Unbiased Estimation of Sinusoidal Signal Parameters via Discrete-Time Frequency-Locked-Loop Filters

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***Abstract*—A novel class of discrete-time Frequency-Locked-Loop nonlinear (FLL) filters is introduced and their relevance for the on-line estimation of the parameters of a possibly time-varying sinusoidal signal is discussed. Continuous-time nonlinear FLL filters have been proposed in the literature because of their circuital simplicity and ability to provide unbiased estimates of the instantaneous frequency, phase and amplitude of a time-varying sinusoidal signal simultaneously. It is shown in this note that standard discretization techniques may fail to generate discrete-time FLL filters exhibiting the same good properties of the continuous-time domain. In particular, biased frequency estimates are usually produced by these discrete-time filters. In this note, such a drawback is overcome because the proposed discrete-time FLL filters are proved to enjoy semi-globally exponential stability and their estimates are unbiased. A final example is presented for assessment purposes where also comparisons with other discrete-time filters, among which some synthesized by means of standard discretization techniques, are provided.**

I. Introduction

Frequency estimation of time-varying sinusoidal signals is an important topic in science and engineering that has found applications in many fields such as control engineering, signal processing, biomedical engineering, navigation, instrumentation and measurement and power engineering to mention a few, see e.g. [1], [2], [3], [4] and references therein. In the scientific literature, a large variety of algorithms for frequency estimation have been proposed, e.g. multiple integrals methods [5], adaptive filters ([6], [7], [8], [9], [10]), time-frequency representation (TFR) methods [12], phase-locked-loop (PLL) methods, eigenspace tracking methods [13], extended Kalman filters [14], internal model based methods ([2], [15]) and hybrid observers ([16]).

In [17], a class of continuous-time nonlinear frequency-locked-loop filters (FLL) has been presented that, unlike the aforementioned methods, are capable of estimating all parameters of a time-varying sinusoidal signal at the same time, that is its instantaneous frequency, phase and amplitude. The FLL is composed of a Quadrature-Signals Generator (QSG) equipped with an adaptive tuning mechanism for the related resonant frequency. This approach has found applications in power engineering problems in [18] where it has been used for solving voltage synchronization problems in power-grids. Recent improvements on the estimation accuracy and speed of convergence for this class of filters have been reported in [19]. The method uses modulating functions to adjust the resonant frequency of the FLL filter so as to reduce the convergence time and improve the accuracy of the parameters estimation. Further extensions are reported in [20], where a FLL filter able to deal with biased sinusoidal signals is proposed, and [21] where the structural properties of the filter are discussed providing also a valuable method to estimate the unknown parameters of a signal independently by the FLL filter tuning parameters.

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In this paper, discrete-time versions of the continuous-time FLL filters proposed in [17] are presented and their properties fully analyzed. Discrete-time versions of the filters are more suitable in view of their practical implementation in embedded systems. Although their synthesis from direct discretization of continuous-time filters could be believed a fairly straightforward task, on the contrary in this paper it is shown that the discretization procedures in this case could not be as trivial as expected if the resulting discrete-time filters are required to preserve the same properties (global stability, convergence speed and estimation accuracy) of their continuous-time counterparts and, at the same time, exhibit low computational burdens. This issue has not been dealt with in literature in a formal way. Only in [22], authors address the problem of discretizing the continuous-time version of the same QSG here used. In the latter work, the QSG is enclosed in a control loop designed to track a certain current sinusoidal profile. Several discretization methods have been compared from a numerical perspective but any formal analysis about the stability and convergence properties of the discretized QSGs has not been carried out. Here, it is shown that all standard discretization techniques used in [22] are not effective for discretizing the continuous-time FLL filters presented in [17] because leading to biased estimates of the unknown frequency to be identified. Such a bias is reduced only if the working sampling time of resulting discrete-time filters is very low with respect to the period of the signal to be estimated. As a consequence for high frequency applications very expensive and fast hardware (samplers, cpu, memories) should be used. In [23] such an issue has been addressed and solved for a similar adaptive filter by adding an offset computed on-line to the estimated frequency so as to compensate the undesired error.

