) $B = (L_y \cap G_{x,y})G_y^{[1]} = TG_y^{[1]}$ and so by 6.8(b) $B = TDV$. Since $O_p(D) = Z_0$ we get $B = TV_0D$ and $TV_0 \cong \mathrm{AGL}_1(q)$. As $G_x^{[1]} \leqslant TG_y^{[1]}$ and $T \cap G_x^{[1]} = 1$, $G_x^{[1]}$ is isomorphic to a subgroup of D. On the other hand, by Table 2 of 4.2 $D \leqslant TG_x^{[1]}$, so also D is isomorphic to a subgroup of $G_x^{[1]}$. It follows that $D \cong G_x^{[1]}$ and $B = TV_0G_x^{[1]}$. Hence $B = (L \cap B) \times G_x^{[1]}$. Moreover, by 2.2(iii) $G_x^{[1]} \cong \mathrm{AGL}_1(q)^{(d)}$.

3°. $G_y^* \cong \mathrm{AGL}_2(q, S)^{(d)}$.

By (2°) $B = TV_0 \times G_x^{[1]}$, $TV = C_B(Z_0)$ and $C_B(V_0) = G_x^{[1]}V$, and by (1°) $G_y^* = BS$, $S \in Syl_2(L_y)$. Moreover, by 6.10 G_y^* is isomorphic to a subgroup of $\mathrm{AGL}_2(q)$. We identify G_y^* with its image in $\mathrm{AGL}_2(q)$. Since $[V_0, S] = [V/V_0, S] = 1$, we get

$$C_{G_y^*}(V_0)/S \cong G_x^{[1]}/Z_0 \cong \mathrm{GL}_1(q)^{(d)} \text{ and } C_{G_y^*}(V/V_0)/S \cong T \cong \mathrm{GL}_1(q)$$

and $G_y^* = C_{G_y^*}(V_0)C_{G_y^*}(V/V_0)$. Thus $G_y^* = \mathrm{AGL}_2(q, S)^{(d)}$.

We are now in the position to show that $\mathfrak{A}$ is of shape $\mathcal{A}_{q,d}$.

2.2(1): We have seen already that $q > 2$. By (2°) $|D| = |G_x^{[1]}|$ and so

$$\frac{q-1}{d} = |G_x^{[1]}/Z_0| = |D/Z_0|.$$

Since by 6.10 D/Z_0 is not contained in a subfield of $\mathrm{End}_{G_y^*}(V) \cong \mathbb{F}_q$, $|D/Z_0|$ does not divide $2^i - 1$ for $2^i < q$.

2.2(2): We have $G_x^* = BL$. Now the claim follows from (2°).

2.2(3) and 2.2(4): Recall that $L \cap V = V_0 \trianglelefteq G_y$. Hence (3°) implies the claim.

Case 2. *The case p odd.*

By 6.9 $L \cong \mathrm{PSL}_2(q)$, and by 6.3(c) $|T| = q_y = \frac{q-1}{2}$.

4°. *$G_y^{\Delta(y)}$ is of regular type.*

Assume that $G_y^{\Delta(y)}$ is of simple type. Then by 6.3(e) $L_y^{\Delta(y)} \cong \mathrm{PSL}_2(q_y)$, q_y an odd prime with $q_y \geqslant 5$.

Put $E := G_{u,x,y}$ and $k := |E/C_E(T)|$. By 6.3(f) $C_E(T) = C_E(T^{\Delta(x)}) = C_E(T^{\Delta(y)})$. Since $E = TH$ 6.8(b) gives $E \cap G_y^{[1]} = D$. It follows that

$$k = |E^{\Delta(x)}/C_{E^{\Delta(x)}}(T^{\Delta(x)})| = |E^{\Delta(y)}/C_{E^{\Delta(y)}}(T^{\Delta(y)})|.$$

Now Table 1 of 4.2 shows that k divides r, where $q = p^r$.

Let $X \leqslant E$ such that $X^{\Delta(y)}$ is a torus of $L_y^{\Delta(y)}$ normalizing $T^{\Delta(y)}$. Then $1 \neq |X^{\Delta(y)}| = \frac{q_y-1}{2}$ and $|X^{\Delta(y)}|$ divides k, so $|X^{\Delta(y)}|$ divides r. Hence

$$r > 1 \text{ and } p^r = q \leqslant 4r + 3.$$

Since p is odd this gives $r = 2$ and $q = 3^2$. But then $q_y = 4$, which is not an odd prime. This contradiction shows that $G_y^{\Delta(y)}$ is of regular type.

