

# EXISTENCE AND REGULARITY FOR A GENERAL CLASS OF QUASILINEAR ELLIPTIC PROBLEMS INVOLVING THE HARDY POTENTIAL

GIUSY CHIRILLO, LUIGI MONTORO, LUIGI MUGLIA, BERARDINO SCIUNZI

**ABSTRACT.** In a very general quasilinear setting, we show that the regularizing effect of a first order term causes the existence of energy solutions for problems involving the Hardy potential and  $L^1$  data. In the same setting we study sharp (local and global) integral estimates for the second derivatives of the solutions.

## 1. INTRODUCTION

The aim of this paper is to study the existence and regularity of positive weak solutions to the nonlinear quasilinear elliptic equation

$$\begin{cases} -\operatorname{div}(A(|\nabla u|)\nabla u) + B(|\nabla u|) = \vartheta \frac{u^q}{|x|^p} + f & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$ ,  $0 \in \Omega$ ,  $\vartheta > 0$ ,  $p \in (1, N)$ ,  $q \in (p-1, p)$ ,  $f \geq 0$  and  $f \in L^1(\Omega)$ .

The real function  $A : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is assumed to be of class  $C^1(\mathbb{R}^+)$  and fulfills the following assumptions

$$\bullet \quad t \mapsto tA(t) \text{ is an increasing function;} \quad (1.2)$$

$$\bullet \quad \exists K \geq 1 : \forall \eta \in \mathbb{R}^N \text{ with } |\eta| \geq K, \quad |A(|\eta|)\eta| \leq c_1|\eta|^{p-1}; \quad (1.3)$$

$$\bullet \quad \forall \eta, \eta' \in \mathbb{R}^N, [A(|\eta|)\eta - A(|\eta'|)\eta'] \cdot [\eta - \eta'] \geq c_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2. \quad (1.4)$$

We set  $\overline{\mathbb{R}}^+ := \mathbb{R}^+ \cup \{0\}$ . The real function  $B : \overline{\mathbb{R}}^+ \rightarrow \overline{\mathbb{R}}^+$  is of class  $C^1(\overline{\mathbb{R}}^+)$  and it satisfies

$$\bullet \quad B(0) = 0; \quad (1.5)$$

$$\bullet \quad B(t) \geq \sigma t^p \text{ for some } \sigma > 0; \quad (1.6)$$

$$\bullet \quad B'(t) \leq \hat{C}tA(t) \text{ for some } \hat{C} > 0, \forall t \in \mathbb{R}^+. \quad (1.7)$$

Let us start making explicit what we means by positive weak solution and stating our main result:

---

1991 *Mathematics Subject Classification.* 35J62, 35B65, 35J70.

*Key words and phrases.* Quasilinear elliptic equations; Hardy potential; Supercritical problems; Existence and Regularity.

Corresponding author: Berardino Sciunzi, Università della Calabria, via P. Bucci, 87036, Arcavacata di Rende (CS), Italy.

**Definition 1.1.** We say that  $u$  is a weak solution to

$$-\operatorname{div}(A(|\nabla u|)\nabla u) + B(|\nabla(u)|) = \vartheta \frac{u^q}{|x|^p} + f \quad \text{in } \Omega,$$

if  $u \in W_0^{1,p}(\Omega)$  and

$$\begin{aligned} \int_{\Omega} A(|\nabla u|)(\nabla u, \nabla \varphi) dx + \int_{\Omega} B(|\nabla u|)\varphi dx &= \vartheta \int_{\Omega} \frac{u^q}{|x|^p} \varphi dx + \int_{\Omega} f \varphi dx \\ \forall \varphi &\in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega). \end{aligned}$$

**Theorem 1.2.** Let  $f \in L^1(\Omega)$  a positive function; then for every  $\vartheta > 0$  there exists a weak solution  $u \in W_0^{1,p}(\Omega)$  to (1.1).

This result is proved in the Section 3.

Let us observe that one immediately recognized that the choices  $A(t) = t^{p-2}$  and  $B(t) = t^p$  leads to the supercritical problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.8)$$

for which existence and qualitative properties are proved in [22]. Moreover taking into account [2, 12, 13] we remark that our positive weak solutions are *solutions obtained as limits of approximations* (SOLA) and therefore (in our case) equivalent to the entropy solutions.

The main elements of the proof are summarized in what follows.

- In order to prove the existence of a solution  $u_k$  for the truncated problem

$$-\operatorname{div}(A(|\nabla u_k|)\nabla u_k) + B(|\nabla u_k|) = \vartheta T_k \left( \frac{u_k^q}{|x|^p} \right) + T_k(f) \quad \text{in } \Omega.$$

we construct a sequence  $(w_n)_{n \in \mathbb{N}}$  of solutions to the problem

$$-\operatorname{div}(A(|\nabla w_n|)\nabla w_n) + \frac{B(|\nabla w_n|)}{1 + \frac{1}{n}B(|\nabla w_n|)} = \vartheta T_k \left( \frac{w_{n-1}^q}{|x|^p} \right) + T_k(f) \quad \text{in } \Omega$$

and we prove that  $(w_n)_{n \in \mathbb{N}}$   $W_0^{1,p}(\Omega)$ -converges to  $u_k$ .

- We prove that the sequence  $(u_k)_{k \in \mathbb{N}}$  of solutions of the family of truncated problem weak converges in  $W_0^{1,p}(\Omega)$  to a function  $u$ . This will imply the  $W_0^{1,p}(\Omega)$ -convergence of  $T_m(u_k)$  to  $T_m(u)$  uniformly on  $m$  and finally this will permit to get an a.e.- pointwise convergence of  $\nabla u_k$  to  $\nabla u$ .
- We prove that  $(u_k)_{k \in \mathbb{N}}$  strongly converges to  $u$  and moreover that  $u$  is the positive weak solution to our problem.

Solvability of Problem (1.1) already in the  $p$ -Laplacian case depends of the position of the origin with respect to the domain. This is proved in [15, 23] for  $p = 2$  and in [1, 21] for  $p > 1$ . It is proved in [1] that if the origin belongs to the interior of  $\Omega$  we have no solutions even in entropic sense; for this reason in [21] the authors consider  $0 \in \partial\Omega$ . In this case the existence of solutions turns to be depending on the presence of the source and then on the geometry of the domain.

In this paper we consider  $0 \in \Omega$  but the attendance of a suitable term involving the gradient, leads a regularizing effect that permits to prove the existence of a solution.

The second part of the paper (developed in Section 4) is devoted to obtain some summability properties of the gradient as well as the second derivatives of the solutions of (1.1). In this section we need a further hypothesis on  $A(t)$  that is

$$-1 < \inf_{t>0} \frac{tA'(t)}{A(t)} =: m_A \leq M_A := \sup_{t>0} \frac{tA'(t)}{A(t)} < +\infty. \quad (1.9)$$

Therefore the main results of this section can be included in the following

**Theorem 1.3.** *Assume  $1 < p < N$  and that (1.2)-(1.4), (1.5)-(1.7) and (1.9) are in force. Consider  $u \in C_{loc}^{1,\alpha}(\bar{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus (Z_u \cup \{0\}))$  a solution to (4.3), where  $f \in W^{1,\infty}(\bar{\Omega})$ . We have*

$$\int_{E \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \mathcal{C} \quad \forall i = 1, \dots, N$$

for any  $E \Subset \Omega \setminus \{0\}$  and uniformly for any  $y \in E$ , with

$$\mathcal{C} := \mathcal{C}(\gamma, m_A, M_A, \beta, h, \|\nabla u\|_\infty, \rho)$$

for  $0 \leq \beta < 1$  and  $\gamma < (N-2)$  if  $N \geq 3$  ( $\gamma = 0$  if  $N = 2$ ).

Moreover, if we also assume that  $\Omega$  is a smooth domain and  $f$  is nonnegative in  $\Omega$ , we have that

$$\int_{\Omega \setminus (Z_u \cup \{0\})} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \mathcal{C} \quad \forall i = 1, \dots, N.$$

Here the set  $Z_u$  represents the set of critical points of  $u$  and we refer the readers to Section 3.2 for a discussion regarding the assumption  $u \in C_{loc}^{1,\alpha}(\bar{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus (Z_u \cup \{0\}))$ . Furthermore we emphasize that in all our regularity results we require  $f \in W^{1,\infty}(\bar{\Omega})$ .

As a corollary of this result we will prove that the Lebesgue measure of  $Z_u$  is zero, as in the natural case of  $p$ -Laplacian.

**Theorem 1.4.** *Assume  $1 < p < N$  and that (1.2)-(1.4), (1.5)-(1.7) and (1.9) are in force.*

*Let  $u \in C^{1,\alpha}(\Omega \setminus \{0\}) \cap C^2(\Omega \setminus (Z_u \cup \{0\}))$  be a solution to (4.3) with  $f \in W^{1,\infty}(\Omega)$  and  $f(x) > 0$  in  $B_{2\rho}(x_0) \subset \Omega \setminus \{0\}$ . Then*

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \mathcal{C}$$

for any  $y \in B_\rho(x_0)$ , with  $\alpha := \frac{p-1}{p-2}$  if  $p > 2$  and  $\alpha := \frac{m_A+1}{m_A}$  if  $p \in (1, 2)$ ,  $r \in (0, 1)$ ,  $\gamma < N-2$  if  $N \geq 3$ ,  $\gamma = 0$  if  $N = 2$  and

$$\mathcal{C} = \mathcal{C}(\gamma, \eta, h, \|\nabla u\|_\infty, \rho, x_0, \alpha, M_A, c_2, \tau, \hat{C}).$$

If  $\Omega$  is a smooth domain and  $f$  is nonnegative in  $\Omega$

$$\int_{\bar{\Omega}} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \mathcal{C}.$$

where  $\tilde{\Omega} \Subset \Omega \setminus \{0\}$  and  $y \in \tilde{\Omega}$ .

These results can be framed in the research topic introduced in [14]; here the authors proved summability properties for the gradient and second derivative for positive solutions of  $p$ -Laplacian equations. The aim is obtain Sobolev and Poincaré type inequalities in weighted Sobolev spaces with weight  $A(|\nabla u|)$  and to apply them to the study of monotonicity and symmetry of solutions based on moving plane method. Since then, these ideas have been exploited, developed and detailed in many papers as [7, 17, 22] and much more. Our results can be easily compared with those in [7, 17].

From Theorem 1.3 and 1.4, we obtain this result:

**Theorem 1.5.** *Assume  $1 < p < N$  and that (1.2)-(1.4), (1.5)-(1.7) and (1.9) are in force. Let  $\Omega$  be a smooth domain,  $u \in C^1(\bar{\Omega} \setminus \{0\})$  be a weak solution to (4.3),  $f \in W^{1,\infty}(\bar{\Omega})$  and  $f$  nonnegative function. Then if  $1 < p < 3$ ,  $u_i \in W_{loc}^{1,2}(\Omega \setminus \{0\})$ , while if  $p \geq 3$  then  $u_i \in W_{loc}^{1,q}(\Omega \setminus \{0\})$ , for every  $i = 1, \dots, N$  and for every  $q < \frac{p-1}{p-2}$ . Moreover the generalized derivatives of  $u_i$  coincide with the classical ones, both denoted with  $u_{ij}$  almost everywhere in  $\Omega$ .*

The additional hypothesis (1.9) is well known and we can find it, for instance, in [7, 9, 10, 11, 17]. Let us remark that our setting (1.3)-(1.4)-(1.9) on  $A(t)$  is satisfied, as for example, by  $A(t) = t^{p-2} + bt^{q-2}$  with  $1 < q < p$  and  $b > 0$ ; this particular choice for the operator  $A(t)$  finds application to the study of double-phase equations

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u + b|\nabla u|^{q-2}\nabla u) + B(|\nabla u|) = \vartheta \frac{u^q}{|x|^p} + f \quad \text{in } \Omega.$$

## 2. PRELIMINARIES

In this section, briefly, we enclose some definitions, results and remarks that it will be useful in the rest of the paper. From now on, in order to get a readable notation, generic numerical constant will be denoted by  $c$  and will be allowed to vary within a single line or formula. Moreover, we denote with  $f^+ := \max\{f, 0\}$  and  $f^- := \min\{f, 0\}$ .

*Remark 2.1.* With respect to the setting (1.2)-(1.7), we observe that:

- If  $\eta' = 0$ , the inequality (1.4) becomes

$$A(|\eta|)|\eta|^2 \geq c_2|\eta|^p. \quad (2.1)$$

Moreover, if  $\eta, \eta' \in \mathbb{R}^N$ , since  $|\eta - \eta'| \leq |\eta| + |\eta'|$ , by (1.4) it follows

$$[A(|\eta|)\eta - A(|\eta'|)\eta'] \cdot [\eta - \eta'] \geq c_2|\eta - \eta'|^p \quad \text{if } p \geq 2. \quad (2.2)$$

- By (1.2), we get that

$$\lim_{t \rightarrow 0^+} tA(t) < +\infty. \quad (2.3)$$

Then, if  $0 < t < K$  there exists a constant  $C_K := C(K) > 0$  such that

$$|tA(t)| \leq C_K. \quad (2.4)$$

- By (1.2), (1.5) and (1.7), we have that for  $\xi \in [0, t]$

$$B(t) = \int_0^t B'(s) ds \leq \hat{C}t\xi A(\xi) \leq \hat{C}t^2 A(t). \quad (2.5)$$

**Definition 2.2.** We say that  $u$  is a weak supersolution to Problem (1.1) if  $u \in W_0^{1,p}(\Omega)$  and

$$\begin{aligned} \int_{\Omega} A(|\nabla u|)(\nabla u, \nabla \varphi) dx + \int_{\Omega} B(|\nabla u|)\varphi dx &\geq \vartheta \int_{\Omega} \frac{u^q}{|x|^p} \varphi dx + \int_{\Omega} f \varphi dx \\ \forall \varphi &\in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega), \varphi \geq 0. \end{aligned}$$

We say that  $u$  is a weak subsolution if  $u \in W_0^{1,p}(\Omega)$  and

$$\begin{aligned} \int_{\Omega} A(|\nabla u|)(\nabla u, \nabla \varphi) dx + \int_{\Omega} B(|\nabla u|)\varphi dx &\leq \vartheta \int_{\Omega} \frac{u^q}{|x|^p} \varphi dx + \int_{\Omega} f \varphi dx \\ \forall \varphi &\in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega), \varphi \geq 0. \end{aligned}$$

A main tool will be the following

**Lemma 2.3.** (*Hardy-Sobolev inequality*) Suppose  $1 < p < N$  and  $u \in W^{1,p}(\mathbb{R}^N)$ . Then we have

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla u|^p dx$$

with  $C_{N,p} = \left(\frac{p}{N-p}\right)^p$  optimal and not achieved constant.

### 3. EXISTENCE OF AN ENERGY SOLUTION

#### 3.1. Existence of solutions to the truncated problem.

In this subsection, we are going to study the existence of a solution  $u_k \in W_0^{1,p}(\Omega)$  to the truncated problem

$$-\operatorname{div}(A(|\nabla u_k|)\nabla u_k) + B(|\nabla u_k|) = \vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f) \quad \text{in } \Omega, \quad (3.1)$$

where

$$T_k(s) = \max\{\min\{k, s\}, -k\}, \quad k > 0. \quad (3.2)$$

We observe that, since (1.5) holds,  $\phi \equiv 0$  is a subsolution to (3.1).

By [3, Theorem 4.1] and by classical Stampacchia argument, there exists a solution  $\psi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$  to the following problem

$$\begin{cases} -\operatorname{div}(A(|\nabla \psi|)\nabla \psi) = \vartheta k + T_k(f) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.3)$$

that turns to be a supersolution to the problem (3.1). In fact, let  $\varphi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$  and  $\varphi \geq 0$ , we have

$$\begin{aligned} \int_{\Omega} A(|\nabla \psi|)(\nabla \psi, \nabla \varphi) dx + \int_{\Omega} B(|\nabla \psi|)\varphi dx &\geq \int_{\Omega} A(|\nabla \psi|)(\nabla \psi, \nabla \varphi) dx \\ &= \vartheta \int_{\Omega} k \varphi dx + \int_{\Omega} T_k(f) \varphi dx \geq \vartheta \int_{\Omega} T_k\left(\frac{\psi^q}{|x|^p}\right) \varphi dx + \int_{\Omega} T_k(f) \varphi dx \end{aligned}$$

In order to prove the existence of solutions to the problem (1.1) is useful to consider a sequence of approximated problems. We take as starting point  $w_0 = 0$  and consider iteratively the problems

$$\begin{cases} -\operatorname{div}(A(|\nabla w_n|)\nabla w_n) + \frac{B(|\nabla w_n|)}{1 + \frac{1}{n}B(|\nabla w_n|)} = \vartheta T_k\left(\frac{w_{n-1}^q}{|x|^p}\right) + T_k(f) & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

We observe that  $\phi \equiv 0$  and the supersolution  $\psi$  to problem (3.1) are respectively subsolution and supersolution to the problem (3.4).

Then, using the same argument in [5, Theorem 2.1], it can be proved the following:

**Proposition 3.1.** *There exists  $w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  solution of the problem (3.4). Moreover,  $0 \leq w_n \leq \psi \quad \forall n \in \mathbb{N}$ .*

Next, we prove the following theorem:

**Theorem 3.2.** *There exists a positive solution to the problem*

$$-\operatorname{div}(A(|\nabla u_k|)\nabla u_k) + B(|\nabla u_k|) = \vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f) \quad \text{in } \Omega.$$

*Proof.* We proceed in two steps.