Here such a problem is solved from the outset by introducing a novel discrete-time FLL filter. Such a filter keeps the same structure of its continuous-time counterpart. Moreover, it preserves semi-globally exponential stability and is able to converge to the true values of the signal parameters regardless of their initial conditions when both initial estimates and true values are contained in a prefixed but arbitrarily large compact set. On the contrary, it is shown that FLL filters discretized via the standard discretization techniques, are stable but give rise to biased estimates. Such an approach has been already presented in a preliminary form in [26], where, however, several important details and formal proofs were missing. Here all theoretical results and proofs have been included along with a numerical analysis of the robustness of the filter estimates against the presence of measurement noise. For a recent application of the proposed class of filters please refer to [27].

The note is organized as follows. In section II, we introduce the discrete-time QSG. In section III the overall structure of the FLL is presented and its main properties investigated. In Section IV, the effectiveness of the proposed filter is shown in a final example where also comparisons of filters designed with both other discretization techniques and approaches are reported.

II. A Discrete-Time QSG

The proposed FLL filter consists of the discrete-time QSG with gain K_s and a resonant frequency equal to ω_s shown in Fig. 1 and an adaptive tuning mechanism introduced in next section. In Figure 1, z denotes the forward-shift operator for signal sequences $y(kT_s)$, namely $y((k+1)T_s) := zy(kT_s)$. The QSG is a four-order dynamical system characterized by a resonant frequency ω_s and gain K_s. It is assumed that the filter is fed by the following sinusoidal input

$$v(kT_s) = A_c \sin(\omega_c kT_s + \phi_c) \tag{1}$$

where T_s is the sampling period chosen according to the following requirement

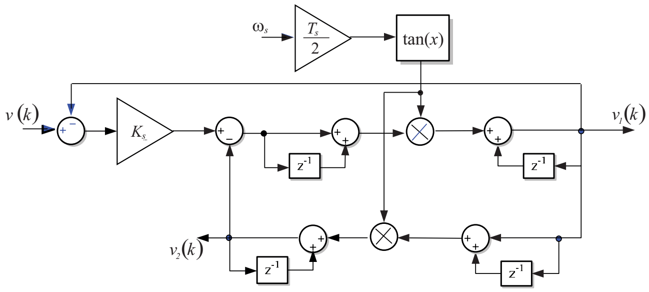

Fig. 1. Discrete-time QSG.

Assumption *1:* According to the Nyquist-Shannon Theorem, the sampling period T_s and frequencies ω_c, and ω_s satisfy the following conditions

$$\omega_c < \pi/T_s, \quad \omega_s < \pi/T_s$$

□

Notice that the choice of the sampling period implicitly defines also the workable range of frequencies $(0, \frac{\pi}{T_s})$ for the allowable sinusoidal signals which can be handled by the discrete-time filter without aliasing problems.

For input signals of type (1) the QSG provides two orthogonal signals $v_1(k)$ and $v_2(k)$ according to the following difference equations

$$v_1(k+1) = -v_1(k) + \mu(k) \tag{2}$$

$$v_2(k+1) = v_2(k) + \tan\left(\frac{\omega_s T_s}{2}\right)\mu(k) \tag{3}$$

with

$$\mu(k) = \frac{\tan\left(\frac{\omega_s T_s}{2}\right)(K_s(v(k)+v(k+1)) - 2v_2(k)) + 2v_1(k)}{1+\tan\left(\frac{\omega_s T_s}{2}\right)\left(K_s + \tan\left(\frac{\omega_s T_s}{2}\right)\right)} \tag{4}$$

and $\tan(\cdot)$, $\cot(\cdot)$ representing the tangent and the cotangent trigonometric functions respectively. Pleas notice that the trigonometric term $\tan(\frac{\omega_s T_s}{2})$ is always finite thanks to Assumption 1. The above model has been derived from a second-order continuous-time QSG ([17]) by means of Tustin discretization (bilinear) transform with pre-warping frequency ω_s presented in [26].