5°. *$q = 7$ and $L_y^{\Delta(y)} \cong \mathrm{Alt}(4)$ or $q = 5$ and $L_y^{\Delta(y)} \cong \mathrm{Sym}(3)$. In both cases V is an irreducible $\mathbb{F}_q L_y$-module.*

Let $N \trianglelefteq L_y$ such that $N^{\Delta(y)}$ is a regular elementary abelian normal subgroup of order $q_y + 1$. Then $q_y + 1$ is a power of a prime different from p. Hence

$$|T| + 1 = q_y + 1 = \frac{q+1}{2}.$$

Since $L \cong \mathrm{PSL}_2(q)$, T acts as multiplication with squares (of $\mathbb{F}_q$) on V_0 and centralizes Z_0. Whence T, and thus also L_y, is isomorphic to a subgroup of $\mathrm{AGL}_2(q)^{(2)}$. Hence $L_y^{\Delta(y)}$ is isomorphic to a subgroup of $\mathrm{PSL}_2(q)$. The list of subgroups of $\mathrm{PSL}_2(q)$, see for example [14, Theorem 6.25], shows that either $q = 7$ and $L_y^{\Delta(y)} \cong \mathrm{Alt}(4)$ or $q = 5$ and $L_y^{\Delta(y)} \cong \mathrm{Sym}(3)$. In both cases V is an irreducible $\mathbb{F}_q L_y$-module.

6°. *$\mathfrak{A}$ is of shape* $\mathrm{L}_2(\mathrm{q}, 2)$.

By 6.13 $\mathfrak{A}$ is of shape $L_2(q,d)$, and we have to show that $d = 2$. Since $G_x^{[1]} \leqslant B = TG_y^{[1]}$ and $G_x^{[1]} \cap G_y^{[1]} = G_x^{[2]} = Z_0$, we get

$$|G_x^{[1]}/Z_0| \leqslant |T| = \frac{q-1}{2} = 2 \text{ or } 3.$$

By 6.6(b) $G_x^{[1]} \nleqslant G_y^{[1]}$ and so $|G_x^{[1]}/Z_0| = |T|$ and $d = \frac{q(q-1)}{|G_x^{[1]}|} = 2$.

7°. *Suppose that $q = 7$. Then (b) holds.*

By (6°) $\mathfrak{A}$ is of shape $L_2(7,2)$. It remains to show 2.3(1) –(3).

By (5°) $L_y^{\Delta(y)} \cong \mathrm{Alt}(4)$. Then $L_y/V \cong \mathrm{SL}_2(3)$, since $\mathrm{SL}_2(7)$ has quaternion Sylow 2-subgroups. It follows that $L_y \cap G_y^{[1]} = V\langle t\rangle$, where t is an involution inverting V. Hence $[t, Z_0] \neq 1$. Since $|G_x^{[1]}/Z_0| = 3$, t is not in $L \times G_x^{[1]}$. This gives 2.3(1). Moreover, $G_y = G_y^* = V \rtimes (\mathrm{C}_6 \curlyvee \mathrm{SL}_2(3))$ and $G_y^{[1]} \cong V \rtimes C_6$. This is 2.3(2).

Finally, 2.3(3) again follows from (5°).

8°. *Suppose that $q = 5$. Then (c) holds.*

By (6°) $\mathfrak{A}$ is of shape $L_2(7,2)$. It remains to show 2.4(1) –(3). By (5°) $L_y^{\Delta(y)} \cong \mathrm{Sym}(3)$. It follows that $L_y^{\Delta(y)} = G_y^{\Delta(y)}$. Hence $G_y = G_y^*$, $B = G_{x,y}$ and $G_x = G_x^*$.

Since $|T| = 2$, L_y is generated by involutions. Hence $L_y/V \cong \mathrm{Sym}(3)$ and so $L_y \cap G_y^{[1]} = V$. Now $G_y^*/V \cong \mathrm{C}_4 \times \mathrm{Sym}(3)$ or $\mathrm{C}_2 \times \mathrm{Sym}(3)$. In the latter case $|H| = 2 \cdot 5$, and $q_x = 4$ does not divide $|H|$, a contradiction. Thus, we have $G_y^* = V \rtimes (\mathrm{C}_4 \times \mathrm{Sym}(3))$ and $G_y^{[1]} \cong V \rtimes C_4$. This is 2.4(2).