*Step 1:* We show the weak convergence of  $\{w_n\}_{n \in \mathbb{N}}$  in  $W_0^{1,p}(\Omega)$ .

Let us set

$$H_n(B(|\nabla w_n|)) := \frac{B(|\nabla w_n|)}{1 + \frac{1}{n}B(|\nabla w_n|)}.$$

Considering that  $w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , we can take  $w_n$  as a test function in the approximated problems (3.4) obtaining

$$\begin{aligned} & \int_{\Omega} A(|\nabla w_n|)|\nabla w_n|^2 dx + \int_{\Omega} H_n(B(|\nabla w_n|))w_n dx \\ &= \vartheta \int_{\Omega} T_k\left(\frac{w_{n-1}^q}{|x|^p}\right) w_n dx + \int_{\Omega} T_k(f)w_n dx \\ &\leq \vartheta \int_{\Omega} kw_n dx + \int_{\Omega} fw_n dx \leq C(k, f, \psi, \vartheta, \Omega) \end{aligned}$$

where  $C(k, f, \psi, \vartheta, \Omega)$  is a positive constant.

Since  $H_n(B(|\nabla w_n|))$  is a positive function, from inequality (2.1), we have that

$$c_2 \int_{\Omega} |\nabla w_n|^p dx \leq \int_{\Omega} A(|\nabla w_n|)|\nabla w_n|^2 dx \leq C(k, f, \psi, \vartheta, \Omega). \quad (3.5)$$

Then  $\|w_n\|_{W_0^{1,p}(\Omega)}$  is uniformly bounded so, up to a subsequence, we have that

$$\begin{aligned} w_n &\rightharpoonup \bar{u}_k \quad \text{in } W_0^{1,p}(\Omega) \text{ and } w_n \rightharpoonup \bar{u}_k \quad \text{in } L^p(\Omega), \\ w_n &\overset{*}{\rightharpoonup} u_k \quad \text{in } L^\infty(\Omega). \end{aligned}$$

We observe that  $u_k = \bar{u}_k$ : in fact, by definition of weak and weak-\* convergence we get that

$$\begin{aligned} \int_{\Omega} w_n \varphi dx &\xrightarrow{n \rightarrow +\infty} \int_{\Omega} \bar{u}_k \varphi dx \quad \forall \varphi \in L^{p'}(\Omega) \\ \int_{\Omega} w_n \varphi dx &\xrightarrow{n \rightarrow +\infty} \int_{\Omega} u_k \varphi dx \quad \forall \varphi \in L^1(\Omega) \end{aligned}$$

but, since  $p' = \frac{p}{p-1} > 1$ , the last limit holds also for  $\varphi \in L^{p'}(\Omega)$ , so  $u_k = \bar{u}_k$  and  $u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ .

*Step 2:* We show the strong convergence of  $\{w_n\}_{n \in \mathbb{N}}$  in  $W_0^{1,p}(\Omega)$ .

To get the strong convergence in  $W_0^{1,p}(\Omega)$ , using Minkowski inequality, it follows that

$$\|w_n - u_k\|_{W_0^{1,p}(\Omega)} \leq \|(w_n - u_k)^+\|_{W_0^{1,p}(\Omega)} + \|(w_n - u_k)^-\|_{W_0^{1,p}(\Omega)}. \quad (3.6)$$

We first consider the asymptotic behaviour of  $\|(w_n - u_k)^+\|_{W_0^{1,p}(\Omega)}$ .

Choosing  $(w_n - u_k)^+$  as test function in (3.4), we have

$$\begin{aligned} & \int_{\Omega} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^+) dx + \int_{\Omega} H_n(B(|\nabla w_n|))(w_n - u_k)^+ dx \\ &= \vartheta \int_{\Omega} T_k \left( \frac{w_{n-1}^q}{|x|^p} \right) (w_n - u_k)^+ dx + \int_{\Omega} T_k(f)(w_n - u_k)^+ dx. \end{aligned} \quad (3.7)$$

Since  $w_n \rightharpoonup u_k$  in  $W_0^{1,p}(\Omega)$ , using the compact Sobolev embedding we obtain  $w_n \rightarrow u_k$  in  $L^p(\Omega)$  with  $p \in [1, p^*)$  and then, up to subsequence we obtain that  $w_n \rightarrow u_k$  a.e. in  $\Omega$ . Moreover  $(w_n - u_k)^+ \rightarrow 0$  a.e. in  $\Omega$ .

By dominated convergence theorem, the right-hand side of (3.7) goes to zero when  $n$  goes to infinity. Since  $\int_{\Omega} H_n(B(|\nabla w_n|))(w_n - u_k)^+ dx \geq 0$ , (3.7) becomes

$$\int_{\Omega} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^+) dx \leq o(1) \quad \text{as } n \rightarrow +\infty. \quad (3.8)$$

Furthermore, let us observe that  $(w_n - u_k)^+ \rightharpoonup 0$  in  $W_0^{1,p}(\Omega)$  implies that

$$\int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla(w_n - u_k)^+) dx = o(1) \quad \text{as } n \rightarrow +\infty. \quad (3.9)$$

In fact, the operator

$$v \mapsto \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla v) dx$$

is linear and continuous because, by (1.3) and (2.4), we have

$$\begin{aligned} & \left| \int_{\Omega} (A(|\nabla u_k|)\nabla u_k, \nabla v) dx \right| \leq \left( \int_{\Omega} |A(|\nabla u_k|)\nabla u_k|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla v|^p dx \right)^{\frac{1}{p}} \\ & \leq \left( C_K^{\frac{p}{p-1}} |\Omega \cap \{|\nabla u_k| < K\}| + c \int_{\Omega \cap \{|\nabla u_k| \geq K\}} |\nabla u_k|^p dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\nabla v|^p dx \right)^{\frac{1}{p}} < +\infty. \end{aligned}$$

Then, using (3.9), we write (3.8) as

$$\int_{\Omega} [A(|\nabla w_n|)\nabla w_n - A(|\nabla u_k|)\nabla u_k] \cdot \nabla(w_n - u_k)^+ dx \leq o(1).$$

Using (1.4), we get

$$\begin{aligned} 0 &\leq c_2 \int_{\Omega} (|\nabla w_n| + |\nabla u_k|)^{p-2} |\nabla(w_n - u_k)^+|^2 dx \\ &\leq \int_{\Omega} [A(|\nabla w_n|) \nabla w_n - A(|\nabla u_k|) \nabla u_k] \cdot \nabla(w_n - u_k)^+ dx \leq o(1), \end{aligned}$$

hence

$$\int_{\Omega} [A(|\nabla w_n|) \nabla w_n - A(|\nabla u_k|) \nabla u_k] \cdot \nabla(w_n - u_k)^+ dx = o(1) \text{ as } n \rightarrow +\infty.$$

By (1.4) and (2.2), we obtain

$$\begin{aligned} o(1) &= \int_{\Omega} [A(|\nabla w_n|) \nabla w_n - A(|\nabla u_k|) \nabla u_k] \cdot \nabla(w_n - u_k)^+ dx \\ &\geq \begin{cases} c_2 \int_{\Omega} \frac{|\nabla(w_n - u_k)^+|^2}{(|\nabla w_n| + |\nabla u_k|)^{2-p}} dx & \text{if } 1 < p < 2, \\ c_2 \int_{\Omega} |\nabla(w_n - u_k)^+|^p dx & \text{if } p \geq 2. \end{cases} \end{aligned} \quad (3.10)$$

We consider two different case:

- If  $p \geq 2$ , by (3.10) we have

$$\|(w_n - u_k)^+\|_{W_0^{1,p}(\Omega)} = o(1) \text{ as } n \rightarrow +\infty. \quad (3.11)$$

- If  $1 < p < 2$ ,

$$\int_{\Omega} |\nabla(w_n - u_k)^+|^p dx = \int_{\Omega} \frac{|\nabla(w_n - u_k)^+|^p}{(|\nabla w_n| + |\nabla u_k|)^{\frac{p(2-p)}{2}}} (|\nabla w_n| + |\nabla u_k|)^{\frac{p(2-p)}{2}} dx.$$

By Hölder inequality and (3.10), we obtain

$$\begin{aligned} &\int_{\Omega} |\nabla(w_n - u_k)^+|^p dx \\ &\leq \left( \int_{\Omega} \frac{|\nabla(w_n - u_k)^+|^2}{(|\nabla w_n| + |\nabla u_k|)^{2-p}} dx \right)^{\frac{p}{2}} \left( \int_{\Omega} (|\nabla w_n| + |\nabla u_k|)^p dx \right)^{\frac{2-p}{p}} = o(1). \end{aligned}$$

Then

$$\|(w_n - u_k)^+\|_{W_0^{1,p}(\Omega)} = o(1) \text{ as } n \rightarrow +\infty. \quad (3.12)$$

In order to complete the asymptotic behaviour of (3.6), let us consider  $e^{-\hat{C}w_n}[(w_n - u_k)^-]$  as test function in (3.4), where  $\hat{C}$  is the positive constant in (1.7).

$$\begin{aligned} &\int_{\Omega} A(|\nabla w_n|) (\nabla w_n, \nabla(e^{-\hat{C}w_n}[(w_n - u_k)^-])) dx \\ &+ \int_{\Omega} H_n(B(|\nabla w_n|)) e^{-\hat{C}w_n} [(w_n - u_k)^-] dx \\ &= \vartheta \int_{\Omega} e^{-\hat{C}w_n} T_k \left( \frac{w_{n-1}^q}{|x|^p} \right) (w_n - u_k)^- dx + \int_{\Omega} e^{-\hat{C}w_n} T_k(f) (w_n - u_k)^- dx. \end{aligned}$$

Then

$$\begin{aligned}
& \int_{\Omega} e^{-\hat{C}w_n} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^-) dx \\
& + \int_{\Omega} e^{-\hat{C}w_n} \left[ H_n(B(|\nabla w_n|)) - \hat{C}A(|\nabla w_n|)|\nabla w_n|^2 \right] (w_n - u_k)^- dx \\
& = \vartheta \int_{\Omega} e^{-\hat{C}w_n} T_k \left( \frac{w_n^q}{|x|^p} \right) (w_n - u_k)^- dx + \int_{\Omega} e^{-\hat{C}w_n} T_k(f)(w_n - u_k)^- dx.
\end{aligned} \tag{3.13}$$

By (2.5), we have that

$$H_n(B(|\nabla w_n|)) - \hat{C}A(|\nabla w_n|)|\nabla w_n|^2 \leq B(|\nabla w_n|) - \hat{C}A(|\nabla w_n|)|\nabla w_n|^2 \leq 0$$

then

$$\int_{\Omega} e^{-\hat{C}w_n} \left[ H_n(B(|\nabla w_n|)) - \hat{C}A(|\nabla w_n|)|\nabla w_n|^2 \right] (w_n - u_k)^- dx \geq 0.$$

As above, since  $(w_n - u_k)^- \rightarrow 0$ , by dominated convergence theorem, the right-hand side of (3.13) tends to zero when  $n$  goes to infinity.

So, the equation (3.13) becomes

$$\int_{\Omega} e^{-\hat{C}w_n} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^-) dx \leq o(1). \tag{3.14}$$

Since  $(w_n - u_k)^- \rightarrow 0$  for  $n \rightarrow +\infty$ ,

$$\begin{aligned}
& \int_{\Omega} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^-) dx \\
& = \int_{\Omega} (A(|\nabla w_n|)\nabla w_n - A(|\nabla u_k|)\nabla u_k, \nabla(w_n - u_k)^-) dx \\
& + \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla(w_n - u_k)^-) dx \\
& = \int_{\Omega} e^{\hat{C}w_n} e^{-\hat{C}w_n} (A(|\nabla w_n|)\nabla w_n - A(|\nabla u_k|)\nabla u_k, \nabla(w_n - u_k)^-) dx + o(1).
\end{aligned} \tag{3.15}$$

By Proposition 3.1,  $0 \leq w_n \leq \psi$ , therefore one has that  $e^{\hat{C}w_n} \leq e^{\hat{C}\psi} =: \gamma > 0$  uniformly on  $n$  being  $\psi \in L^\infty(\Omega)$ . Then, using (1.4) and (3.14), we estimate (3.15) as

$$\begin{aligned}
& \int_{\Omega} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^-) dx \\
& = \int_{\Omega} e^{\hat{C}w_n} e^{-\hat{C}w_n} (A(|\nabla w_n|)\nabla w_n - A(|\nabla u_k|)\nabla u_k, \nabla(w_n - u_k)^-) dx + o(1) \\
& \leq \gamma \int_{\Omega} e^{-\hat{C}w_n} (A(|\nabla w_n|)\nabla w_n - A(|\nabla u_k|)\nabla u_k, \nabla(w_n - u_k)^-) dx + o(1) \\
& \leq -\gamma \int_{\Omega} e^{-\hat{C}w_n} A(|\nabla u_k|)(\nabla u_k, \nabla(w_n - u_k)^-) dx + o(1).
\end{aligned} \tag{3.16}$$

Now, we estimate the right hand-side of (3.16) in this way:

$$\begin{aligned}
 & \int_{\Omega} e^{-\hat{C}w_n} A(|\nabla u_k|)(\nabla u_k, \nabla(w_n - u_k)^-) dx \\
 &= \int_{\Omega} e^{-\hat{C}(w_n - u_k)^-} e^{-\hat{C}u_k} A(|\nabla u_k|)(\nabla u_k, \nabla(w_n - u_k)^-) dx \\
 &= -\frac{1}{\hat{C}} \int_{\Omega} e^{-\hat{C}u_k} A(|\nabla u_k|)(\nabla u_k, \nabla(e^{-\hat{C}(w_n - u_k)^-} - 1)) dx
 \end{aligned} \tag{3.17}$$

Since  $\{e^{-\hat{C}(w_n - u_k)^-} - 1\}_{n \in \mathbb{N}}$  is uniformly bounded in  $W_0^{1,p}(\Omega)$ , up to a subsequence there exists  $g \in W_0^{1,p}(\Omega)$  such that

$$e^{-\hat{C}(w_n - u_k)^-} - 1 \rightharpoonup g \text{ as } n \rightarrow +\infty. \tag{3.18}$$

In particular we note that  $g \equiv 0$  since  $w_n \rightarrow u_k$  a.e. in  $\Omega$  as  $n \rightarrow +\infty$ . Hence, by (3.18), for  $n \rightarrow +\infty$

$$\int_{\Omega} e^{-\hat{C}u_k} A(|\nabla u_k|)(\nabla u_k, \nabla(e^{-\hat{C}(w_n - u_k)^-} - 1)) dx \rightarrow 0. \tag{3.19}$$

Then, using (3.17) and (3.19) in (3.16), one has

$$\int_{\Omega} A(|\nabla w_n|)(\nabla w_n, \nabla(w_n - u_k)^-) dx \leq o(1).$$

Arguing in the same way as we have done from equation (3.8) to (3.12), we obtain that

$$\|(w_n - u_k)^-\|_{W_0^{1,p}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow +\infty. \tag{3.20}$$

Summarizing up, from (3.6) we get that

$$\|w_n - u_k\|_{W_0^{1,p}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow +\infty \tag{3.21}$$

Up to a subsequence,  $|\nabla w_n| \rightarrow |\nabla u_k|$  a.e. in  $\Omega$  and there exists  $v(x) \in L^1(\Omega)$  such that  $|\nabla w_n|^p \leq v(x)$  a.e. in  $\Omega$ .

Since  $B$  is continuous, we get that

$$\lim_{n \rightarrow +\infty} H_n(B(|\nabla w_n|)) = \lim_{n \rightarrow +\infty} \frac{B(|\nabla w_n|)}{1 + \frac{1}{n}B(|\nabla w_n|)} = B(|\nabla u_k|) \text{ a.e. in } \Omega.$$

Moreover, if  $E \subset \Omega$  is a measurable set, by (2.5), (1.3) and (2.4), we have

$$\begin{aligned}
 \int_E H_n(B(|\nabla w_n|)) dx &\leq \int_E B(|\nabla w_n|) dx \leq \hat{C} \int_E |A(|\nabla w_n|)| |\nabla w_n|^2 dx \\
 &\leq KC_K \hat{C} |E \cap \{|\nabla w_n| < K\}| + c_1 \hat{C} \int_{E \cap \{|\nabla w_n| \geq K\}} |\nabla w_n|^p dx \\
 &\leq KC_K \hat{C} |E| + c_1 \hat{C} \int_E v(x) dx
 \end{aligned}$$

and, since  $v(x) \in L^1(\Omega)$ , we get that  $H_n(B(|\nabla w_n|))$  is uniformly integrable.

Using Vitali Theorem we have

$$H_n(B(|\nabla w_n|)) \rightarrow B(|\nabla u_k|) \text{ in } L^1(\Omega).$$

To conclude the proof, passing to the limit in weak formulation of problem (3.4), we obtain that  $u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  satisfies

$$\begin{aligned} & \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla \varphi) dx + \int_{\Omega} B(|\nabla u_k|)\varphi dx \\ &= \vartheta \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) \varphi dx + \int_{\Omega} T_k(f)\varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{aligned}$$

hence  $u_k$  is a positive weak solution to the problem (3.1).  $\square$

### 3.2. Existence of solutions to the problem (1.1).

In order to prove the existence of solution to the problem (1.1), we exploit the following Lemma (see [22]).