In the z-domain the above filter can be described by the following transfer functions, mapping the input sequence $v(k)$ in $v_1(k)$ and $v_2(k)$ respectively.

$$F_1(z) := \frac{K_s\tan(\frac{\omega_s T_s}{2})\, z^2 - K_s\tan(\frac{\omega_s T_s}{2})}{D(z)} \tag{5}$$

$$F_2(z) := \frac{K_s\tan^2(\frac{\omega_s T_s}{2})\,(z+1)^2}{D(z)} \tag{6}$$

and where

$$D(z) := \left(1+K_s\tan(\tfrac{\omega_s T_s}{2})+\tan^2(\tfrac{\omega_s T_s}{2})\right)z^2 + \left(2\tan^2(\tfrac{\omega_s T_s}{2})-2\right)z + 1 - K_s\tan^2(\tfrac{\omega_s T_s}{2}) + \tan^2(\tfrac{\omega_s T_s}{2})$$

Thus, if the sinusoidal excitation (1) were applied under a constant resonant frequency ω_s, the QSG would generate the following two orthogonal sinusoidal signals under permanent conditions

$$v_1^\infty(k) = \left|F_1\left(e^{j\omega_c T_s}\right)\right| A_c \sin(\omega_c k + \phi_c + \phi) \tag{7}$$

$$v_2^\infty(k) = -\cot\left(\frac{\omega_c T_s}{2}\right)\tan\left(\frac{\omega_s T_s}{2}\right)\left|F_1\left(e^{j\omega_c T_s}\right)\right| A_c \cos(\omega_c k + \phi_c + \phi) \tag{8}$$

where

$$\phi := \arg\left\{F_1\left(e^{j\omega_c T_s}\right)\right\} = \arctan\left\{\frac{2(\cos(T_s\omega_c) - \cos(T_s\omega_s))}{K_s\sin(T_s\omega_c)\sin(T_s\omega_s)}\right\} \tag{9}$$

Moreover, if the resonant frequency ω_s were the same of the unknown ω_c, then the signal $v_1^\infty(k)$ would coincide with the unbiased sinusoidal input, while $v_2^\infty(k)$ would be a sinusoidal signal with a phase shift of $\frac{\pi}{2}$ with respect to $v_1^\infty(k)$. That is, in such a case, the QSG output sequences would assume the following form

$$v_1^\infty(k) = \left|F_1\left(e^{j\omega_c T_s}\right)\right| A_c \sin(\omega_c k + \phi_c) \tag{10}$$

$$v_2^\infty(k) = -\left|F_1\left(e^{j\omega_c T_s}\right)\right| A_c \cos(\omega_c k + \phi_c) \tag{11}$$

III. Adaptive Tuning of Resonant Frequency ω_s and Parameters Estimation

In this section a discrete-time adaptive updating law for the resonant frequency $\omega_s(k)$ of the QSG filter is investigated, along with its exponential stability property. Such an adaptive updating law is based on the following function to be evaluated at each time instant

$$\beta(\omega_s(k)) := \omega_s(k) - \gamma\tan\left(\frac{\omega_s(k)T_s}{2}\right)(v(k)-v_1(k))v_2(k) \tag{12}$$

and it is given by

$$\omega_s(k+1) = \max\{\varepsilon, \beta(\omega_s(k))\} \tag{13}$$

with $\omega_s(0) > 0$ and $\varepsilon \in (0, \omega_c)$ an arbitrarily small scalar. Please notice that (12)-(13) account for the following recurrent tuning law for ω_s

$$\omega_s(k+1) = \omega_s(k) - \gamma\tan\left(\frac{\omega_s(k)T_s}{2}\right)(v(k)-v_1(k))v_2(k) \tag{14}$$

incorporating in addition a condition for avoiding $\omega_s(k)$ to become either 0 or negative when starting from a positive initial value. This event, although rare in practice, could occur if the parameter γ is badly chosen. In such undesiderable case (13) simply re-initializes the adaptation law (14) to the safe value $\omega_s(k) = \varepsilon$ and it allows $\omega_s(k)$ to converge to the correct value ω_c when $k \to \infty$ as proved in the next section.