Let t be an element of order 4 in $G_y^{[1]}$. Since $q_x = 5$ we may assume that $t \in D$. Since t is not in $G_x^{[1]}$ and t^2 inverts the elements of $V = V_0 \times Z_0$, we get that $t^2 \notin L$ and $t^2 \notin G_x^{[1]}$. Moreover, since $L \cong \mathrm{PSL}_2(5)$ and $G_x^{[1]} \cong \mathrm{AGL}_1(5)^{(2)}$, we have $t^2 \in L \times G_x^{[1]}$. It follows that $G_x^* \cong (\mathrm{PSL}_2(5) \times \mathrm{AGL}_1(5)^{(2)}) \cdot \mathrm{C}_2 \cong (\mathrm{PGL}_2(5) \curlywedge \mathrm{AGL}_1(5))$ and $B \cong \mathrm{AGL}_1(5) \curlywedge \mathrm{AGL}_1(5)$. This gives 2.4(1).

As by (5°) L_y acts irreducible on V, so also 2.4(3) holds. ☐

Lemma 6.15. *Suppose that $p = 2$. Then H is 2-transitive on $\Delta(z)\backslash\{y\}$.*

Proof. Note that $Z_0 \leqslant H$ and that Z_0 is regular on $\Delta(z)\backslash\{y\}$. Then H is 2-transitive on $\Delta(z)\backslash\{y\}$ if and only if H is transitive on Z_0. Thus, it suffices to show that H is transitive on Z_0.

Let $a \in \Delta(z)\backslash\{y\}$. Then $H = Z_0 H_a$ and $Z_0 \cap H_a = 1$. We now use 6.14(a). Let D_0 be a $2'$-Hall subgroup of D. By 6.8(c) D_0 acts fixed-point-freely on V. Since $O_2(L_y) \in Syl_2(\mathrm{AGL}(q,S))$, there exists a $t \in C_{O_2(L_y)}(D_0)$ of order 2 such that $z^t = x$. Let $a := u^t$. Then $H_a^t = H_a$.

By [3, 2.4] $[t, H_a] \leqslant G_y^{[1]} \cap L_y \cap H_a = V \cap H_a$. Since $V \cap G_u = Z_0$ and $V \cap G_a = Z_0^t$, we get that $V \cap H_a = V \cap Z_0 \cap Z_0^t = 1$. Hence $V \cap H_a = 1$ and so $[H_a, t] = 1$.

Now let $b \in Z_0^\sharp$, $K := C_{H_a}(b)$ and $U := C_{Z_0}(K)$. Then $UU^t \leqslant C_V(K)$. Since $|Z_0| = q$, it suffices to show that $|H_a : K| \geqslant q-1$. Since t acts $\mathbb{F}_q$-linear on the 2-dimensional $\mathbb{F}_q L_y$-module V and normalizes V_0, we get $C_V(t) = V_0$ and $bb^t \in V_0$. This shows that $C_{V_0}(K) \neq 1$.

Pick $v \in C_{V_0}(K) \backslash Z_0^\sharp$ and put $W_0 := O_2(L \cap G_u)$. Then $[K, W_0^v] \leqslant W_0^v$ and so by the action of W_0 on $\Delta(x)\backslash\{u\}$ there exists $w \in W_0^\sharp$ such that $V_0^w = W_0^v$. It follows that K and K^w normalizes V_0^w. Hence $[w, K] \leqslant W_0 \cap N_{G_x}(V_0^w) = 1$. That is, K centralizes wU in $W_0 Z_0$. Since $Z_0^{G_u}$ together with W_0 form a partition of the elements of $W_0 Z_0$ which is invariant under the action of G_u, there exists $c \in G_u$ such that $wU \cap Z_0^c \neq 1$ and so $K \leqslant N_{G_u}(Z_0^c) = G_{ux'}$, where $x' = x^c$. On the other hand, since $s \geqslant 4$, H is transitive on $\Delta(u)\backslash\{x\}$ and $H = Z_0 H_a$. Thus, also H_a is transitive on $\Delta(u)\backslash\{x\}$. Hence $q-1 = |H_a / H_a \cap G_{x'}| \leqslant |H_a/K|$, and so H_a is transitive on Z_0. □

6.16. Proof of Theorem 1:

We may assume that the G-graph Δ is not of local characteristic p. Then by 3.2 Δ satisfies Hypothesis 5.1. Now 6.14 gives Theorem 1(1). Moreover, 6.15 gives Theorem 1(3).

As Z_0 is transitive on $\Delta(z)\backslash\{y\}$, the 3-arc (u, x, y, z) satisifies the hypothesis of 3.6. Hence $s \geqslant 5$. In order to prove Theorem 1(2) it remains to show that $s \leqslant 5$. We will do this be applying 3.7.