**Lemma 3.3.** *Let  $\psi_n(s)$  defined as*

$$\psi_n(s) = \int_0^s T_n(t)^{\frac{1}{p}} dt. \quad (3.22)$$

For fixed  $q \in [p-1, p)$ ,  $\forall \varepsilon > 0$  and  $\forall n > 0$ , there exists  $C_\varepsilon$  such that

$$s^q T_n(s) \leq \varepsilon \psi_n^p(s) + C_\varepsilon, \quad s \geq 0. \quad (3.23)$$

Next we prove our main Theorem 1.2.

**Theorem 1.2.** *Let  $f \in L^1(\Omega)$  a positive function; then for every  $\vartheta > 0$  there exists a weak solution  $u \in W_0^{1,p}(\Omega)$  to (1.1).*

*Proof.* Again our proof is divided in steps.

*Step 1:* We show the weak convergence of  $\{u_k\}_{k \in \mathbb{N}}$  in  $W_0^{1,p}(\Omega)$ .

Taking as function test  $T_n(u_k)$  in the truncated problem (3.1), we obtain

$$\begin{aligned} & \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla T_n(u_k)) dx + \int_{\Omega} B(|\nabla u_k|)T_n(u_k) dx \\ &= \vartheta \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) T_n(u_k) dx + \int_{\Omega} T_k(f)T_n(u_k) dx. \end{aligned}$$

By (3.22) we have that  $|\nabla \psi_n(u_k)|^p = T_n(u_k)|\nabla u_k|^p$  and since  $u_k$  is a positive function, we get

$$T_k \left( \frac{u_k^q}{|x|^p} \right) \leq \frac{u_k^q}{|x|^p}.$$

Then

$$\begin{aligned} & \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla T_n(u_k)) dx + \int_{\Omega} B(|\nabla u_k|)T_n(u_k) dx \\ & \leq \vartheta \int_{\Omega} \frac{u_k^q}{|x|^p} T_n(u_k) dx + n \int_{\Omega} f dx. \end{aligned} \quad (3.24)$$

Using Lemma 3.3 and Lemma 2.3, the equation (3.24) becomes

$$\begin{aligned} & \int_{\Omega} A(|\nabla T_n(u_k)|)|\nabla T_n(u_k)|^2 dx + \int_{\Omega} B(|\nabla u_k|)T_n(u_k) dx \\ & \leq \vartheta \int_{\Omega} (\varepsilon \psi_n^p(u_k) + C_\varepsilon) \frac{1}{|x|^p} dx + n \|f\|_1 \\ & \leq \vartheta \varepsilon C_{N,p} \int_{\Omega} |\nabla \psi_n(u_k)|^p dx + \vartheta C_\varepsilon \int_{\Omega} \frac{1}{|x|^p} dx + n \|f\|_1. \end{aligned} \quad (3.25)$$

Using (1.6) and (2.1), equation (3.25) becomes

$$\begin{aligned} & c_2 \int_{\Omega} |\nabla T_n(u_k)|^p dx + \sigma \int_{\Omega} |\nabla \psi_n(u_k)|^p dx \\ & \leq \vartheta \varepsilon C_{N,p} \int_{\Omega} |\nabla \psi_n(u_k)|^p dx + \vartheta C_{\varepsilon} \int_{\Omega} \frac{1}{|x|^p} dx + n \|f\|_1 \end{aligned}$$

and since  $\Omega$  is a bounded set and  $p < N$  we get

$$c_2 \int_{\Omega} |\nabla T_n(u_k)|^p dx + (\sigma - \vartheta \varepsilon C_{N,p}) \int_{\Omega} |\nabla \psi_n(u_k)|^p dx \leq C(n, \varepsilon, p, f, \vartheta, \Omega).$$

If we choose  $0 < \varepsilon < \frac{\sigma}{\vartheta C_{N,p}}$ , by setting  $\tilde{C} := \sigma - \vartheta \varepsilon C_{N,p} > 0$ , we get

$$\int_{\Omega} |\nabla T_n(u_k)|^p dx + \frac{\tilde{C}}{c_2} \int_{\Omega} |\nabla \psi_n(u_k)|^p dx \leq C(n, f, p, c_2, \sigma, \vartheta, C_{N,p}, \Omega). \quad (3.26)$$

Fixed  $l \geq \frac{c_2}{\tilde{C}}$ , since  $|\nabla \psi_l(u_k)|^p = T_l(u_k) |\nabla u_k|^p = l |\nabla u_k|^p$  on  $\Omega \cap \{u_k \geq l\}$  and using (3.26), one has

$$\begin{aligned} \int_{\Omega} |\nabla u_k|^p dx &= \int_{\Omega \cap \{u_k \leq l\}} |\nabla u_k|^p dx + \int_{\Omega \cap \{u_k \geq l\}} |\nabla u_k|^p dx \\ &= \int_{\Omega} |\nabla T_l(u_k)|^p dx + \frac{1}{l} \int_{\Omega \cap \{u_k \geq l\}} |\nabla \psi_l(u_k)|^p dx \\ &\leq \int_{\Omega} |\nabla T_l(u_k)|^p dx + \frac{1}{l} \int_{\Omega} |\nabla \psi_l(u_k)|^p dx \\ &\leq \int_{\Omega} |\nabla T_l(u_k)|^p dx + \frac{\tilde{C}}{c_2} \int_{\Omega} |\nabla \psi_l(u_k)|^p dx \\ &\leq C(l, f, p, c_2, \sigma, \vartheta, C_{N,p}, \Omega). \end{aligned} \quad (3.27)$$

Then,  $\|u_k\|_{W_0^{1,p}(\Omega)}$  is uniformly bounded on  $k$ . Therefore, up to a subsequence, it follows that  $u_k \rightharpoonup u$  in  $W_0^{1,p}(\Omega)$  and a.e. in  $\Omega$ .

*Step 2:* We show the strong convergence in  $L^1(\Omega)$  of the singular term.

By Hölder and Hardy inequalities we have

$$\begin{aligned} \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) dx &\leq \int_{\Omega} \frac{u_k^q}{|x|^p} dx = \int_{\Omega} \frac{u_k^q}{|x|^q} \frac{1}{|x|^{p-q}} dx \\ &\leq \left( \int_{\Omega} \frac{u_k^p}{|x|^p} dx \right)^{\frac{q}{p}} \left( \int_{\Omega} \frac{1}{|x|^p} dx \right)^{\frac{p-q}{p}} \leq c \left( \int_{\Omega} |\nabla u_k|^p dx \right)^{\frac{q}{p}} \leq \bar{C}, \end{aligned} \quad (3.28)$$

where  $\bar{C}$  is a positive constant that does not depend on  $k$ . Hence we deduce that  $T_k \left( \frac{u_k^q}{|x|^p} \right)$  is bounded in  $L^1(\Omega)$ .

Since  $u_k \rightarrow u$  a.e. in  $\Omega$ , by definition of  $T_k(s)$ , we get that

$$T_k \left( \frac{u_k^q}{|x|^p} \right) \rightarrow \frac{u^q}{|x|^p} \text{ a.e. in } \Omega.$$

Noting that  $T_k \left( \frac{u_k^q}{|x|^p} \right) \geq 0$ , we can use Fatou Lemma and obtain

$$\int_{\Omega} \frac{u^q}{|x|^p} dx \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) dx,$$

then  $\frac{u^q}{|x|^p} \in L^1(\Omega)$ .

We now consider a measurable set  $E \subset \Omega$ . Arguing as in (3.28), using again Fatou Lemma and Hölder inequality, we have

$$\int_E T_k \left( \frac{u_k^q}{|x|^p} \right) dx \leq \int_E \frac{u_k^q}{|x|^p} dx \leq c \|\nabla u_k\|_{\frac{q}{p}}^{\frac{q}{p}} \left( \int_E \frac{1}{|x|^p} dx \right)^{\frac{p-q}{p}} < \delta(\mu(E))$$

uniformly on  $k$ , where  $\lim_{s \rightarrow 0} \delta(s) = 0$ .

Thus, from Vitali Theorem, we get

$$T_k \left( \frac{u_k^q}{|x|^p} \right) \rightarrow \frac{u^q}{|x|^p} \text{ in } L^1(\Omega). \quad (3.29)$$

*Step 3:* We show the strong convergence of  $|\nabla u_k|^p \rightarrow |\nabla u|^p$  in  $L^1(\Omega)$ . We need two preliminary results.

**Lemma 3.4.** *Let  $u_k$  be a weak solution to the problem (3.1). Then*

$$\lim_{n \rightarrow +\infty} \int_{\{u_k \geq n\}} |\nabla u_k|^p dx = 0$$

*uniformly on  $k$ .*

*Proof.* Let us consider the functions

$$G_n(s) = s - T_n(s) \quad \text{and} \quad \tau_{n-1}(s) = T_1(G_{n-1}(s)).$$

We get that

$$\tau_{n-1}(u_k) = T_1(u_k - T_{n-1}(u_k)) = \begin{cases} 0 & \text{if } u_k < n-1, \\ u_k - (n-1) & \text{if } n-1 \leq u_k < n, \\ 1 & \text{if } u_k \geq n. \end{cases} \quad (3.30)$$

Using  $\tau_{n-1}(u_k)$  as a function test in (3.1), we get

$$\begin{aligned} & \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla \tau_{n-1}(u_k)) dx + \int_{\Omega} B(|\nabla u_k|) \tau_{n-1}(u_k) dx \\ &= \vartheta \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) \tau_{n-1}(u_k) dx + \int_{\Omega} T_k(f) \tau_{n-1}(u_k) dx. \end{aligned} \quad (3.31)$$

We want to estimate the left-hand side of (3.31). Using the definition of  $\tau_{n-1}(u_k)$  and (2.1), we get

$$\begin{aligned} & \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla \tau_{n-1}(u_k)) dx = \int_{\{n-1 \leq u_k < n\}} A(|\nabla u_k|)(\nabla u_k, \nabla u_k) dx \\ &= \int_{\{n-1 \leq u_k < n\}} A(|\nabla u_k|) |\nabla u_k|^2 dx \geq c_2 \int_{\{n-1 \leq u_k < n\}} |\nabla u_k|^p dx \\ &= c_2 \int_{\Omega} |\nabla \tau_{n-1}(u_k)|^p dx \end{aligned} \quad (3.32)$$

while from (1.6) one has

$$\int_{\Omega} B(|\nabla u_k|) \tau_{n-1}(u_k) dx \geq \sigma \int_{\Omega} |\nabla u_k|^p \tau_{n-1}(u_k) dx. \quad (3.33)$$

Then, using (3.32) and (3.33) in (3.31), one has

$$\begin{aligned} c_2 \int_{\Omega} |\nabla \tau_{n-1}(u_k)|^p dx + \sigma \int_{\Omega} |\nabla u_k|^p \tau_{n-1}(u_k) dx \\ \leq \vartheta \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) \tau_{n-1}(u_k) dx + \int_{\Omega} T_k(f) \tau_{n-1}(u_k) dx. \end{aligned} \quad (3.34)$$

Moreover, from (3.30)

$$\tau_{n-1}(u_k) |\nabla u_k|^p \geq |\nabla u_k|^p \chi_{\{u_k \geq n\}}, \quad (3.35)$$

hence (3.34) becomes

$$\begin{aligned} \sigma \int_{\{u_k \geq n\}} |\nabla u_k|^p dx \leq c_2 \int_{\Omega} |\nabla \tau_{n-1}(u_k)|^p dx + \sigma \int_{\Omega} |\nabla u_k|^p \tau_{n-1}(u_k) dx \\ \leq \vartheta \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) \tau_{n-1}(u_k) dx + \int_{\Omega} T_k(f) \tau_{n-1}(u_k) dx. \end{aligned} \quad (3.36)$$

By (3.27),  $\{u_k\}_{k \in \mathbb{N}}$  is uniformly bounded in  $W_0^{1,p}(\Omega)$  on  $k$  so, up to a subsequence,  $u_k$  weakly converges in  $W_0^{1,p}(\Omega)$ , strongly converges in  $L^p(\Omega)$  with  $1 \leq p < p^*$  and a.e. in  $\Omega$ .

Then

$$\begin{aligned} |\{x \in \Omega : n-1 \leq u_k(x) < n\}| &\rightarrow 0 \quad \text{if } n \rightarrow +\infty, \\ |\{x \in \Omega : u_k(x) \geq n\}| &\rightarrow 0 \quad \text{if } n \rightarrow +\infty \end{aligned} \quad (3.37)$$

uniformly on  $k$ .

By (3.30) and (3.36) we get

$$\begin{aligned} \sigma \int_{\{u_k \geq n\}} |\nabla u_k|^p dx &\leq \vartheta \int_{\{n-1 \leq u_k < n\}} T_k \left( \frac{u_k^q}{|x|^p} \right) \tau_{n-1}(u_k) dx \\ &+ \vartheta \int_{\{u_k \geq n\}} T_k \left( \frac{u_k^q}{|x|^p} \right) \tau_{n-1}(u_k) dx + \int_{\{n-1 \leq u_k < n\}} T_k(f) \tau_{n-1}(u_k) dx \\ &+ \int_{\{u_k \geq n\}} T_k(f) \tau_{n-1}(u_k) dx. \end{aligned}$$

Then, by (3.37) and dominated convergence theorem, we have

$$\lim_{n \rightarrow +\infty} \int_{\{u_k \geq n\}} |\nabla u_k|^p dx = 0.$$

□

**Lemma 3.5.** *Consider  $u_k \rightharpoonup u$  in  $W_0^{1,p}(\Omega)$ . Then one has for every  $m$*

$$T_m(u_k) \rightarrow T_m(u) \text{ in } W_0^{1,p}(\Omega) \quad \text{for } k \rightarrow +\infty.$$

*Proof.* Notice that

$$\begin{aligned} \|T_m(u_k) - T_m(u)\|_{W_0^{1,p}(\Omega)} &\leq \|(T_m(u_k) - T_m(u))^+\|_{W_0^{1,p}(\Omega)} \\ &\quad + \|(T_m(u_k) - T_m(u))^- \|_{W_0^{1,p}(\Omega)}. \end{aligned} \quad (3.38)$$

First of all we study the asymptotic behaviour of  $\|(T_m(u_k) - T_m(u))^+\|_{W_0^{1,p}(\Omega)}$ .

If we take  $(T_m(u_k) - T_m(u))^+$  as a test function in (3.1) we get

$$\begin{aligned} &\int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+) dx + \int_{\Omega} B(|\nabla u_k|)(T_m(u_k) - T_m(u))^+ dx \\ &= \int_{\Omega} \left( \vartheta T_k \left( \frac{u_k^q}{|x|^p} \right) + T_k(f) \right) (T_m(u_k) - T_m(u))^+ dx. \end{aligned} \quad (3.39)$$

Since  $T_m(u_k) \rightharpoonup T_m(u)$  in  $W_0^{1,p}(\Omega)$  and  $T_m(u_k) \rightarrow T_m(u)$  a.e. in  $\Omega$ , we have  $(T_m(u_k) - T_m(u))^+ \rightarrow 0$  a.e. in  $\Omega$  and  $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$  in  $W_0^{1,p}(\Omega)$ . Thus, the right-hand side of (3.39), using (3.29) and dominated convergence theorem, tends to zero as  $k$  goes to infinity.