Since $\omega_s(k)$ is a time-varying parameter, the QSG filter described by equations (2)-(4) becomes a nonlinear filter. For small enough values of γ, with $\gamma > 0$, it is possible to prove the exponential convergence property of such a law by using the averaging theory [25]. Once ω_s reaches its equilibrium point, i.e. ω_s is tuned on ω_c, the amplitude of $v(t)$ can be estimated by considering the following function

$$\alpha(k) := -v_2(k) + jv_1(k)\cot\left(\frac{\omega_c T_s}{2}\right)\tan\left(\frac{\omega_s T_s}{2}\right) \tag{15}$$

It easily results that

$$|\alpha(k)| = \cot\left(\frac{\omega_c T_s}{2}\right)\tan\left(\frac{\omega_s T_s}{2}\right)\left|F_1\left(e^{j\omega_c T_s}\right)\right| A_c \tag{16}$$

$$\arg\{\alpha(k)\} = \omega_c k + \phi_c + \phi \tag{17}$$

From the above expressions, once an estimate of ω_c is obtained, it is straightforward to compute estimates of A_c and ϕ_c as

$$\hat{A}_c = \frac{|\alpha(t)|}{\cot(\frac{\omega_c T_s}{2})\tan(\frac{\omega_s T_s}{2})|F_1(e^{j\omega_c T_s})|}, \tag{18}$$

$$\hat{\phi}_c = \arg\alpha(t) - \hat{\omega}_c t - \phi \tag{19}$$

In fact, it can be proved that $\left|F_1\left(e^{j\omega_c T_s}\right)\right| \to 1$ when $\omega_s \to \omega_c$.

Remark 1: It is worth commenting that the above proposed filter has been derived from the continuous-time FLL filter of [17] by means of the particular Tustin (bilinear) discretization method with an adaptive pre-warping mechanism presented in [26]. Note also that by strictly following that approach the updating equation would result different from (14). In fact, in [26] the following discrete-time equivalent tuning law was achieved

$$\omega_s(k+1) = \omega_s(k)\frac{\omega_s(k)\cot\left(\frac{\omega_s(k)T_s}{2}\right) - \xi(k)}{\omega_s(k)\cot\left(\frac{\omega_s(k)T_s}{2}\right) + \xi(k+1)} \tag{20}$$

with $\xi(k) = \gamma(v(k) - v_1(k))v_2(k)$, that however may present a singularity whenever $\omega_s(k)\cot\left(\frac{\omega_s(k)T_s}{2}\right) = -\xi(k+1)$. In order to remove from the outset both such a singularity problem and the occurrence of zero or negative values for $\omega_s(k)$, the slightly modified adaptive tuning law (12)-(13) is here proved to enjoy nice convergence properties free of singularities.

Remark 2: The introduction of equation (14) is needed in the special cases where either $\omega(0)$ or ω_c are very close to 0. In these situations, if γ is badly chosen, the updating law (14) can output during the iterations either a negative or 0 value leading the QSG system to instability. There is not any particular algorithm to choose ε, but it is convenient in practice to select values for it that are three or four order of magnitudes lower than the possible spanning range for ω_c, usually known in many applications.

A. Averaging discrete-time methods

For the convergence and stability analysis of system (2),(3),(13) we resort to the averaging theory for discrete-time systems [25]. In particular, recurrent equations (2), (3) and (13) can be seen to be in the form

$$y(k+1) = A(x(k))y(k) + \gamma g(k, x(k), y(k), \gamma) \tag{21}$$

$$x(k+1) = x(k) + \gamma f(k, x(k), y(k)) \tag{22}$$

where $x(k) \in \mathbb{R}^n$ stands for the slow state and $y(k) \in \mathbb{R}^m$ for the fast one. Notice that in our case $v_1(k)$ and $v_2(k)$ behave as fast oscillatory dynamics and hence they will assume the role of fast states. On the contrary, the evolution of $\omega_s(k)$ may be made arbitrarily slow by acting on the parameter γ that can be chosen arbitrarily close to 0.