Let D_0 be a p'-Hall subgroup of D. By 6.8(b) $G_y^{[1]} = D_0 V$. Suppose that $\mathfrak{A}$ is of shape $\mathcal{A}_{q,d}$. Then $C_{O_2(G_y^*)}(D_0)$ is a regular 2-group on $\Delta(y)$ and $O_2(G_y^*) = V C_{O_2(G_y^*)}(D_0)$. Thus, we can choose $t \in O_2(G_y^*)$ such that $x^t = z$ and $t^2 = 1$. Then t normalizes $G_{x,y,z}$ and $[t, G_{x,y,z}] \leqslant O_2(G_y^*) \cap G_{x,y,z} \leqslant V$.

Suppose that $\mathfrak{A}$ is of shape $\mathcal{B}$ or $\mathcal{C}$. Then $G_{x,y,z} = G_y^{[1]} = D_0 V$. In this case we can choose $t \in L_y$ such that $x^t = z$ and $t^2 \in D_0$. Thus, also in this case t normalizes $G_{x,y,z}$ and by 6.8(b) $[t, G_{x,y,z}] \leqslant V$. Note that in all cases $t^2 \leqslant D_0 \leqslant G_u$.

Put $K := G_{u,x,y,z,u^t}$. Since $t^2 \in G_u$, t normalizes K and so

$$[t, K] \leqslant V \cap K = V \cap G_u \cap G_{u^t} = 1.$$

Hence the hypothesis of 3.7 is fulfilled and so $s \leqslant 5$. □

6.17. Proof of Theorem 2:

This follows from Theorem 1(1), 2.6 and 2.7. □

7. References

[1] J. van Bon, Thompson-Wielandt like theorems revisited, *Bull. Lond. Math. Soc.* **35**, 2003, 30–36.

[2] J. van Bon, On locally s-arc transitive graphs with trivial edge kernel, *Bull. Lond. Math. Soc.* **43**, 2011, 799–804.

[3] J. van Bon & B. Stellmacher, Locally s-transitive graphs, *J. Algebra* **441**, 2015, 243–293.

[4] P. Cameron, *Permutation Groups*, LMS Student Text 45, Cambridge University Press, Cambridge, 1999.

[5] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker & R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[6] C.W. Curtis, W.M. Kantor & G.M. Seitz, The 2-Transitive Permutation Representations of the Finite Chevalley Groups, *Trans. American Math. Soc.* **218**, 1976, pp. 1–59.

[7] A. Delgado & B. Stellmacher, Weak BN-pairs of rank 2, in A. Delgado, D. Goldschmidt, B. Stellmacher, *Groups and graphs: new results and methods*, DMV Seminar, 6. Birkhäuser Verlag, Basel, 1985.

[8] D. Foulser, The flag-transitive collineation groups of the finite Desarguesian affine planes, *Canad. J. Math.* **16**, 1964, 443–472

[9] M. Giudici, C-H. Li & C.E. Praeger, A new family of locally 5-arc transitive graphs, *European J. Combin.* **28**, 2007, 533–548.

[10] D. Goldschmidt, Automorphism of trivalent graphs, *Annals of Math.* **111**, 1980, 377–406.

[11] D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968.

[12] H. Kurzweil & B. Stellmacher *The Theory of Finite Groups: An Introduction*, Springer-Verlag, Berlin Heidelberg New York, 2004.

[13] M. Suzuki, On a Class of Doubly Transitive Groups, *Annals of Math.*, **75**, 1962, 105–145

[14] M. Suzuki, *Group Theory 1*, Grundlehren Der Mathematischen Wissenschaften 247, Springer-Verlag, Berlin Heidelberg New York, 1982.

[15] D.E. Taylor, *The Geometry of the Classical groups* Helderman Verlag, Berin, 1992.

[16] J.G. Thompson, Bounds on the orders of maximal subgroups, *J. Algebra* **14**, 1970, 135–138.

[17] W. Tutte, A family of cubic graphs, *Proc. Camb. Math.Soc.* **43** 1947, 449–474.

[18] W. Tutte, On the symmetry of cubic graphs, *Canad. J. Math.* **11** 1959, 621–624.

[19] H.N. Ward, On Ree's Series of Simple Groups, *Trans. Am. Math. Soc.* **121**, 1966, 62–89

[20] H. Wielandt, Subnormal subgroups and permutation groups, Lecture Notes (by: F. Demana, W. McWorter & S. Seghal), Ohio State University, Columbus, Ohio, 1971.

[21] R. Weiss, Groups with a (B,N)-pair and locally transitive graphs. *Nagoya Math. J.* **74** (1979), 1–21.

[22] R. Weiss, The non existence of 8-transitive graphs, *Combinatorica* **1**, 1980, pp. 309-311.

[23] K. Zsigmondy, Zur Theorie der Potenzresten, *Monatsh. Math. Phys.* **3**, 1892, 265–284.