Then, for  $k \rightarrow +\infty$

$$\begin{aligned} &\int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+) dx \\ &\quad + \int_{\Omega} B(|\nabla u_k|)(T_m(u_k) - T_m(u))^+ dx = o(1). \end{aligned} \quad (3.40)$$

Let us note that equation (3.40) is equivalent to

$$\begin{aligned} &\int_{\Omega} (A(|\nabla u_k|)\nabla u_k - A(|\nabla u|)\nabla u, \nabla(T_m(u_k) - T_m(u))^+) dx \\ &\quad + \int_{\Omega} B(|\nabla u_k|)(T_m(u_k) - T_m(u))^+ dx \\ &= - \int_{\Omega} A(|\nabla u|)(\nabla u, \nabla(T_m(u_k) - T_m(u))^+) dx + o(1) = o(1), \end{aligned} \quad (3.41)$$

since by  $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$  in  $W_0^{1,p}(\Omega)$  one has that

$$\int_{\Omega} A(|\nabla u|)(\nabla u, \nabla(T_m(u_k) - T_m(u))^+) dx = o(1) \text{ as } k \rightarrow +\infty.$$

We have that

$$\begin{aligned} (T_m(u_k) - T_m(u))^+ &= \begin{cases} (u_k - T_m(u))^+ & \text{if } 0 < u_k \leq m, \\ m - T_m(u) & \text{if } u_k > m, \end{cases} \quad (3.42) \\ \nabla(T_m(u_k) - T_m(u))^+ &= \begin{cases} \nabla(u_k - u)^+ & \text{if } u_k \leq m, u \leq m, \\ 0 & \text{if } u_k \leq m, u > m, \\ 0 & \text{if } u_k > m, u \geq m, \\ -\nabla u & \text{if } u_k > m, u < m. \end{cases} \end{aligned}$$

So, using (3.42) and the fact that  $B$  is a positive function, (3.41) becomes

$$\begin{aligned} & \int_{\Omega \cap \{u_k \leq m, u \leq m\}} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla(u_k - u)^+ \right) dx \\ & + \int_{\Omega \cap \{u_k > m, u < m\}} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, -\nabla u \right) dx \\ & = \int_{\Omega} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla(T_m(u_k) - T_m(u))^+ \right) dx \leq o(1). \end{aligned} \quad (3.43)$$

We set:

$$\begin{aligned} D &:= \int_{\Omega} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla(T_m(u_k) - T_m(u))^+ \right) dx, \\ D_1 &:= \int_{\Omega \cap \{u_k \leq m, u \leq m\}} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla(u_k - u)^+ \right) dx, \\ D_2 &:= \int_{\Omega \cap \{u_k > m, u < m\}} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, -\nabla u \right) dx. \end{aligned}$$

- For the term  $D_1$ , using (1.4), we have

$$\begin{aligned} D_1 &= \int_{\Omega \cap \{u_k \leq m, u \leq m\}} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla(u_k - u)^+ \right) dx \\ &\geq c_2 \int_{\Omega \cap \{u_k \leq m, u \leq m\}} (|\nabla u_k| + |\nabla u|)^{p-2} |\nabla(u_k - u)^+|^2 dx \geq 0. \end{aligned} \quad (3.44)$$

- We write  $D_2$  as

$$\begin{aligned} D_2 &= D_{2,1} + D_{2,2} \\ &= \int_{\Omega \cap \{u_k > m, u < m\}} A(|\nabla u|) |\nabla u|^2 dx \\ &\quad + \int_{\Omega \cap \{u_k > m, u < m\}} A(|\nabla u_k|) (\nabla u_k, -\nabla u) dx. \end{aligned}$$

Then, by (2.4) and (1.3), we have

$$\begin{aligned} |D_{2,1}| &\leq \int_{\Omega \cap \{u_k > m, u < m\}} |A(|\nabla u|)| |\nabla u|^2 dx \\ &\leq K C_K |\Omega \cap \{u_k > m, u < m\} \cap \{|\nabla u| < K\}| \\ &\quad + c_1 \int_{\Omega} |\nabla u|^p \chi_{\Omega \cap \{u_k > m, u < m\} \cap \{|\nabla u| \geq K\}} dx. \end{aligned}$$

Since  $u_k \rightarrow u$  a.e. in  $\Omega$  for  $k \rightarrow +\infty$ , we get that  $\chi_{\Omega \cap \{u_k > m, u < m\}} \rightarrow 0$ . Moreover,  $|\nabla u|^p \in L^1(\Omega)$ , then using Dominated Convergence, one has that

$$|D_{2,1}| = o(1).$$

Analogously, using Hölder inequality, (2.4), (1.3), dominated convergence theorem and (3.27)

$$\begin{aligned}
|D_{2,2}| &= \left| \int_{\Omega \cap \{u_k > m, u < m\}} (A(|\nabla u_k|) \nabla u_k, \nabla u) dx \right| \\
&\leq \left( \int_{\Omega} |A(|\nabla u_k|) \nabla u_k|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega \cap \{u_k > m, u < m\}} |\nabla u|^p dx \right)^{\frac{1}{p}} \\
&\leq \left( C_K^{\frac{p}{p-1}} |\Omega \cap \{|\nabla u_k| < K\}| + c \int_{\Omega \cap \{|\nabla u_k| \geq K\}} |\nabla u_k|^p dx \right)^{\frac{p-1}{p}} \\
&\quad \cdot \left( \int_{\Omega \cap \{u_k > m, u < m\}} |\nabla u|^p dx \right)^{\frac{1}{p}}
\end{aligned}$$

Arguing for that last integral as in  $|D_{2,1}|$  and using the fact that  $\Omega$  is a bounded set and  $u_k$  is uniformly bounded in  $W_0^{1,p}(\Omega)$ , we get that

$$|D_{2,2}| = o(1) \text{ as } k \rightarrow +\infty.$$

Then

$$|D_2| \leq |D_{2,1}| + |D_{2,2}| = o(1) \Rightarrow D_2 = o(1). \quad (3.45)$$

By (3.43), (3.44), (3.45) we obtain

$$o(1) = D_2 < D_1 + D_2 = D \leq o(1),$$

that is

$$D = \int_{\Omega} \left( A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla (T_m(u_k) - T_m(u))^+ \right) dx = o(1). \quad (3.46)$$

Since  $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$  in  $W_0^{1,p}(\Omega)$ , from (3.46) we get

$$\int_{\Omega} A(|\nabla u_k|) (\nabla u_k, \nabla (T_m(u_k) - T_m(u))^+) dx = o(1) \text{ as } k \rightarrow +\infty. \quad (3.47)$$

Since  $T_m(u_k) = u_k$  on  $\{0 < u_k \leq m\}$ , we estimate the left-hand side of (3.47) as

$$\begin{aligned}
o(1) &= \int_{\Omega} A(|\nabla u_k|) (\nabla u_k, \nabla (T_m(u_k) - T_m(u))^+) dx \\
&= \int_{\Omega \cap \{u_k > m\}} A(|\nabla u_k|) (\nabla u_k, \nabla (T_m(u_k) - T_m(u))^+) dx \\
&\quad + \int_{\Omega \cap \{u_k \leq m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx \\
&\quad + \int_{\Omega \cap \{u_k \leq m\}} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+ \right) dx.
\end{aligned} \quad (3.48)$$

Let us denote

$$\chi_m := \chi_{\Omega \cap \{u_k > m\}} \quad (3.49)$$

and consider the first term of the right-hand side of (3.48). Using (3.42), Hölder inequality (2.4) and (1.3), we get

$$\begin{aligned}
 & \left| \int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \chi_m \nabla T_m(u)) dx \right| \\
 & \leq \left( \int_{\Omega} |A(|\nabla u_k|) \nabla u_k|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |\chi_m \nabla T_m(u)|^p dx \right)^{\frac{1}{p}} \\
 & \leq \left( C_K^{\frac{p}{p-1}} |\Omega \cap \{|\nabla u_k| < K\}| + c \int_{\Omega \cap \{|\nabla u_k| \geq K\}} |\nabla u_k|^p dx \right)^{\frac{p-1}{p}} \|\chi_m \nabla T_m(u)\|_p \\
 & \leq c(|\Omega| + \|u_k\|_{W_0^{1,p}(\Omega)}^p)^{\frac{p}{p-1}} \|\chi_m \nabla T_m(u)\|_p.
 \end{aligned} \tag{3.50}$$

Since  $\lim_{k \rightarrow +\infty} \chi_m \nabla T_m(u) = 0$  a.e. in  $\Omega$ , using dominated convergence theorem, we get

$$\|\chi_m \nabla T_m(u)\|_p \rightarrow 0 \text{ as } k \rightarrow +\infty$$

and since  $u_k$  is uniformly bounded in  $W_0^{1,p}(\Omega)$  on  $k$  by (3.27), we obtain

$$\left| \int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \chi_m \nabla T_m(u)) dx \right| = o(1) \text{ as } k \rightarrow +\infty$$

and in particular

$$\int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \chi_m \nabla T_m(u)) dx = o(1) \text{ as } k \rightarrow +\infty. \tag{3.51}$$

For the second term of the right-hand side of (3.48), we rewrite it as

$$\begin{aligned}
 & \int_{\Omega \cap \{u_k \leq m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx \\
 & = \int_{\Omega} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx \\
 & \quad - \int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx.
 \end{aligned} \tag{3.52}$$

Since  $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$  in  $W_0^{1,p}(\Omega)$ , the first term of the right-hand side goes to zero when  $k \rightarrow +\infty$ . Instead, for the second term, arguing as in (3.50), we obtain that for  $k \rightarrow +\infty$

$$\begin{aligned}
 & \left| \int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx \right| \\
 & \leq c(|\Omega| + \|T_m(u)\|_{W_0^{1,p}(\Omega)}^p)^{\frac{p}{p-1}} \|\chi_m \nabla T_m(u)\|_p = o(1) \text{ as } k \rightarrow +\infty.
 \end{aligned} \tag{3.53}$$

Then

$$\int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx = o(1)$$

as  $k \rightarrow +\infty$  and equation (3.52) becomes

$$\int_{\Omega \cap \{u_k \leq m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+) dx = o(1). \tag{3.54}$$

Now, we have to estimate the last term in the right-hand side of (3.48). We write

$$\begin{aligned}
 & \int_{\Omega \cap \{u_k \leq m\}} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx \\
 &= \int_{\Omega} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx \\
 & - \int_{\Omega \cap \{u_k > m\}} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx
 \end{aligned} \tag{3.55}$$

By (3.42), the second term of the right-hand side of (3.55) becomes

$$\int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|) |\nabla T_m(u)|^2 dx \tag{3.56}$$

We know that

$$\int_{\Omega \cap \{u_k > m\}} |\nabla T_m(u)|^p dx = o(1) \text{ for } k \rightarrow +\infty,$$

while using (2.4) and (1.3) we get

$$\begin{aligned}
 & \int_{\Omega \cap \{u_k > m\}} (A(|\nabla T_m(u)|) |\nabla T_m(u)|)^{\frac{p}{p-1}} dx \\
 & \leq C_K^{\frac{p}{p-1}} |\Omega| + c_1^{\frac{p}{p-1}} \int_{\Omega \cap \{u_k > m\} \cap \{|\nabla T_m(u)| \geq K\}} |\nabla T_m(u)|^p dx < +\infty
 \end{aligned} \tag{3.57}$$

Then using Holder inequality in (3.56), we obtain

$$\begin{aligned}
 & \left| \int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|) |\nabla T_m(u)|^2 dx \right| \\
 & \leq \left( \int_{\Omega \cap \{u_k > m\}} (A(|\nabla T_m(u)|) |\nabla T_m(u)|)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega \cap \{u_k > m\}} |\nabla T_m(u)|^p dx \right)^{\frac{1}{p}} = o(1)
 \end{aligned}$$

as  $k \rightarrow +\infty$ .

Therefore

$$\int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|) |\nabla T_m(u)|^2 dx = o(1) \quad \text{as } k \rightarrow +\infty. \tag{3.58}$$

Using (3.58), equation (3.55) becomes

$$\begin{aligned}
 & \int_{\Omega \cap \{u_k \leq m\}} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx \\
 &= \int_{\Omega} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx + o(1)
 \end{aligned} \tag{3.59}$$

Then, using (3.51), (3.54), (3.59), the equation (3.48) becomes

$$\begin{aligned} o(1) &= \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+) dx \\ &= \int_{\Omega} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx + o(1) \\ &\text{as } k \rightarrow +\infty. \end{aligned}$$

Using equations (1.4) and (2.2),

$$\begin{aligned} o(1) &= \int_{\Omega} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+ \right) dx \\ &\geq \begin{cases} c_2 \int_{\Omega} \frac{|\nabla(T_m(u_k) - T_m(u))^+|^2}{(|\nabla T_m(u_k)| + |\nabla T_m(u)|)^{2-p}} dx & \text{if } 1 < p < 2, \\ c_2 \int_{\Omega} |\nabla(T_m(u_k) - T_m(u))^+|^p dx & \text{if } p \geq 2. \end{cases} \end{aligned} \quad (3.60)$$

Arguing as we have done from (3.10) to (3.12), we obtain

$$\|(T_m(u_k) - T_m(u))^+\|_{W_0^{1,p}(\Omega)} \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (3.61)$$

For the study of the asymptotic behaviour of  $\|(T_m(u_k) - T_m(u))^- \|_{W_0^{1,p}(\Omega)}$ , we use  $e^{-\hat{C}T_m(u_k)}(T_m(u_k) - T_m(u))^-$  as a test function in weak formulation of (3.1) obtaining

$$\begin{aligned} &\int_{\Omega} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla(T_m(u_k) - T_m(u))^-) dx \\ &\quad - \int_{\Omega} \hat{C} e^{-\hat{C}T_m(u_k)} \left( A(|\nabla u_k|) \nabla u_k, \nabla T_m(u_k) \right) (T_m(u_k) - T_m(u))^- dx \\ &\quad + \int_{\Omega} B(|\nabla u_k|) e^{-\hat{C}T_m(u_k)} (T_m(u_k) - T_m(u))^- dx \\ &= \int_{\Omega} \left( \vartheta T_k \left( \frac{u_k^q}{|x|^p} \right) + T_k(f) \right) e^{-\hat{C}T_m(u_k)} (T_m(u_k) - T_m(u))^- dx \end{aligned} \quad (3.62)$$

As above, since  $(T_m(u_k) - T_m(u))^- \rightharpoonup 0$  in  $W_0^{1,p}(\Omega)$  and  $(T_m(u_k) - T_m(u))^- \rightarrow 0$  a.e. in  $\Omega$ , by dominated convergence theorem, the right-hand side of (3.62) tends to zero when  $k$  goes to infinity. Moreover, splitting the domains of the integrals in the left-hand side of (3.62), we get

$$\begin{aligned}
 & \int_{\Omega} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla (T_m(u_k) - T_m(u))^-) dx \\
 & - \int_{\Omega \cap \{u_k > m\}} \hat{C} e^{-\hat{C}T_m(u_k)} \left( A(|\nabla u_k|) \nabla u_k, \nabla T_m(u_k) \right) (T_m(u_k) - T_m(u))^- dx \\
 & - \int_{\Omega \cap \{u_k \leq m\}} \hat{C} e^{-\hat{C}T_m(u_k)} \left( A(|\nabla u_k|) \nabla u_k, \nabla T_m(u_k) \right) (T_m(u_k) - T_m(u))^- dx \\
 & + \int_{\Omega \cap \{u_k > m\}} B(|\nabla u_k|) e^{-\hat{C}T_m(u_k)} (T_m(u_k) - T_m(u))^- dx \\
 & + \int_{\Omega \cap \{u_k \leq m\}} B(|\nabla u_k|) e^{-\hat{C}T_m(u_k)} (T_m(u_k) - T_m(u))^- dx = o(1) \text{ as } k \rightarrow +\infty.
 \end{aligned} \tag{3.63}$$

We have that

$$\begin{aligned}
 (T_m(u_k) - T_m(u))^- &= \begin{cases} (u_k - T_m(u))^- & \text{if } 0 < u_k \leq m, \\ 0 & \text{if } u_k > m, \end{cases} \tag{3.64} \\
 \nabla (T_m(u_k) - T_m(u))^- &= \begin{cases} \nabla (u_k - u)^- & \text{if } u_k \leq m, u \leq m, \\ \nabla u_k & \text{if } u_k \leq m, u > m, \\ 0 & \text{if } u_k > m. \end{cases}
 \end{aligned}$$

Recalling the definition of  $\chi_m$  given in (3.49), one has that  $(T_m(u_k) - T_m(u))^- \chi_m = 0$  and  $T_m(u_k) = u_k$  on  $\{0 < u_k \leq m\}$ , hence (3.63) becomes

$$\begin{aligned}
 & \int_{\Omega} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla (T_m(u_k) - T_m(u))^-) dx \\
 & + \int_{\Omega \cap \{u_k \leq m\}} B(|\nabla T_m(u_k)|) e^{-\hat{C}T_m(u_k)} (T_m(u_k) - T_m(u))^- dx \\
 & - \int_{\Omega \cap \{u_k \leq m\}} \hat{C} A(|\nabla T_m(u_k)|) |\nabla T_m(u_k)|^2 e^{-\hat{C}T_m(u_k)} (T_m(u_k) - T_m(u))^- dx \\
 & = o(1) \text{ as } k \rightarrow +\infty.
 \end{aligned} \tag{3.65}$$

Moreover, using (2.5), we have that

$$\int_{\Omega} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla (T_m(u_k) - T_m(u))^-) dx \leq o(1) \text{ as } k \rightarrow +\infty. \tag{3.66}$$

Recalling (3.64), we define  $A_1 := \{x \in \Omega : u_k \leq m, u \leq m\}$  and  $A_2 := \{x \in \Omega : u_k \leq m, u > m\}$  and we split the integral in (3.66) as follows:

$$\begin{aligned}
 & \int_{\Omega} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla (T_m(u_k) - T_m(u))^-) dx \\
 & = \int_{\Omega \cap A_1} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla (u_k - u)^-) dx \\
 & + \int_{\Omega \cap A_2} e^{-\hat{C}T_m(u_k)} A(|\nabla u_k|) |\nabla u_k|^2 dx \leq o(1).
 \end{aligned} \tag{3.67}$$

Since  $(T_m(u_k) - T_m(u))^- \rightarrow 0$  in  $W_0^{1,p}(\Omega)$  for  $k \rightarrow +\infty$ ,

$$\begin{aligned}
& \int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \nabla (T_m(u_k) - T_m(u))^-) dx \\
&= \int_{\Omega} (A(|\nabla u_k|) \nabla u_k - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^-) dx \\
&\quad + \int_{\Omega} (A(|\nabla T_m(u)|) \nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^-) dx \\
&= \int_{\Omega} (A(|\nabla u_k|) \nabla u_k - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^-) dx + o(1). \\
&= \int_{\Omega \cap A_1} (A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla (u_k - u)^-) dx \\
&\quad + \int_{\Omega \cap A_2} A(|\nabla u_k|) |\nabla u_k|^2 dx + o(1)
\end{aligned} \tag{3.68}$$