The averaged version of equation (22) turns out to be as follows

$$x_{av}(k+1) = x_{av}(k) + \gamma f_{av}(x_{av}(k)) \tag{23}$$

where f_{av} is the limit

$$f_{av}(x) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=s+1}^{s+T} f(k, x, 0) \tag{24}$$

Standard methods may be employed to analyze the stability properties of the equilibrium points of such a system. Stability of the averaged and the original systems are related by the following Proposition (Please refer to [25] for details)

Proposition 1: Let $g(k, x, y, \gamma)$ and $f(k, x, y)$ be Lipschitz functions for $(k, x, y, \gamma) \in [0, \infty) \times X_0 \times Y_0 \times [0, \gamma_0]$ in every compact sets $X_0 \subset X \subset \mathbb{R}^n$ and $Y_0 \subset Y \subset \mathbb{R}^m$. Assume also that f and g are T-periodic in k for some $T > 0$. Let $x(k)$ and $x_{av}(k)$ denote the solutions for the original and, respectively, the averaged systems. Then

1) for any $\bar{k} \in \mathbb{Z}_+$, there exist $b_{\bar{k}} > 0$, $0 < \gamma_{\bar{k}} < \gamma^*$ and a class K function $\Psi(\gamma)$ such that

$$\|x(k) - x_{av}(k)\| \le \Psi(\gamma)b_{\bar{k}} \tag{25}$$

for all $k \in [0, \bar{k}/\gamma]$, $0 < \gamma < \gamma_{\bar{k}}$ ([25], Theorem 2.2.3).

2) Assume that $f_{av}(x)$ has continous and bounded derivative in x. If x_{eq} is an exponential stable equilibrium for the averaged system (23), then there exists $0 < \gamma_2 < \gamma_0$ such that x_{eq} is also an exponential stable equilibrium for the original system (21)-(22) for all $0 < \gamma < \gamma_2$ ([25], Theorem 2.2.4).
3) Assume that $f_{av}(x)$ has continous and bounded derivative in x. If x_{eq} is an unstable equilibrium for the averaged system (23), then x_{eq} is also an unstable equilibrium for the original system (21)-(22) provided that $0 < \gamma < \gamma_0$ ([25], Theorem 4.2.2).

B. Main Results

Before stating the main properties of the proposed FLL filter, it is worth observing that when the evolution of $\omega_s(k)$ is remarkably slower than those of the time-varying signals $v_1(k)$ and $v_2(k)$, the following equivalent expression can be derived for the evolution of $\beta(k)$ in (12)

$$\beta(\omega_s(k)) = \omega_s(k) - \gamma\tan\left(\frac{\omega_s(k)T_s}{2}\right)(1 - F_1(z))F_2(z)v^2(k) \tag{26}$$

In this case the averaged version of (26) takes this structure

$$\beta_{av}(\omega_s(k)) = \left(1 - \gamma\frac{A_c^2}{2}\mathrm{Re}\left\{\left(1 - F_1\left(e^{j\omega_c T_s}\right)\right)\tilde{F}_2\left(e^{-j\omega_c T_s}\right)\right\}\right)\omega_s(k) \tag{27}$$

with

$$\tilde{F}_2\left(e^{-j\omega_c T_s}\right) = F_2\left(e^{-j\omega_c T_s}\right)\frac{\tan\left(\frac{\omega_s(k)T_s}{2}\right)}{\omega_s}$$

that can be recast into the following different equation

$$\beta_{av}(k) = \big(1 + \gamma q(\omega_s(k))h(\omega_s(k))\big)\omega_s(k) \tag{28}$$

where

$$h(\omega_s) := -\frac{8A_c^2K_s\cos^2\left(\frac{T_s\omega_c}{2}\right)\sin^2\left(\frac{T_s\omega_s}{2}\right)\tan\left(\frac{T_s\omega_s}{2}\right)}{4\omega_s q^2(\omega_s) + K_s^2\omega_s\sin^2(T_s\omega_c)\sin^2(T_s\omega_s)} \tag{29}$$

and

$$q(\omega_s) := (\cos(T_s\omega_c) - \cos(T_s\omega_s)) \tag{30}$$

From equation (28) it is trivial to see that ω_c is an equilibrium solution in $(\varepsilon, \frac{\pi}{T_s})$ for the following average version of recurrent equation (13)

$$\omega_{s,av}(k+1) = \max\{\varepsilon, \beta_{av}(\omega_{s,av}(k))\} \tag{31}$$

Moving from the above considerations, we present the following Lemma 1 that is instrumental for the main result of this work

Lemma 1: Consider the following function

$$V(\omega_s) := (\omega_s - \omega_c)^2 \tag{32}$$

and its increment evaluated on β_{av}

$$\Delta V(\beta_{av}(\omega_s), \omega_s) := V(\beta_{av}(\omega_s)) - V(\omega_s) \tag{33}$$

Then, there exists positive scalars ϵ_1, ϵ_2, such that

$$\epsilon_1(\omega_s - \omega_c)^2 \le V(\omega_s) \le \epsilon_2(\omega_s - \omega_c)^2 \tag{34}$$

Moreover, there exists scalars γ^* and ϵ_3 such that for all $0 < \gamma < \gamma^*$, and $\omega_s, \omega_c > 0$ satisfying Assumption 1

$$\Delta V(\beta_{av}(\omega_s), \omega_s) \le -\epsilon_3(\omega_s - \omega_c)^2, \tag{35}$$

Proof: It is trivial to see that $0 < \epsilon_1 < 1$ and $\epsilon_2 > 1$ satisfy (34), while, in order to prove the existence of ϵ_3, a deeper investigation is required. To this end, please notice that the increment $\Delta V(\beta(\omega_s), \omega_s)$ takes the following form

$$\Delta V(\beta_{av}(\omega_s), \omega_s) = \gamma h(\omega_s)\omega_s\left[2(\omega_s - \omega_c)q(\omega_s) + \gamma h(\omega_s)\omega_s q^2(\omega_s)\right] \tag{36}$$

and $h(\omega_s) < 0$ under Assumption 1. Then, a necessary and sufficient condition to ensure (35) to hold true is that

$$2\left(\omega_s - \omega_c\right) q(\omega_s) + \gamma h(\omega_s)\omega_s q^2(\omega_s) \geq -\frac{\epsilon_3}{\gamma h(\omega_s)}\frac{(\omega_s-\omega_c)^2}{\omega_s} \qquad (37)$$

In fact, if (37) holds true, a trivial way to obtain (35) is to multiply both sides of (37) by the negative term $\gamma h(\omega_{s,av})\omega_{s,av}$. Next, because we want to investigate the function (32) for $\omega_s \neq \omega_c$, things can be simplified by dividing all factors in the inequality (37) by the positive term $\omega_s q^2(\omega_s)$. As a result, after further simple algebraic manipulations, one arrives to the following equivalent inequality to be satisfied

$$-\gamma h(\omega_s) \; - \frac{\epsilon_3}{\gamma h(\omega_s) q^2(\omega_s)}\frac{(\omega_s-\omega_c)^2}{\omega_s^2} \leq \frac{2}{q(\omega_s)}\frac{\omega_s-\omega_c}{\omega_s} \qquad (38)$$

Moreover, because ω_s and ω_c are scalar quantities, we can redefine ω_s as

$$\omega_s := \rho\omega_c \qquad (39)$$

for $0 < \rho < \frac{\pi}{T_s\omega_c}$. In this way, if we impose $\epsilon_3 = \gamma^2$, inequality (38) takes the following form

$$-\gamma\Big(h(\rho\omega_c) + \frac{1}{h(\rho\omega_c)q^2(\rho\omega_c)}\Big(\frac{\rho-1}{\rho}\Big)^2\Big) \leq \frac{2}{q(\rho\omega_c)}\frac{\rho-1}{\rho} \qquad (40)$$

Notice that the term $q(\rho\omega_c)$ is strictly negative if $\rho < 1$ while strictly positive if $\rho > 1$. Furthermore,

$$\lim_{\rho\to 1}\frac{1}{q(\rho\omega_c)}\frac{\rho-1}{\rho} = \frac{1}{T_s\omega_c\sin\left(T_s\omega_c\right)} > 0 \qquad (41)$$

Hence, the right-hand side of (40) is always strictly positive. Therefore, there surely exists an arbitrarily small γ^* satisfying (40). □

The following result, representing the main contribution of this note, ensures that the tuning law (13) for ω_s semi-globally exponentially converges to the actual signal frequency ω_c.