Moreover, one has that  $e^{\hat{C}T_m(u_k)} \leq e^{m\hat{C}} =: \gamma_m$  uniformly on  $k$ . Then, using (1.4) and (3.67), we estimate (3.68) as

$$\begin{aligned}
& \int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \nabla (T_m(u_k) - T_m(u))^-) dx \\
&= \int_{\Omega \cap A_1} e^{\hat{C}T_m(u_k)} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla (u_k - u)^-) dx \\
&\quad + \int_{\Omega \cap A_2} e^{\hat{C}T_m(u_k)} e^{-\hat{C}T_m(u_k)} A(|\nabla u_k|) |\nabla u_k|^2 dx + o(1) \\
&\leq \gamma_m \int_{\Omega \cap A_1} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k - A(|\nabla u|) \nabla u, \nabla (u_k - u)^-) dx \\
&\quad + \gamma_m \int_{\Omega \cap A_2} e^{-\hat{C}T_m(u_k)} A(|\nabla u_k|) |\nabla u_k|^2 dx + o(1) \\
&\leq \gamma_m \left[ \int_{\Omega \cap A_1} e^{-\hat{C}T_m(u_k)} (A(|\nabla u_k|) \nabla u_k, \nabla (u_k - u)^-) dx \right. \\
&\quad \left. + \int_{\Omega \cap A_2} e^{-\hat{C}T_m(u_k)} A(|\nabla u_k|) |\nabla u_k|^2 dx \right] \\
&\quad - \gamma_m \int_{\Omega \cap A_1} e^{-\hat{C}T_m(u_k)} (A(|\nabla u|) \nabla u, \nabla (u_k - u)^-) dx + o(1) \\
&\leq -\gamma_m \int_{\Omega \cap A_1} e^{-\hat{C}T_m(u_k)} (A(|\nabla u|) \nabla u, \nabla (u_k - u)^-) dx + (\gamma_m + 1)o(1).
\end{aligned} \tag{3.69}$$

Now, we estimate the right hand-side of (3.69) in this way:

$$\begin{aligned}
 & \int_{\Omega \cap A_1} e^{-\hat{C}T_m(u_k)} (A(|\nabla u|) \nabla u, \nabla(u_k - u)^-) dx \\
 &= \int_{\Omega} e^{-\hat{C}T_m(u_k)} (A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^-) dx \\
 &= \int_{\Omega} e^{-\hat{C}(T_m(u_k) - T_m(u))^-} e^{-\hat{C}T_m(u)} (A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^-) dx \\
 &= -\frac{1}{\hat{C}} \int_{\Omega} e^{-\hat{C}T_m(u)} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla(e^{-\hat{C}(T_m(u_k) - T_m(u))^-} - 1)) dx.
 \end{aligned} \tag{3.70}$$

Since  $\{e^{-\hat{C}(T_m(u_k) - T_m(u))^-} - 1\}_{k \in \mathbb{N}}$  is uniformly bounded in  $W_0^{1,p}(\Omega)$ , up to a subsequence there exists  $g \in W_0^{1,p}(\Omega)$  such that

$$e^{-\hat{C}(T_m(u_k) - T_m(u))^-} - 1 \rightharpoonup g \text{ as } n \rightarrow +\infty. \tag{3.71}$$

In particular we note that  $g \equiv 0$  since  $T_m(u_k) \rightarrow T_m(u)$  a.e. in  $\Omega$  as  $k \rightarrow +\infty$ . Hence, by (3.71), for  $k \rightarrow +\infty$

$$\int_{\Omega} e^{-\hat{C}T_m(u)} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla(e^{-\hat{C}(T_m(u_k) - T_m(u))^-} - 1)) dx \rightarrow 0. \tag{3.72}$$

Then, using (3.70) and (3.72) in (3.69), one has that

$$\int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \nabla(T_m(u_k) - T_m(u))^-) dx \leq o(1) \text{ as } k \rightarrow +\infty. \tag{3.73}$$

Arguing in the similar way as we have done from (3.40) to (3.47), we get

$$\int_{\Omega} (A(|\nabla u_k|) \nabla u_k, \nabla(T_m(u_k) - T_m(u))^-) dx = o(1) \text{ as } k \rightarrow +\infty. \tag{3.74}$$

In order to obtain the desired result, we proceed writing the left-hand side of (3.74) as

$$\begin{aligned}
 o(1) &= \int_{\Omega} A(|\nabla u_k|) (\nabla u_k, \nabla(T_m(u_k) - T_m(u))^-) dx \\
 &= \int_{\Omega \cap \{u_k > m\}} A(|\nabla u_k|) (\nabla u_k, \nabla(T_m(u_k) - T_m(u))^-) dx \\
 &+ \int_{\Omega \cap \{u_k \leq m\}} A(|\nabla T_m(u)|) (\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^-) dx \\
 &+ \int_{\Omega \cap \{u_k \leq m\}} \left( A(|\nabla T_m(u_k)|) \nabla T_m(u_k) - A(|\nabla T_m(u)|) \nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^- \right) dx.
 \end{aligned} \tag{3.75}$$

The first term of the right-hand side of (3.75), by (3.64), is zero since  $(T_m(u_k) - T_m(u))^- \chi_m = 0$ .

The second term of the right-hand side of (3.75) can be rewritten as

$$\begin{aligned} & \int_{\Omega \cap \{u_k \leq m\}} A(|\nabla T_m(u)|)(\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^-) dx \\ &= \int_{\Omega} A(|\nabla T_m(u)|)(\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^-) dx \\ & \quad - \int_{\Omega \cap \{u_k > m\}} A(|\nabla T_m(u)|)(\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^-) dx. \end{aligned} \quad (3.76)$$

Since  $(T_m(u_k) - T_m(u))^- \rightarrow 0$  in  $W_0^{1,p}(\Omega)$ , the first term of the right-hand side goes to zero when  $k \rightarrow +\infty$ . While, the second term, is zero since  $(T_m(u_k) - T_m(u))^- \chi_m = 0$ .

Then, the equation (3.75) becomes

$$\begin{aligned} o(1) &= \int_{\Omega \cap \{u_k \leq m\}} \left( A(|\nabla T_m(u_k)|)\nabla T_m(u_k) - A(|\nabla T_m(u)|)\nabla T_m(u), \right. \\ & \quad \left. \dots \nabla(T_m(u_k) - T_m(u))^- \right) dx. \end{aligned} \quad (3.77)$$

Now, we estimate the the right-hand side of (3.77).

We write

$$\begin{aligned} & \int_{\Omega \cap \{u_k \leq m\}} \left( A(|\nabla T_m(u_k)|)\nabla T_m(u_k) - A(|\nabla T_m(u)|)\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^- \right) dx \\ &= \int_{\Omega} \left( A(|\nabla T_m(u_k)|)\nabla T_m(u_k) - A(|\nabla T_m(u)|)\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^- \right) dx \\ & \quad - \int_{\Omega \cap \{u_k > m\}} \left( A(|\nabla T_m(u_k)|)\nabla T_m(u_k) - A(|\nabla T_m(u)|)\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^- \right) dx. \end{aligned} \quad (3.78)$$

By (3.64), the second term of the right-hand side is zero then equation (3.77) becomes

$$o(1) = \int_{\Omega} \left( A(|\nabla T_m(u_k)|)\nabla T_m(u_k) - A(|\nabla T_m(u)|)\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^- \right) dx. \quad (3.79)$$

From equation (1.4) and (2.2),

$$\begin{aligned} o(1) &= \int_{\Omega} \left( A(|\nabla T_m(u_k)|)\nabla T_m(u_k) - A(|\nabla T_m(u)|)\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^- \right) dx \\ &\geq \begin{cases} c_2 \int_{\Omega} \frac{|\nabla(T_m(u_k) - T_m(u))^-|^2}{(|\nabla T_m(u_k)| + |\nabla T_m(u)|)^{2-p}} dx & \text{if } 1 < p < 2, \\ c_2 \int_{\Omega} |\nabla(T_m(u_k) - T_m(u))^-|^p dx & \text{if } p \geq 2. \end{cases} \end{aligned} \quad (3.80)$$

Arguing as we have done from (3.10) to (3.12), we obtain

$$\|(T_m(u_k) - T_m(u))^- \|_{W_0^{1,p}(\Omega)} \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (3.81)$$

Summarizing up, from (3.38) we get

$$\|T_m(u_k) - T_m(u)\|_{W_0^{1,p}(\Omega)} \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (3.82)$$

□

Using Lemma 3.5, up to a subsequence, we get

$$|\nabla T_m(u_k)|^p \rightarrow |\nabla T_m(u)|^p \text{ a.e. in } \Omega, \text{ for } k \rightarrow +\infty. \quad (3.83)$$

For almost everywhere  $x$  fixed in  $\Omega$ , we choose  $\eta \gg |u(x)|$ . Since  $u_k \rightarrow u$  a.e. in  $\Omega$  for  $k \rightarrow +\infty$ , we get that  $|u_k(x)| < \eta$  definitely.

Choosing  $m = \eta$  in (3.83), one has

$$|\nabla u_k|^p \rightarrow |\nabla u|^p \text{ a.e. in } \Omega \text{ for } k \rightarrow +\infty.$$

In order to use Vitali Theorem we need to prove the uniform-integrability of  $|\nabla u_k|^p$ . Let  $E \subset \Omega$  be a measurable set, then

$$\begin{aligned} \int_E |\nabla u_k|^p dx &= \int_{E \cap \{u_k > m\}} |\nabla u_k|^p dx + \int_{E \cap \{u_k \leq m\}} |\nabla u_k|^p dx \\ &= \int_{E \cap \{u_k > m\}} |\nabla u_k|^p dx + \int_E |\nabla T_m(u_k)|^p dx. \end{aligned} \quad (3.84)$$

For fixed  $\varepsilon > 0$ , using Lemma 3.4 in the first term of (3.84), there exists  $\bar{m}$  that does not depend on  $k$ , such that

$$\int_{E \cap \{u_k > m\}} |\nabla u_k|^p dx \leq \int_{\Omega \cap \{u_k > m\}} |\nabla u_k|^p dx \leq \frac{\varepsilon}{2}.$$

By (3.82) and up to a subsequence, there exists  $v(x) \in L^1(\Omega)$  such that  $|\nabla T_{\bar{m}}(u_k)|^p \leq v(x)$  a.e. in  $\Omega$ . So for  $|E|$  small we get

$$\int_E |\nabla T_{\bar{m}}(u_k)|^p dx \leq \frac{\varepsilon}{2}.$$

Therefore,

$$\int_E |\nabla u_k|^p dx \leq \varepsilon.$$

is uniformly integrable on  $k$ .

Collecting the previous result, the left-hand side of (3.84) is uniformly integrable on  $k$ .

Then Vitali Theorem implies that

$$|\nabla u_k|^p \rightarrow |\nabla u|^p \text{ in } L^1(\Omega), \quad (3.85)$$

that is equivalent to

$$\|u_k\|_{W_0^{1,p}(\Omega)} \rightarrow \|u\|_{W_0^{1,p}(\Omega)}. \quad (3.86)$$

*Step 4:* In order to obtain a solution to the problem (1.1), we note that by [6, Theorem 1]

$$\lim_{k \rightarrow +\infty} \|u_k - u\|_{W_0^{1,p}(\Omega)} = 0.$$

Moreover we observe that  $B(|\nabla u_k|) \rightarrow B(|\nabla u|)$  in  $L^1(\Omega)$ .

Since  $B$  is a continuous function

$$B(|\nabla u_k|) \rightarrow B(|\nabla u|) \text{ a.e. in } \Omega.$$

From (3.85), up to a subsequence, there exists  $v(x) \in L^1(\Omega)$  such that  $|\nabla u_k|^p \leq v(x)$  a.e. in  $\Omega$ .

Then, if  $E \subset \Omega$  is a measurable set, using (2.5), (2.4) and (1.3), we have that

$$\begin{aligned} \int_E B(|\nabla u_k|) dx &\leq \hat{C} \int_E |A(|\nabla u_k|)| |\nabla u_k|^2 dx \\ &\leq KC_K \hat{C} |E \cap \{|\nabla u_k| < K\}| + c_1 \hat{C} \int_{E \cap \{|\nabla u_k| \geq K\}} |\nabla u_k|^p dx \\ &\leq KC_K |E| + c_1 \hat{C} \int_E v(x) dx \end{aligned}$$

and, since  $v(x) \in L^1(\Omega)$ , we get that  $B(|\nabla u_k|)$  is uniformly integrable.

Using Vitali Theorem we have

$$B(|\nabla u_k|) \rightarrow B(|\nabla u|) \text{ in } L^1(\Omega).$$

Then, collecting all the previous results, by dominate convergence, passing to the limit for  $k \rightarrow +\infty$  in

$$\begin{aligned} \int_{\Omega} A(|\nabla u_k|)(\nabla u_k, \nabla \varphi) dx + \int_{\Omega} B(|\nabla u_k|) \varphi dx &= \vartheta \int_{\Omega} T_k \left( \frac{u_k^q}{|x|^p} \right) \varphi dx + \int_{\Omega} T_k(f) \varphi dx \\ &\quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned}$$

we obtain

$$\begin{aligned} \int_{\Omega} A(|\nabla u|)(\nabla u, \nabla \varphi) dx + \int_{\Omega} B(|\nabla u|) \varphi dx &= \vartheta \int_{\Omega} \frac{u^q}{|x|^p} \varphi dx + \int_{\Omega} f \varphi dx \\ &\quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned}$$

□

#### 4. REGULARITY OF THE ENERGY SOLUTION

In all this section we shall assume that the solution is bounded far from the origin, up to the boundary. In some cases this can be deduced by construction. In general, if the problem is at most critical the local boundedness of the solution follows by Theorem 1 in Serrin [29] taking into account remark page 261, that in our case is useful to deal with the first order term. Once we are considering locally bounded solutions we deduce that  $u \in C_{loc}^{1,\alpha}(\Omega \setminus \{0\})$  by [16, 19, 24, 25, 26, 27, 31, 32]. The boundedness of the solution up to the boundary can be proved via rectification arguments or via scaling arguments; supposing that  $\Omega$  is smooth,  $C_{loc}^{1,\alpha}(\bar{\Omega} \setminus \{0\})$  regularity follows by [20]. Finally outside  $Z_u$  and far from the origin, our equation can be rewritten as

$$-\operatorname{div}(a(x)\nabla u) = c(x)$$

where  $a(\cdot)$  and  $c(\cdot)$  are locally Hölder continuous. We are therefore in position to apply standard regularity theory (see e.g. [18]-Chapter 6) to deduce that  $u \in C^2(\Omega \setminus (Z_u \cup \{0\}))$ .

For this reason in what follows we will indicate by

$$\tilde{u}_{ij}(x) := \begin{cases} u_{x_i x_j}(x) & x \in \Omega \setminus (Z_u \cup \{0\}), \\ 0 & x \in Z_u \end{cases} \quad (4.1)$$

and  $\tilde{\nabla} u_i$  stands for "gradient" of  $(\tilde{u}_{i1}, \dots, \tilde{u}_{iN})$ .

Next results give us the summability properties of the second derivative, main

tool for the analysis of the qualitative properties of solutions of elliptic problems. The proofs follow the ideas introduced in [7, 14, 17] therefore we include only some details.

We will assume that  $A$  satisfies (1.3), (1.4) and also

$$-1 < \inf_{t>0} \frac{tA'(t)}{A(t)} =: m_A \leq M_A := \sup_{t>0} \frac{tA'(t)}{A(t)} < +\infty. \quad (4.2)$$

*Remark 4.1.* Recalling [10, Proposition 4.1] it is already proved that

$$A(1) \min\{t^{m_A}, t^{M_A}\} \leq A(t) \leq A(1) \max\{t^{m_A}, t^{M_A}\}.$$

Since  $m_A > -1$  there exists  $\eta \in [0, 1)$  such that  $m_A + \eta > 0$ ; hence

$$\lim_{t \rightarrow 0} t^\eta A(t) = 0$$

and, moreover, it is easy to check that  $tA(t)$  is non-decreasing on  $[0, +\infty)$ .

The following results deal with the study of regularity of solutions to the following problem

$$\begin{cases} -\operatorname{div}(A(|\nabla u|)\nabla u) + B(|\nabla u|) = h(x, u), & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.3)$$

where we set  $h(x, u) := \vartheta \frac{u^q}{|x|^p} + f(x)$  and  $u_i := \frac{\partial u}{\partial x_i}$ .

**Theorem 4.2.** *Assume that  $\Omega$  is a bounded smooth domain and  $1 < p < N$ . Consider  $u \in C_{loc}^{1,\alpha}(\bar{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus (Z_u \cup \{0\}))$  a solution to (4.3), where  $f \in W^{1,\infty}(\bar{\Omega})$ . We have*

$$\int_{E \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \mathcal{C} \quad \forall i = 1, \dots, N \quad (4.4)$$

for any  $E \Subset \Omega \setminus \{0\}$  and uniformly for any  $y \in E$ , with

$$\mathcal{C} := \mathcal{C}(\gamma, m_A, M_A, \beta, h, \|\nabla u\|_\infty, \rho)$$

for  $0 \leq \beta < 1$  and  $\gamma < (N-2)$  if  $N \geq 3$  ( $\gamma = 0$  if  $N = 2$ ).

Moreover, if we also assume that  $f$  is nonnegative in  $\Omega$ , we have that

$$\int_{\Omega \setminus (Z_u \cup \{0\})} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \mathcal{C} \quad \forall i = 1, \dots, N \quad (4.5)$$

*Proof. I step:* Since  $u > 0$  in  $\Omega$ , we have that  $h(x, u)$  is locally Lipschitz with respect to  $u$ , uniformly in  $x$  and  $h(\cdot, u)$  is locally Lipschitz. Following [14, Lemma 2.1], we can state that  $u_i$  solves the linearized problem

$$\begin{aligned} & \int_{\Omega} A(|\nabla u|)(\tilde{\nabla} u_i, \nabla \psi) dx + \int_{\Omega} \frac{A'(|\nabla u|)}{|\nabla u|} (\nabla u, \tilde{\nabla} u_i) (\nabla u, \nabla \psi) dx \\ & + \int_{\Omega} \frac{B'(|\nabla u|)}{|\nabla u|} (\nabla u, \tilde{\nabla} u_i) \psi dx - \int_{\Omega} [h_i(x, u) + h_u(x, u)u_i] \psi dx = 0, \end{aligned} \quad (4.6)$$

for any  $\psi \in C_c^\infty(\Omega \setminus (Z_u \cup \{0\}))$ .