Theorem 1: Assume $v(k)$ is as in (1) and Assumption 1 holding true. Let the resonant frequency $\omega_s(k)$ be updated on the basis of (2), (3) and (13). Then for all $0 < \omega_s(0) < \frac{\pi}{T_s}$ and $0 < \omega_c < \frac{\pi}{T_s}$ the constant solution $\omega_s(k) \equiv \omega_c$ is the unique exponentially stable equilibrium in $(0, \frac{\pi}{T_s})$.

Proof: In order to investigate the exponential stability property of the equilibrium ω_c for system (31), consider the candidate Lyapunov function $V(\omega_{s,av}(k))$ with $V(\omega_s)$ already defined in (32) and try to determine positive scalars ϵ_1, ϵ_3, ϵ_3 such that

$$\epsilon_1(\omega_{s,av}(k) - \omega_c)^2 \leq V(\omega_{s,av}(k)) \leq \epsilon_2(\omega_{s,av}(k) - \omega_c)^2 \qquad (42)$$

$$\Delta V(\max\{\varepsilon, \beta_{av}(\omega_{s,av}(k))\}, \omega_{s,av}(k)) \;\leq\; -\epsilon_3(\omega_{s,av}(k) - \omega_c)^2 \qquad (43)$$

Lemma 1 guarantees that scalars ϵ_1 and ϵ_2 exist satisfying condition (42). Moreover when $\beta_{av}(\omega_{s,av}(k)) \geq \varepsilon$, leading to the one-step ahead evolution for $\omega_{s,av}(k)$ given by $\omega_{s,av}(k+1) = \beta_{av}(\omega_{s,av}(k))$, condition (43) takes the same form of condition (35). Then, thanks to Lemma 1, the existence of a scalar ϵ_3 fulfilling (43) is ensured. The same ϵ_3 can be proved to satisfy condition (43) also in the case $\beta_{av}(\omega_{s,av}(k)) < \varepsilon$, corresponding to a state transition of the type $\omega_{s,av}(k+1) = \varepsilon$. To this end, since

$$\Delta V(\beta_{av}(\omega_{s,av}(k)), \omega_{s,av}(k)) < -\epsilon_3(\omega_{s,av}(k) - \omega_c)^2$$

it is sufficient to prove that

$$\Delta V(\varepsilon, \omega_{s,av}(k)) < \Delta V(\beta_{av}(\omega_{s,av}(k)), \omega_{s,av}(k)) \qquad (44)$$

The above inequality holds true because

$$\beta_{av}(\omega_{s,av}(k)) - \omega_c < \varepsilon - \omega_c \qquad (45)$$

In fact, because $\beta_{av}(\omega_{s,av}(k)) < \varepsilon \leq \omega_c$, both sides of the latter inequality are negative. Then, if we square them, the following new inequality is obtained

$$\left(\varepsilon - \omega_c\right)^2 < \left(\beta_{av}(\omega_{s,av}(k)) - \omega_c\right)^2 \qquad (46)$$

and, in turn,

$$\begin{aligned}\left(\varepsilon - \omega_c\right)^2 - &\left(\omega_{s,av}(k) - \omega_c\right)^2 < \\ &\left(\beta_{av}(\omega_{s,av}(k)) - \omega_c\right)^2 - \left(\omega_{s,av}(k) - \omega_c\right)^2\end{aligned} \qquad (47)$$

Then, (44) simply follows. Finally, the results stated in item 1)-2) of Proposition 1 complete the proof. □

IV. Simulation Results

A. Example 1: Comparison with discretized FLL filters

The aim of this section is to show the effectiveness of the above proposed discrete-time FLL filter. The simulation results will involve the following time-varying sinusoidal signal whose instantaneous frequency has to be estimated (see Figure 2 black solid line)

$$v(t) = A\sin\left(2\pi\int_0^{3.5} f_c(t)dt + \frac{\pi}{2}\right) \qquad (48)$$

where $A = 10$ and

$$f_c(t) = \begin{cases} 20 \text{ Hz}, & 0s \leq t \leq 0.5s, \\ 16t + 16 \text{ Hz}, & 0.5s < t \leq 3s, \\ 60 \text{ Hz}, & 3s < t \leq 3.5s \end{cases}$$