For every  $\varepsilon, \delta > 0$ , we consider the following function:

$$\begin{aligned} G_\varepsilon(t) &:= (2t - 2\varepsilon)\chi_{[\varepsilon, 2\varepsilon]}(t) + t\chi_{[2\varepsilon, \infty)}(t) \quad \text{for } t > 0, \\ T_\varepsilon(t) &:= \frac{G_\varepsilon(t)}{|t|^\beta}, \quad H_\delta(t) := \frac{G_\delta(t)}{|t|^{\gamma+1}} \end{aligned}$$

Then, if we consider  $x_0 \in \Omega \setminus \{0\}$ ,  $B_{2\rho}(x_0) \subset \Omega \setminus \{0\}$  and  $\varphi_\rho \in C_c^\infty(B_{2\rho}(x_0))$  such that  $\varphi_\rho = 1$  in  $B_\rho(x_0)$  and  $|\nabla \varphi_\rho| \leq \frac{2}{\rho}$ , arguing as in [17, Theorem 1.2] and using the test function

$$\psi = T_\varepsilon(u_i)H_\delta(|x - y|)\varphi_\rho^2,$$

we get that

$$\begin{aligned} & \min\{1, 1 + m_A\} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\tilde{\nabla} u_i|^2}{|x - y|^\gamma} \left( \frac{G'_\varepsilon(u_i)}{|u_i|^\beta} - \beta \frac{G_\varepsilon(u_i)}{|u_i|^{1+\beta}} \right) \varphi_\rho^2 dx \\ & - 3\tilde{\vartheta} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\tilde{\nabla} u_i|^2}{|x - y|^\gamma |u_i|^\beta} \chi_{\{|u_i| \geq \varepsilon\}} \varphi_\rho^2 dx \leq C_{3,4,5,6,7} \\ & + \limsup_{\delta \rightarrow 0} \int_{B_{2\rho}(x_0)} [h_i(x, u) + h_u(x, u)u_i] T_\varepsilon(u_i) H_\delta(|x - y|) \varphi_\rho^2 dx. \end{aligned} \quad (4.7)$$

Exploiting the definition of  $T_\varepsilon(t)$  and  $H_\delta(t)$ , we can estimate the last term of the right-hand side of (4.7) as

$$\begin{aligned} & \left| \int_{B_{2\rho}(x_0)} [h_i(x, u) + h_u(x, u)u_i] T_\varepsilon(u_i) H_\delta(|x - y|) \varphi_\rho^2 dx \right| \\ & \leq \int_{B_{2\rho}(x_0)} \frac{|h_i(x, u)||u_i|^{1-\beta}}{|x - y|^\gamma} \varphi_\rho^2 dx + \int_{B_{2\rho}(x_0)} \frac{|h_u(x, u)||u_i|^{2-\beta}}{|x - y|^\gamma} \varphi_\rho^2 dx \\ & \leq \sup_{x \in B_{2\rho}(x_0)} |h_i(x, u)| \|\nabla u\|_\infty^{1-\beta} \int_{B_{2\rho}(x_0)} \frac{1}{|x - y|^\gamma} dx \\ & \quad + \|h_u(x, u)\|_\infty \|\nabla u\|_\infty^{2-\beta} \int_{B_{2\rho}(x_0)} \frac{1}{|x - y|^\gamma} dx \leq C_8. \end{aligned}$$

Passing to the limit for  $\varepsilon \rightarrow 0$  and applying Fatou Lemma in (4.7), we obtain

$$\begin{aligned} & \left( \min\{1, 1 + m_A\}(1 - \beta) - 3\tilde{\vartheta} \right) \int_{B_{2\rho}(x_0) \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x - y|^\gamma |u_i|^\beta} \varphi_\rho^2 dx \\ & \leq C_{3,4,5,6,7} + C_8. \end{aligned}$$

Now we fix  $\tilde{\vartheta}$  sufficiently small such that

$$\min\{1, 1 + m_A\}(1 - \beta) - 3\tilde{\vartheta} > 0$$

so that,

$$\int_{B_\rho(x_0) \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x - y|^\gamma |u_i|^\beta} dx \leq \int_{B_{2\rho}(x_0) \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x - y|^\gamma |u_i|^\beta} \varphi_\rho^2 dx \leq C_0$$

where  $C_0 = C_0(\gamma, m_A, M_A, \beta, h, \|\nabla u\|_\infty, \rho, x_0)$ , for all  $i = 1, \dots, N$ .

**II step:** Let us consider  $E \Subset \Omega \setminus \{0\}$ . Since  $\overline{E}$  is a compact set, there exists a finite number of  $B_\rho(x_i)$  with  $x_i \in \overline{E}$  such that

$$E \subset \overline{E} \subset \bigcup_{i=1}^M B_\rho(x_i).$$

Hence, repeating the covering arguments exploited in [7, Theorem 4.1] and by using the estimate obtained in Step I, we get

$$\int_{E \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \sum_{i=1}^M \int_{B_\rho(x_i) \setminus Z_u} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx + c \leq \sum_{i=1}^M \mathcal{C}_i + c =: \mathcal{C}. \quad (4.8)$$

**III step:** If  $\Omega$  is a smooth domain and  $f$  is a nonnegative function in  $\Omega$ , by Höpf Lemma (see [28]) we get that  $Z_u \cap \partial\Omega = \emptyset$ . Then, we can take  $\delta > 0$  such that

$$|\nabla u| \neq 0 \text{ in } I_{3\delta}(\partial\Omega) \text{ and } A(|\nabla u|) > 0 \text{ in } I_{3\delta}(\partial\Omega).$$

Since  $u \in C^2(I_{3\delta}(\partial\Omega))$ , arguing as in [7, Theorem 4.1] and using (4.8) we get (4.5).  $\square$

**Corollary 4.3.** *Let  $u \in C^1(\Omega \setminus \{0\})$  be a weak solution to (4.3) with  $f \in W^{1,\infty}(\Omega)$ ,  $1 < p < \infty$ . Then  $A(|\nabla u|)|\nabla u| \in W_{loc}^{1,2}(\Omega \setminus \{0\}, \mathbb{R}^N)$ . If  $\Omega$  is smooth,  $u \in C^1(\overline{\Omega} \setminus \{0\})$ ,  $f$  is a nonnegative function and  $f \in W^{1,\infty}(\overline{\Omega})$ , then  $A(|\nabla u|)|\nabla u| \in W_{loc}^{1,2}(\overline{\Omega} \setminus \{0\}, \mathbb{R}^N)$ .*

*Proof.* Since  $h(x, u)$  is locally Lipschitz continuous,  $u \in C^2(\Omega \setminus (Z_u \cup \{0\}))$  (see [18]). Let us consider a compact set  $E \subset \Omega \setminus \{0\}$  and let us set

$$\phi_n \equiv G_{\frac{1}{n}}(A(|\nabla u|)u_i), \quad i = 1, \dots, N,$$

where  $G_{\frac{1}{n}}$  is defined as in Theorem 4.2.

Using definition of  $G_{\frac{1}{n}}$ , we get that  $G'_{\frac{1}{n}}(A(|\nabla u|)u_i) \equiv 0$  on  $Z_u$ , then

$$\frac{\partial \phi_n}{\partial x_j} = G'_{\frac{1}{n}}(A(|\nabla u|)u_i) \left[ \frac{A'(|\nabla u|)}{|\nabla u|} \langle \nabla u, \nabla u_j \rangle u_i + A(|\nabla u|)u_{ij} \right] \text{ on } E \setminus Z_u.$$

We observe that  $\phi_n \in W^{1,2}(E)$ .

In fact, choosing  $0 < \eta < 1$ , using (4.2), Remark 4.1 and Theorem 4.2 we get

$$\begin{aligned}
\int_E |\nabla \phi_n|^2 dx &\leq 4 \int_{E \setminus \{|A(|\nabla u|)u_i| \leq \frac{1}{n}\}} \sum_{j=1}^N \left[ \frac{\partial}{\partial x_j} A(|\nabla u|) u_i \right]^2 dx \\
&\leq 4 \int_{E \setminus Z_u} \sum_{j=1}^N \left[ \frac{A'(|\nabla u|)}{|\nabla u|} \langle \nabla u, \nabla u_j \rangle u_i + A(|\nabla u|) u_{ij} \right]^2 dx \\
&\leq 4 \int_{E \setminus Z_u} \sum_{j=1}^N \left[ \frac{A'(|\nabla u|)|\nabla u|}{|\nabla u|} |\nabla u_j| |\nabla u| + A(|\nabla u|) u_{ij} \right]^2 dx \\
&\leq 4 \int_{E \setminus Z_u} \sum_{j=1}^N [M_A A(|\nabla u|) \|D^2 u\| + A(|\nabla u|) \|D^2 u\|]^2 dx \\
&\leq 4N \int_{E \setminus Z_u} (1 + M_A)^2 A^2(|\nabla u|) \|D^2 u\|^2 dx \\
&= 4N(1 + M_A)^2 \int_{E \setminus Z_u} A(|\nabla u|) |\nabla u|^\eta \frac{A(|\nabla u|) \|D^2 u\|^2}{|\nabla u|^\eta} dx \\
&\leq 4N(1 + M_A)^2 \sup_{E \setminus Z_u} A(|\nabla u|) |\nabla u|^\eta \int_{E \setminus Z_u} \frac{A(|\nabla u|) \|D^2 u\|^2}{|\nabla u|^\eta} dx \leq K_1 \quad \forall n \in \mathbb{N}.
\end{aligned}$$

In the same way we can prove that  $\|\phi_n\|_2^2 \leq K_2 \quad \forall n \in \mathbb{N}$ . Since  $W^{1,2}(E)$  is a reflexive space and  $\|\phi_n\|_{W^{1,2}(E)} \leq K := \sqrt{K_1 + K_2} \quad \forall n \in \mathbb{N}$ , up to a subsequence, there exists  $h \in W^{1,2}(E)$  such that

$$\phi_n \rightharpoonup h \text{ in } W^{1,2}(E)$$

Moreover, using Sobolev compact embedding, we get

$$\begin{aligned}
\phi_n &\rightarrow h \text{ in } L^2(E) \\
\phi_n &\rightarrow h \text{ almost everywhere in } E.
\end{aligned}$$

Since  $\phi_n \xrightarrow{n \rightarrow +\infty} A(|\nabla u|)u_i$  almost everywhere in  $E$ , we get

$$A(|\nabla u|)u_i \equiv h \in W^{1,2}(E)$$

and by arbitrariness of  $i \in \{1, \dots, N\}$  we have that  $A(|\nabla u|)\nabla u \in W_{loc}^{1,2}(\Omega \setminus \{0\}, \mathbb{R}^N)$ .

If moreover  $\Omega$  is smooth,  $u \in C^1(\bar{\Omega} \setminus \{0\})$ ,  $f$  is a nonnegative function and  $f \in W^{1,\infty}(\Omega)$ , we get that  $u \in C^2(\bar{\Omega} \setminus (Z_u \cup \{0\}))$  and  $A(|\nabla u|)\nabla u \in W_{loc}^{1,2}(\bar{\Omega} \setminus \{0\}, \mathbb{R}^N)$ .  $\square$

*Remark 4.4.* If  $u$  is a weak solution of Problem (1.1) with  $f$  nonnegative, by Corollary 4.3, since  $A(|\nabla u|)|\nabla u| \in W_{loc}^{1,2}(\bar{\Omega} \setminus \{0\})$ , we get that  $u$  solves Problem (1.1) almost everywhere in  $\Omega$ . In particular, since  $B(0) = 0$ , we get that  $-\operatorname{div}(A(|\nabla u|)\nabla u) = 0$  on  $Z_u$ , that is absurd since  $f$  is nonnegative. Then

$$|Z_u| = 0.$$

*Remark 4.5.* Using Remark 4.4, since  $f$  is a nonnegative function, (4.5) becomes

$$\int_{\tilde{\Omega}} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \mathcal{C} \quad \forall i = 1, \dots, N$$

for any  $\tilde{\Omega} \Subset \Omega \setminus \{0\}$  and uniformly for any  $y \in \tilde{\Omega}$ , with

$$\mathcal{C} := \mathcal{C}(\gamma, m_A, M_A, \beta, h, \|\nabla u\|_\infty, \rho)$$

for  $0 \leq \beta < 1$  and  $\gamma < (N-2)$  if  $N \geq 3$  ( $\gamma = 0$  if  $N = 2$ ).

Using (2.1) and the fact that  $|u_i| \leq |\nabla u|$ , we also obtain

$$c_2 \int_{\tilde{\Omega}} \frac{|\nabla u|^{p-2-\beta} |\nabla u_i|^2}{|x-y|^\gamma} dx \leq \int_{\tilde{\Omega}} \frac{A(|\nabla u|)|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta} dx \leq \mathcal{C} \quad \forall i = 1, \dots, N.$$

Next Theorems concern the summability of  $(A(|\nabla u|))^{-1}$ . The first is obtained supposing that  $p > 2$ .

**Theorem 4.6.** *Let  $u \in C^{1,\alpha}(\Omega \setminus \{0\}) \cap C^2(\Omega \setminus (Z_u \cup \{0\}))$  be a solution to (4.3) with  $f \in W^{1,\infty}(\Omega)$  and  $f(x) > 0$  in  $B_{2\rho}(x_0) \subset \Omega \setminus \{0\}$  (where  $B_{2\rho}(x_0)$  is defined in the proof of Theorem 4.2). Let us suppose that  $p > 2$ . Then*

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \mathcal{C} \quad (4.9)$$

for any  $y \in B_\rho(x_0)$ , with  $\alpha := \frac{p-1}{p-2}$ ,  $r \in (0, 1)$ ,  $\gamma < N-2$  if  $N \geq 3$ ,  $\gamma = 0$  if  $N = 2$  and

$$\mathcal{C} = \mathcal{C}(\gamma, \eta, h, \|\nabla u\|_\infty, \rho, x_0, \alpha, M_A, c_2, \tau, \hat{C}).$$

If  $\Omega$  is a smooth domain and  $f$  is nonnegative in  $\Omega$

$$\int_{\tilde{\Omega}} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \mathcal{C}, \quad (4.10)$$

for any  $\tilde{\Omega} \Subset \Omega \setminus \{0\}$ ,  $y \in \tilde{\Omega}$  and  $\alpha := \frac{p-1}{p-2}$ ,  $r \in (0, 1)$ .

*Proof.* In this proof, in order to simplify the notation we will indicate by  $\{A > 1\} := \{x \in B_{2\rho}(x_0) : A(|\nabla u(x)|) > 1\}$  and by  $\{A < 1\} := \{x \in B_{2\rho}(x_0) : A(|\nabla u(x)|) < 1\}$ . Moreover, in the sequel

$$I_F := \int_{B_{2\rho}(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx$$

Following [7, 14, 17], for  $\varepsilon > 0$ ,  $\varphi_\rho$ ,  $H_\delta$  defined as in Theorem 4.2, let us define

$\phi = \frac{H_\delta(|x-y|)\varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}}$ . Note that  $\phi$  is a good test function, then

$$\begin{aligned}
& \int_{B_{2\rho}(x_0)} \frac{h(x,u)H_\delta \varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}} dx \\
&= -\alpha r \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)A'(|\nabla u|)H_\delta \varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r+1}} (\nabla u, \tilde{\nabla}|\nabla u|) dx \\
&+ \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)\varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}} (\nabla u, \nabla_x H_\delta) dx \\
&+ 2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)H_\delta \varphi_\rho}{(\varepsilon + A(|\nabla u|))^{\alpha r}} (\nabla u, \nabla \varphi_\rho) dx + \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)H_\delta \varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}} dx.
\end{aligned} \tag{4.11}$$

Since the source term  $h$  is positive, calling  $c_h(\rho) := \min_{B_{2\rho}} h(x,u)$  we deduce that

$$c_h(\rho) \int_{B_{2\rho}(x_0)} \frac{H_\delta \varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}} dx \leq |I_1| + |I_2| + |I_3| + |I_B|. \tag{4.12}$$

where

$$\begin{aligned}
I_1 &= -\alpha r \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)A'(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r+1}} (\nabla u, \tilde{\nabla}|\nabla u|) H_\delta \varphi_\rho^2 dx \\
I_2 &= \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} (\nabla u, \nabla_x H_\delta) \varphi_\rho^2 dx \\
I_3 &= 2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} (\nabla u, \nabla \varphi_\rho) H_\delta \varphi_\rho dx \\
I_B &= \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} H_\delta \varphi_\rho^2 dx.
\end{aligned} \tag{4.13}$$

The idea of the proof is to show that all above integrals are bounded and then our estimate can be obtained by using Fatou Lemma.

By Young inequality  $ab \leq \tau a^2 + \frac{b^2}{4\tau}$ , we observe that

$$\begin{aligned}
& \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
&\leq \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}} |x-y|^{\frac{\gamma}{2}}} \frac{B(|\nabla u|)\varphi_\rho}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}} |x-y|^{\frac{\gamma}{2}}} dx \\
&\leq \tau I_F + \frac{1}{4\tau} \int_{B_{2\rho}(x_0)} \frac{B^2(|\nabla u|)}{(A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
&\leq \tau I_F + \frac{\hat{C}^2}{4\tau} \int_{B_{2\rho}(x_0)} \frac{A^2(|\nabla u|)|\nabla u|^4}{(A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
&\leq \tau I_F + \frac{\hat{C}^2}{4\tau} \left[ \int_{\{A < 1\}} \frac{A^2(|\nabla u|)|\nabla u|^4}{(A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx + \int_{\{A > 1\}} \frac{A^2(|\nabla u|)|\nabla u|^4}{(A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \right].
\end{aligned} \tag{4.14}$$

Since  $p > 2$  and  $\alpha r > r > 0$ , we get that  $A(|\nabla u|) < A(|\nabla u|)^r$  on the set  $\{A < 1\}$  therefore by (2.1)

$$\begin{aligned}
& \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \leq \tau I_F + \frac{\hat{C}^2}{4\tau} \int_{\{A < 1\}} \frac{A(|\nabla u|)|\nabla u|^4}{(A(|\nabla u|))^{\frac{r}{p-2}}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \quad + \frac{\hat{C}^2}{4\tau} \int_{\{A > 1\}} A^2(|\nabla u|)|\nabla u|^4 \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \leq \tau I_F + \frac{\hat{C}^2}{4\tau c_2^{\frac{r}{p-2}}} \int_{\{A < 1\}} A(|\nabla u|)|\nabla u|^{4-r} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \quad + \frac{\hat{C}^2}{4\tau} \sup_{B_{2\rho}(x_0)} (|\nabla u|^4 A^2(|\nabla u|)) \int_{\{A > 1\}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \leq \tau I_F + \frac{\hat{C}^2}{4\tau c_2^{\frac{r}{p-2}}} \sup_{B_{2\rho}(x_0)} (|\nabla u|^{4-r} A(|\nabla u|)) \int_{\{A < 1\}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \quad + \frac{\hat{C}^2}{4\tau} \sup_{B_{2\rho}(x_0)} (|\nabla u|^4 A^2(|\nabla u|)) \int_{\{A > 1\}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx.
\end{aligned}$$

To conclude, we can assert that there exists a positive  $C_B$  such that

$$\int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \leq \tau I_F + \frac{C_B}{4\tau}$$

hence by Fatou Lemma

$$\limsup_{\delta \rightarrow 0} |I_B| = \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \leq \tau I_F + \frac{C_B}{4\tau}. \quad (4.15)$$

Similarly, recalling that  $|\nabla \varphi_\rho| \leq \frac{2}{\rho}$ , we get

$$2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u||\nabla \varphi|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx \leq \frac{4}{\rho} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx. \quad (4.16)$$

Again we divide the last integral on the set  $\{A > 1\}$  and  $\{A < 1\}$  respectively and we get that

$$\int_{\{A > 1\}} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx \leq \sup_{B_{2\rho}(x_0)} (|\nabla u| A(|\nabla u|)) \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho}{|x-y|^\gamma} dx$$

while, following arguing as for  $I_B$ , being  $\alpha r > r$  and  $A(|\nabla u|) < A(|\nabla u|)^r$

$$\int_{\{A < 1\}} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx \leq \sup_{B_{2\rho}(x_0)} (|\nabla u|^{1-r}) \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho}{|x-y|^\gamma} dx.$$

Therefore we can state that there exists  $C_3 > 0$  such that

$$2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u||\nabla \varphi|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} \varphi_\rho dx \leq C_3$$

and then

$$\limsup_{\delta \rightarrow 0} |I_3| \leq C_3.$$

Now we proceed further observing that, with the same procedure, we can show that there exists  $C_2 > 0$  such that

$$\limsup_{\delta \rightarrow 0} |I_2| \leq C_2.$$

It remain to consider  $I_1$ : again Young inequality give us that

$$\begin{aligned} \limsup_{\delta \rightarrow 0} |I_1| &\leq \alpha r \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|) A'(|\nabla u|) |\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r + 1}} \frac{\|D^2 u\|}{|x - y|^\gamma} \varphi_\rho^2 dx \\ &= \alpha r \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}} |x - y|^{\frac{\gamma}{2}}} \frac{A(|\nabla u|) A'(|\nabla u|) |\nabla u|}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2} + 1}} \frac{\|D^2 u\|}{|x - y|^{\frac{\gamma}{2}}} \varphi_\rho dx \\ &\leq \alpha \tau I_F + \frac{\alpha r}{4\tau} \int_{B_{2\rho}(x_0)} \frac{(A(|\nabla u|))^2 (A'(|\nabla u|))^2 |\nabla u|^2}{(\varepsilon + A(|\nabla u|))^{\alpha r + 2}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \\ &\leq \alpha r \tau I_F + \frac{\alpha r M_A^2}{4\tau} \int_{B_{2\rho}(x_0)} \frac{(A(|\nabla u|))^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \\ &\leq \alpha r \tau I_F + \frac{\alpha r M_A^2}{4\tau} \int_{B_{2\rho}(x_0)} \frac{(A(|\nabla u|))^2}{A(|\nabla u|)^{\alpha r}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx. \end{aligned}$$

As in previous cases, in the set  $\{A < 1\}$ , we easily get

$$\begin{aligned} \int_{\{A < 1\}} \frac{(A(|\nabla u|))^2}{A(|\nabla u|)^{\alpha r}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx &\leq \int_{\{A < 1\}} \frac{A(|\nabla u|)}{A(|\nabla u|)^{\frac{r}{p-2}}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \\ &\leq \frac{1}{c_2^{\frac{r}{p-2}}} \int_{\{A < 1\}} \frac{A(|\nabla u|)}{|\nabla u|^r} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \end{aligned}$$

so we can use Theorem 4.2 with  $\beta = 0$ . On the set  $\{A > 1\}$ ,

$$\begin{aligned} \int_{\{A > 1\}} \frac{(A(|\nabla u|))^2}{A(|\nabla u|)^{\alpha r}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx &\leq \int_{\{A > 1\}} A(|\nabla u|)^{1-r} A(|\nabla u|) \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \\ &\leq \sup_{B_{2\rho}} (A(|\nabla u|) |\nabla u|^\eta)^{1-r} \int_{\{A > 1\}} \frac{A(|\nabla u|)}{|\nabla u|^{\eta(1-r)}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \end{aligned}$$

hence we can use again Theorem 4.2 with  $\beta = \eta(1 - r)$ . Summarizing we obtain the existence of a constant  $C_1 > 0$  such that

$$\limsup_{\delta \rightarrow 0} |I_1| \leq \alpha r \tau I_F + \frac{\alpha r M_A^2}{4\tau} C_1. \quad (4.17)$$

Collecting all our estimates, from (4.12) and letting  $\delta \rightarrow 0$  we have for every  $r \in (0, 1)$

$$(c_h(\rho) - (\alpha r + 1)\tau) \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\|D^2 u\|^2}{|x - y|^\gamma} \varphi_\rho^2 dx \leq \frac{\alpha r M_A^2 C_1}{4\tau} + C_2 + C_3 + \frac{C_B}{4\tau}.$$

For  $\tau$  sufficient small such that  $(c_h(\rho) - (\alpha r + 1)\tau) > 0$ , letting  $\varepsilon \rightarrow 0$  and observing that  $|Z_u| = 0$ , we get by Fatou Lemma that

$$\int_{B_{2\rho}(x_0)} \frac{\varphi_\rho^2}{A(|\nabla u|)^{\alpha r} |x-y|^\gamma} dx \leq \lim_{\varepsilon \rightarrow 0} \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r} |x-y|^\gamma} dx \leq \mathcal{C} \quad (4.18)$$

where  $\mathcal{C} = \mathcal{C}(\gamma, \beta, h, \|\nabla u\|_\infty, \rho, x_0, \alpha, M_A, c_2, \tau, \hat{C})$ .

Hence, we obtain

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \int_{B_{2\rho}(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r} |x-y|^\gamma} \varphi_\rho^2 dx \leq \mathcal{C}.$$

Then, we obtain that

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \mathcal{C}. \quad (4.19)$$

If  $\Omega$  is a smooth domain and  $f$  is nonnegative in  $\Omega$ , by Höpf Lemma we get that  $Z_u \cap \partial\Omega = \emptyset$ , then in a neighbourhood of  $\partial\Omega$ ,  $|\nabla u|$  is strictly positive. Arguing as in [7, Theorem 4.2] we get (4.10).  $\square$

Similarly to Theorem 4.6, we now show summability properties for  $A(|\nabla u|)^{-1}$  in the case  $1 < p < 2$ .

*Remark 4.7.* If  $1 < p < 2$ , we have that  $m_A < 0$ . In fact, by (2.1), one has that

$$A(|\nabla u|) \geq \frac{c_2}{|\nabla u|^{2-p}} \xrightarrow{|\nabla u| \rightarrow 0} +\infty.$$

Then if  $|\nabla u| \rightarrow 0$ , considering also [10, Proposition 4.1], we get

$$\frac{c_2}{|\nabla u|^{2-p}} \leq A(|\nabla u|) \leq A(1)|\nabla u|^{m_A}$$

and so  $m_A$  has to be necessarily negative. Therefore, from (4.2), we get that  $\alpha := \frac{m_A+1}{m_A} < 0$ .

**Theorem 4.8.** *Let  $u \in C^{1,\alpha}(\Omega \setminus \{0\}) \cap C^2(\Omega \setminus (Z_u \cup \{0\}))$  be a solution to (4.3) with  $f \in W^{1,\infty}(\Omega)$  and  $f(x) > 0$  in  $B_{2\rho}(x_0) \subset \Omega$ . Let us suppose that  $p \in (1, 2)$ . Then*

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} \leq \mathcal{C} \quad (4.20)$$

for any  $y \in B_\rho(x_0)$ , with  $\alpha := \frac{m_A+1}{m_A}$ ,  $r \in (0, 1)$ ,  $\gamma < N-2$  if  $N \geq 3$ ,  $\gamma = 0$  if  $N = 2$  and

$$\mathcal{C} = \mathcal{C}(\gamma, \beta, h, \|\nabla u\|_\infty, \rho, x_0, \alpha, M_A, c_2, \tau, \hat{C}).$$

If  $\Omega$  is a smooth domain and  $f$  is nonnegative in  $\Omega$

$$\int_{\tilde{\Omega}} \frac{1}{(A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} \leq \mathcal{C} \quad (4.21)$$

for any  $\tilde{\Omega} \Subset \Omega \setminus \{0\}$ ,  $y \in \tilde{\Omega}$  and  $\alpha := \frac{m_A+1}{m_A}$ ,  $r \in (0, 1)$ .

*Proof.* Choose the same test function as in Theorem 4.6 (taking into account the new value of  $\alpha$ ) and let us start from the inequality

$$c_h(\rho) \int_{B_{2\rho}(x_0)} \frac{H_\delta \varphi_\rho^2}{(\varepsilon + A(|\nabla u|))^{\alpha r}} dx \leq |I_1| + |I_2| + |I_3| + |I_B|. \quad (4.22)$$

by using the same notations as in proof of Theorem 4.6.

Let  $\bar{\varepsilon} > 0$ . To our goal, we will proceed in two different ways depending on whether we work on the set  $Z_u^{\bar{\varepsilon}} := \{x \in B_{2\rho}(x_0) : d(x, Z_u) < \bar{\varepsilon}\}$  or  $B_{2\rho}(x_0) \setminus Z_u^{\bar{\varepsilon}}$ .

If  $x \in Z_u^{\bar{\varepsilon}}$ , by (2.1), there exists  $M \geq 1$  such that  $A(|\nabla u(x)|) > M$ . Then, we will use later the following estimate

$$(\varepsilon + A(|\nabla u|))^{-\alpha r} < (M + A(|\nabla u|))^{-\alpha r} < (M + A(|\nabla u|))^{-\alpha} < 2^{-\alpha} A(|\nabla u|)^{-\alpha}; \quad (4.23)$$

while, if  $x \in B_{2\rho}(x_0) \setminus Z_u^{\bar{\varepsilon}}$  we have that  $\sup_{B_{2\rho}(x_0) \setminus Z_u^{\bar{\varepsilon}}} A(|\nabla u(x)|) < +\infty$ . Then, there exists  $C \in \mathbb{R}^+$  such that

$$(\varepsilon + A(|\nabla u|))^{-\alpha r} < 2^{-(\alpha r + 1)} (\varepsilon^{-\alpha r} + A(|\nabla u|)^{-\alpha r}) < C. \quad (4.24)$$

By Young inequality  $ab \leq \tau a^2 + \frac{b^2}{4\tau}$ , we observe that

$$\begin{aligned} & \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx \\ & \leq \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}} |x - y|^{\frac{\gamma}{2}}} \frac{\varphi_\rho B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}} |x - y|^{\frac{\gamma}{2}}} dx \\ & \leq \tau I_F + \frac{1}{4\tau} \int_{B_{2\rho}(x_0)} \frac{B^2(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx \\ & \leq \tau I_F + \frac{\hat{C}^2}{4\tau} \int_{B_{2\rho}(x_0)} \frac{A^2(|\nabla u|) |\nabla u|^4}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx \\ & \leq \tau I_F + \frac{\hat{C}^2}{4\tau} \int_{B_{2\rho}(x_0) \setminus Z_u^{\bar{\varepsilon}}} \frac{A^2(|\nabla u|) |\nabla u|^4}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx \\ & \quad + \frac{\hat{C}^2}{4\tau} \int_{Z_u^{\bar{\varepsilon}}} \frac{A^2(|\nabla u|) |\nabla u|^4}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx \\ & \leq \tau I_F + \frac{\hat{C}^2}{4\tau} C \sup_{B_{2\rho}(x_0) \setminus Z_u^{\bar{\varepsilon}}} (|\nabla u|^4 A^2(|\nabla u|)) \int_{B_{2\rho}(x_0) \setminus Z_u^{\bar{\varepsilon}}} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx \\ & \quad + \frac{\hat{C}^2}{4\tau 2^\alpha} \int_{Z_u^{\bar{\varepsilon}}} A^2(|\nabla u|) |\nabla u|^4 A(|\nabla u|)^{-\alpha} \frac{\varphi_\rho^2}{|x - y|^\gamma} dx. \end{aligned} \quad (4.25)$$

By [10, Proposition 4.1] we have that  $A(|\nabla u|) \leq A(1)|\nabla u|^{m_A}$  on  $Z_u^\varepsilon$  then

$$\begin{aligned}
& \int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \leq \tau I_F + \frac{\hat{C}^2}{4\tau} C \sup_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} (|\nabla u|^4 A^2(|\nabla u|)) \int_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \quad + \frac{\hat{C}^2 A(1)^{2-\alpha}}{4\tau 2^\alpha} \int_{Z_u^\varepsilon} \frac{|\nabla u|^{m_A+3} \varphi_\rho^2}{|x-y|^\gamma} dx \\
& \leq \tau I_F + \frac{\hat{C}^2}{4\tau} C \sup_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} (|\nabla u|^4 A^2(|\nabla u|)) \int_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx \\
& \quad + \frac{\hat{C}^2 A(1)^{2-\alpha}}{4\tau 2^\alpha} \sup_{Z_u^\varepsilon} |\nabla u|^{m_A+3} \int_{Z_u^\varepsilon} \frac{\varphi_\rho^2}{|x-y|^\gamma} dx.
\end{aligned} \tag{4.26}$$

To conclude, we can assert that there exists a positive  $C_B$  such that

$$\int_{B_{2\rho}(x_0)} \frac{B(|\nabla u|)}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} \varphi_\rho^2 dx \leq \tau I_F + \frac{C_B}{4\tau}$$

hence by Fatou Lemma

$$\limsup_{\delta \rightarrow 0} |I_B| \leq \tau I_F + \frac{C_B}{4\tau}. \tag{4.27}$$

Similarly, recalling that  $|\nabla \varphi_\rho| \leq \frac{2}{\rho}$ , we get

$$2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u||\nabla \varphi|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx \leq \frac{4}{\rho} \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx. \tag{4.28}$$

Again we divide the last integral on the set  $Z_u^\varepsilon$  and  $B_{2\rho}(x_0) \setminus Z_u^\varepsilon$  respectively and we get, using (4.23), that

$$\begin{aligned}
& \int_{Z_u^\varepsilon} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx \leq 2^{-\alpha} \int_{Z_u^\varepsilon} \frac{A(|\nabla u|)^{1-\alpha} |\nabla u| \varphi_\rho}{|x-y|^\gamma} dx \\
& \leq 2^{-\alpha} A(1)^{1-\alpha} \int_{Z_u^\varepsilon} \frac{|\nabla u|^{m_A(1-\alpha)} |\nabla u| \varphi_\rho}{|x-y|^\gamma} dx = 2^{-\alpha} A(1)^{1-\alpha} \int_{Z_u^\varepsilon} \frac{\varphi_\rho}{|x-y|^\gamma} dx;
\end{aligned}$$

while, using (4.24) we get

$$\begin{aligned}
& \int_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} \frac{A(|\nabla u|)|\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{\varphi_\rho}{|x-y|^\gamma} dx \\
& \leq C \sup_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} (A(|\nabla u|)|\nabla u|) \int_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} \frac{\varphi_\rho}{|x-y|^\gamma} dx.
\end{aligned}$$

Therefore we can state that there exists  $C_3 > 0$  such that

$$2 \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|)|\nabla u||\nabla \varphi|}{(\varepsilon + A(|\nabla u|))^{\alpha r}} \frac{1}{|x-y|^\gamma} \varphi_\rho dx \leq C_3$$

and then

$$\limsup_{\delta \rightarrow 0} |I_3| \leq C_3.$$

Now we proceed further observing that, with the same procedure, we can show that there exists  $C_2 > 0$  such that

$$\limsup_{\delta \rightarrow 0} |I_2| \leq C_2.$$

It remain to consider  $I_1$ : again Young inequality give us that

$$\begin{aligned} \limsup_{\delta \rightarrow 0} |I_1| &\leq \alpha r \int_{B_{2\rho}(x_0)} \frac{A(|\nabla u|) |A'(|\nabla u|)| |\nabla u|}{(\varepsilon + A(|\nabla u|))^{\alpha r+1}} \frac{\|D^2 u\|}{|x-y|^\gamma} \varphi_\rho^2 dx \\ &= \alpha r \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}} |x-y|^{\frac{\gamma}{2}}} \varphi_\rho \frac{A(|\nabla u|) |A'(|\nabla u|)| |\nabla u|}{(\varepsilon + A(|\nabla u|))^{\frac{\alpha r}{2}+1}} \frac{\|D^2 u\|}{|x-y|^{\frac{\gamma}{2}}} \varphi_\rho dx \\ &\leq \alpha \tau I_F + \frac{\alpha r}{4\tau} \int_{B_{2\rho}(x_0)} \frac{(A(|\nabla u|))^2 (A'(|\nabla u|))^2 |\nabla u|^2 \|D^2 u\|^2}{(\varepsilon + A(|\nabla u|))^{\alpha r+2} |x-y|^\gamma} \varphi_\rho^2 dx \\ &\leq \alpha r \tau I_F + \frac{\alpha r M_A^2}{4\tau} \int_{B_{2\rho}(x_0)} \frac{(A(|\nabla u|))^2 \|D^2 u\|^2}{(\varepsilon + A(|\nabla u|))^{\alpha r} |x-y|^\gamma} \varphi_\rho^2 dx. \end{aligned}$$

As in previous cases, using (4.24) and choosing  $\eta$  as in Remark 4.1, we easily get

$$\begin{aligned} &\int_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} \frac{(A(|\nabla u|))^2 \|D^2 u\|^2}{(\varepsilon + A(|\nabla u|))^{\alpha r} |x-y|^\gamma} \varphi_\rho^2 dx \\ &\leq C \int_{B_{2\rho}(x_0) \setminus Z_u^\varepsilon} \frac{A(|\nabla u|) \|D^2 u\|^2}{|\nabla u|^\eta |x-y|^\gamma} A(|\nabla u|) |\nabla u|^\eta \varphi_\rho^2 dx \end{aligned}$$

so we can use Theorem 4.2 and Remark 4.1 to conclude that this term is finite. On the set  $Z_u^\varepsilon$ , by (4.23), one has

$$\begin{aligned} &\int_{Z_u^\varepsilon} \frac{(A(|\nabla u|))^2 \|D^2 u\|^2}{(\varepsilon + A(|\nabla u|))^{\alpha r} |x-y|^\gamma} \varphi_\rho^2 dx \leq 2^{-\alpha r} \int_{Z_u^\varepsilon} \frac{A(|\nabla u|) \|D^2 u\|^2}{|x-y|^\gamma} A(|\nabla u|)^{1-\alpha r} \varphi_\rho^2 dx \\ &\leq 2^{-\alpha r} A(1)^{1-\alpha r} \int_{Z_u^\varepsilon} \frac{A(|\nabla u|) \|D^2 u\|^2}{|x-y|^\gamma} |\nabla u|^{m_A(1-\alpha r)} \varphi_\rho^2 dx \\ &= 2^{-\alpha r} A(1)^{1-\alpha r} \int_{Z_u^\varepsilon} \frac{A(|\nabla u|) \|D^2 u\|^2}{|x-y|^\gamma |\nabla u|^{m_A(\alpha r-1)}} \varphi_\rho^2 dx \end{aligned}$$

hence we can use again Theorem 4.2 with  $\beta = m_A(\alpha r - 1)$ . Summarizing we obtain the existence of a constant  $C_1 > 0$  such that

$$\limsup_{\delta \rightarrow 0} |I_1| \leq \alpha r \tau I_F + \frac{\alpha r M_A^2}{4\tau} C_1. \quad (4.29)$$

Collecting all our estimates, from (4.22) and letting  $\delta \rightarrow 0$  we have for every  $r \in (0, 1)$

$$(c_h(\rho) - (\alpha r + 1)\tau) \int_{B_{2\rho}(x_0)} \frac{1}{(\varepsilon + A(|\nabla u|))^{\alpha r} |x-y|^\gamma} \varphi_\rho^2 dx \leq \tilde{C}_1 + C_2 + C_3 + \tilde{C}_B.$$

For  $\tau$  sufficient small such that  $(c_h(\rho) - (\alpha r + 1)\tau) > 0$ , letting  $\varepsilon \rightarrow 0$  and recalling that  $|Z_u| = 0$ , we get by Fatou Lemma that

$$\int_{B_{2\rho}(x_0)} \frac{\varphi_\rho^2}{A(|\nabla u|^{\alpha r})|x-y|^\gamma} dx \leq \lim_{\varepsilon \rightarrow 0} \int_{B_{2\rho}(x_0)} \frac{\varphi_\rho^2 dx}{(\varepsilon + A(|\nabla u|^{\alpha r})|x-y|^\gamma)} \leq C \quad (4.30)$$

where  $C = C(\gamma, \beta, h, \|\nabla u\|_\infty, \rho, x_0, \alpha, M_A, c_2, \tau, \hat{C})$ .

Therefore, we obtain

$$\int_{B_\rho(x_0)} \frac{1}{(A(|\nabla u|^{\alpha r}))^{\alpha r}} \frac{1}{|x-y|^\gamma} dx \leq \int_{B_{2\rho}(x_0)} \frac{1}{(A(|\nabla u|^{\alpha r}))^{\alpha r}} \varphi_\rho^2 dx \leq C.$$

i.e. the claim.

If  $\Omega$  is a smooth domain and  $f$  is nonnegative in  $\Omega$ , by Höpf Lemma we get that  $Z_u \cap \partial\Omega = \emptyset$ , then in a neighbourhood of  $\partial\Omega$ ,  $|\nabla u|$  is strictly positive. Arguing as in [7, Theorem 4.2] we get (4.21).  $\square$

Theorem 4.6 and 4.8 state that in case of  $A(t) = t^{p-2}$  for  $p > 1$ , we exactly find [14, Theorem 2.3]; the same choice made in [7, Theorem 4.2] for  $p > 2$  give us a closer interval for the power  $\alpha$  and a different interval for  $p \in (1, 2)$ . The summability properties obtained in this section can be exploited e.g. in the study of geometric qualitative properties of the solutions, see [8]. Here we emphasize some further regularity consequences in the following

**Proposition 4.9.** *Let  $\Omega$  be a smooth domain,  $\tilde{\Omega} \Subset \Omega \setminus \{0\}$ ,  $u \in C^1(\Omega \setminus \{0\})$  be a weak solution to (4.3) and suppose that  $f$  is nonnegative and  $f \in W^{1,\infty}(\tilde{\Omega})$ . Then  $|u_{ij}| \in L^2(\tilde{\Omega})$  if  $1 < p < 3$ . If  $p \geq 3$ , then  $|u_{ij}| \in L^q(\tilde{\Omega})$  with  $q < \frac{p-1}{p-2}$ .*

*Proof.* By Remark 4.5, for every  $\beta < 1$  we have

$$|\nabla u|^{\frac{p-2-\beta}{2}} u_{ij} \in L^2(\tilde{\Omega}).$$

Then, if  $1 < p < 3$  we can choose  $\beta$  such that  $p - 2 - \beta < 0$  obtaining that  $u_{ij} \in L^2(\tilde{\Omega})$ .

Instead, for the case  $p \geq 3$ , by Hölder inequality with conjugate exponents  $\frac{2}{q}$  and  $\frac{2}{2-q}$ , for  $\eta \in [0, 1)$  we can apply Remark 4.5, obtaining

$$\begin{aligned} \int_{\tilde{\Omega}} |u_{ij}|^q dx &= \int_{\tilde{\Omega}} \frac{|u_{ij}|^q A(|\nabla u|^{\frac{q}{2}})}{|\nabla u|^{\frac{\eta q}{2}} A(|\nabla u|^{\frac{q}{2}})} |\nabla u|^{\frac{\eta q}{2}} dx \\ &\leq \left( \int_{\tilde{\Omega}} \frac{|u_{ij}|^2 A(|\nabla u|)}{|\nabla u|^\eta} dx \right)^{\frac{q}{2}} \left( \int_{\tilde{\Omega}} \frac{|\nabla u|^{\frac{\eta q}{2-q}}}{A(|\nabla u|^{\frac{q}{2-q}})} dx \right)^{\frac{2-q}{2}} \\ &\leq c C^{\frac{q}{2}} \left( \int_{\tilde{\Omega}} \frac{A(|\nabla u|)^{\frac{\eta q}{(2-q)(p-2)}}}{A(|\nabla u|)^{\frac{q}{2-q}}} dx \right)^{\frac{2-q}{2}} = c C^{\frac{q}{2}} \left( \int_{\tilde{\Omega}} \frac{1}{A(|\nabla u|)^{\frac{q}{2-q}(1-\frac{\eta}{p-2})}} dx \right)^{\frac{2-q}{2}}. \end{aligned}$$

For the last integral, we show that if  $q < \frac{p-1}{p-2}$ , we can choose  $\eta \in [0, 1)$  such that

$$\frac{q}{2-q} \left( 1 - \frac{\eta}{p-2} \right) < \frac{p-1}{p-2}; \quad (4.31)$$

hence, exploiting Theorem 4.6, we get

$$\left( \int_{\tilde{\Omega}} \frac{1}{A(|\nabla u|)^{\frac{q}{2-q}(1-\frac{\eta}{p-2})}} dx \right)^{\frac{2-q}{2}} < +\infty.$$

In fact, let be consider  $\varepsilon > 0$  and  $q = \frac{p-1}{p-2+\varepsilon} < \frac{p-1}{p-2}$  in (4.31). Observing that  $q < \frac{p-1}{p-2} < 2$ , the left-hand side of (4.31) becomes

$$\frac{q}{2-q} \left( 1 - \frac{\eta}{p-2} \right) = \frac{p-1}{p-2} \left( \frac{p-2-\eta}{p-3+2\varepsilon} \right).$$

Choosing  $\eta > 1 - 2\varepsilon$ , we get that

$$\left( \frac{p-2-\eta}{p-3+2\varepsilon} \right) < 1;$$

hence, for this choice of  $\eta$ , (4.31) is verified. Therefore,  $|u_{ij}| \in L^q(\tilde{\Omega})$  with  $q < \frac{p-1}{p-2}$ . □

Arguing as in [14, Proposition 2.2], the following can be proved:

**Theorem 4.10.** *Let  $\Omega$  be a smooth domain,  $u \in C^1(\bar{\Omega} \setminus \{0\})$  be a weak solution to (4.3),  $f \in W^{1,\infty}(\bar{\Omega})$  and  $f$  nonnegative function. Then if  $1 < p < 3$ ,  $u_i \in W_{loc}^{1,2}(\Omega \setminus \{0\})$ , while if  $p \geq 3$  then  $u_i \in W_{loc}^{1,q}(\Omega \setminus \{0\})$ , for every  $i = 1, \dots, N$  and for every  $q < \frac{p-1}{p-2}$ . Moreover the generalized derivatives of  $u_i$  coincide with the classical ones, both denoted with  $u_{ij}$  almost everywhere in  $\Omega$ .*

*Remark 4.11.* Since  $q < \frac{p-1}{p-2} < 2$ , by Theorem 4.10, we have that for every  $i = 1, \dots, N$

$$u_i \in W_{loc}^{1,q}(\Omega \setminus \{0\}) \quad \text{for } 1 < p < \infty.$$

By Stampacchia Theorem [33, Theorem 1.56], we get that

$$\nabla u_i = 0 \text{ a.e. in } \{u_i = 0\}, \text{ for } i = 1, \dots, N.$$

Then, the choice to set zero the second derivatives of  $u$  in (4.1) is completely justified.

## 5. DECLARATIONS

**Competing interests.** The authors declare that they have no competing interests.

**Acknowledgment.** L. Montoro, L. Muglia and B. Sciunzi are partially supported by PRIN project 2017JPCAPN (Italy): Qualitative and quantitative aspects of nonlinear PDEs, and L. Montoro by Agencia Estatal de Investigación (Spain), project PDI2019-110712GB-100. All the authors are partially supported also by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

**Authors' contributions.** Each author equally contributed to this paper, read and approved the final manuscript.

## REFERENCES

- [1] B. Abdellaoui, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential, *Annali di Matematica Pura ed Applicata*, 182 (3), (2003), 247–270.

- [2] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An  $L^1$ -theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 22(2), (1995), 241–273.
- [3] L. Boccardo, G. Croce, *Esistenza e regolarità di soluzioni per alcuni problemi ellittici*, Pitagora, 2010.
- [4] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, 87(1), (1989), 149–169.
- [5] L. Boccardo, F. Murat, J.P. Puel, Résultats d'existence pour certains problèmes elliptiques quasilineaires, *Ann. Sc. Norm. Super. Pisa*, 11(2), (1984), 213–235.
- [6] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proceedings of the American Mathematical Society*, 88(3), (1983), 486–490.
- [7] A. Canino, E. De Giorgio, B. Sciunzi, Second order regularity for degenerate nonlinear elliptic equations, *Discrete Contin. Dyn. Syst.*, 38 (8), (2018), 4231–4242.
- [8] G. Chirillo, L. Montoro, L. Muglia, B. Sciunzi, Symmetry via the moving plane method for a class of quasilinear elliptic problems involving the Hardy potential, *Proc. Royal Soc. Edinburgh: Sec. A Mathematics*, doi:10.1017/prm.2022.74, (2022), 1–25.
- [9] A. Cianchi, V. Maz'ya, Global Lipschitz regularity for a class of quasilinear elliptic equations, *Communications in Partial Differential Equations*, 36, (2011), 100–133.
- [10] A. Cianchi, V. Maz'ya, Global boundedness of the gradient for a class of nonlinear elliptic systems, *Archive for Rational Mechanics and Analysis*, 212 (1), (2014) 129–177.
- [11] A. Cianchi, V. Maz'ya, Second-order two-sided estimates in nonlinear elliptic problems, *Archive for Rational Mechanics and Analysis*, 229 (2), (2018), 569–599.
- [12] A. Dall'Aglio, Approximated solutions of equations with  $L^1$  data. Application to the  $H$ -convergence of quasi-linear parabolic equations, *Ann. Mat. Pura. Appl.*, 170(4), (1996), 207–240.
- [13] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 28 (4), (1999), 741–808.
- [14] L. Damascelli, B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of  $m$ -Laplace equations, *J. Differential Equations*, 206 (2), (2004), 483–515.
- [15] J. Dávila, I. Peral, Nonlinear elliptic problems with a singular weight on the boundary, *Calculus of Variations and Partial Differential Equations*, 41(3), (2011), 567–586.
- [16] E. DiBenedetto,  $C^{1+\alpha}$  local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.*, 7(8), (1983), 827–850.
- [17] F. Esposito, B. Sciunzi, A. Trombetta, Regularity and symmetry results for nonlinear degenerate elliptic equations, *J. Differential Equations*, 336, (2022), 315–333.
- [18] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2nd Edition, 1983.
- [19] T. Kuusi, G. Mingione, Universal potential estimates, *J. Funct. Anal.*, 262, (2012), 4205–4269.
- [20] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* 12 (11), (1988), 1203–1219.
- [21] S. Merçà, L. Montoro, Remarks on the existence of solutions to some quasilinear elliptic problems involving the Hardy-Leray potential, *Annali di Matematica Pura ed Applicata*, 193(2), (2014), 609–632.
- [22] S. Merçà, L. Montoro, I. Peral, B. Sciunzi, Existence and qualitative properties of solutions to a quasilinear elliptic equation involving the Hardy-Leray potential, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 31 (1), (2014), 1–22.
- [23] S. Merçà, I. Peral, Remarks on the solvability of an elliptic equation with a supercritical term involving the Hardy - Leray potential, *Journal of Mathematical Analysis and Applications*, 394 (1), (2012), 347–359.
- [24] G. Mingione, Regularity of minima: An invitation to the dark side of the calculus of variations, *Appl. Math.*, 51(4), (2006), 355–426.
- [25] G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, (5) 6 (2), (2007), 195–261.
- [26] G. Mingione, Gradient estimates below the duality exponent, *Math. Ann.*, 346, (2010), 571–627.
- [27] G. Mingione, Gradient potential estimates, *J. Eur. Math. Soc.*, 13, (2011), 459–486.
- [28] P. Pucci, J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Verlag, Basel, 73, 2007.

- [29] J. Serrin, Local behavior of solutions of quasi-linear equations, *Acta Mathematica*, 111, (1964), 247–302.
- [30] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Ins. Fourier*, 15 (1), (1965), 189–257.
- [31] E. Teixeira, Regularity for quasilinear equations on degenerate singular sets, *Math. Ann.*, 358, (2014), 241–256.
- [32] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Diff. Equations*, 51, (1984), 126 – 150.
- [33] G.M. Troianello, *Elliptic differential equations and obstacle problems*, Springer Science and Business Media, 1987.

GIUSY CHIRILLO, LUIGI MONTORO, LUIGI MUGLIA, BERARDINO SCIUNZI: DIPARTIMENTO DI  
MATEMATICA, UNIVERSITÀ DELLA CALABRIA, 87036 ARCAVACATA DI RENDE (CS), ITALY  
*Email address:* `chirillo@mat.unical.it`; `montoro@mat.unical.it`, `muglia@mat.unical.it`,  
`sciunzi@mat.unical.it`