The above signal has been sampled with a period of $T_s = 2.5\times10^{-3}$s (400 [Hz]) and processed by the proposed discrete-time FLL filters with parameters $K_s = 1.5$, $\gamma = 0.9$ and $\varepsilon = 10^{-5}$. To put in evidence the performance of the proposed filter, comparisons with other discrete-time FLL filters achieved by directly discretizing the continuous-time FLL filter in [17] have been carried out. In particular we have taken into account the following discretization methods

- Forward-Euler ($K_s = 2$, $\gamma = 0.5$);
- Tustin with a pre-warping frequency $\omega_p = \frac{30}{2\pi}rad/s$ ($K_s = 1.5$, $\gamma = 0.5$);
- 4th order Runge-Kutta ($K_s = 1.5$, $\gamma = 0.5$).

In the simulations all filters have been initialized in $\varepsilon = \omega_s(0) = \frac{10}{2\pi}rad/s$. The results are depicted in Figure 2. Except for the transient phases, the graph related to the frequency estimated by the proposed FLL filter (green dashed line), is indistinguishable from the true frequency $f_c(t)$ and in this case the filter achieves the best performance. On the contrary all other filters produce biased estimates making them not suitable to be used in this kind of applications. Notice also that the performance related to these latter methods could be improved if the sampling period T_s were reduced. In this respect, we have carried out a further analysis where the above simulations have been repeated with several increasing sampling frequencies $\frac{1}{T_s}$. In particular, we have quantified the estimation error by introducing the following performance index

$$E_N := \frac{1}{N}\sum_{k=0}^{N} E(k) \qquad (49)$$

where $E(k)$ represents the percentage relative error between the actual frequency $f_c(k)$ and the estimated frequency $f_s(k) = \frac{\omega_s(k)}{2\pi}$ computed with a simulation step $k = t/T_s$, i.e.

$$E(k) := \frac{|f_c(k) - f_s(k)|}{f_c(k)} \times 100 \qquad (50)$$

Table I reports, for each discretization method, the values of E_N for several values of the sampling frequency $\frac{1}{T_s}$. In Tables II and

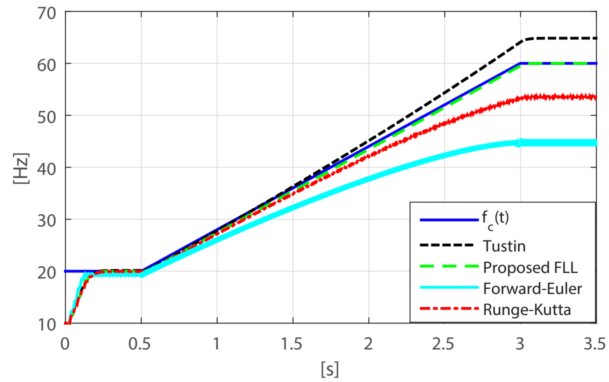

Fig. 2. Example 1: Estimated input signal frequency profile.

$1/T_s[Hz]$	Forward Euler	Tustin pre-warping	Proposed FLL	Runge Kutta
200	49.85	18.96	2.25	17.23
400	14.44	4.04	2.24	4.01
800	5.10	2.24	2.24	2.26
1000	4.03	2.24	2.24	2.26
12000	2.25	2.24	2.24	2.26

TABLE I
EXAMPLE 1: RELATIVE ESTIMATION ERROR $E_N[\%]$

III, the instantaneous values of the error $E(k)$ at time $k = \frac{0.5}{T_s}$ and $k = \frac{3.5}{T_s}$ respectively, are reported in order to investigate the steady-state behavior of the proposed FLL filter. It results that, whenever Assumption 1 hold true, the performance of the proposed discrete-time FLL filter do not seem to depend remarkably on the sampling time T_s. On the contrary, the errors pertaining to all other discrete-time FLL filters, that are considerable especially at high frequencies (Table 2), diminish when $1/T_s$ increases. However, in order to obtain comparable performance, it would be needed to work with a sampling frequency 100 times greater then the Nyquist frequency (120[Hz] in our case), a case which would require considerable computing resources.

Further simulations have been carried out in order to verify the robustness of the proposed filter when the same signal $v(t)$ considered

$1/T_s[Hz]$	Forward Euler	Tustin pre-warping	Proposed FLL	Runge Kutta
200	11.58	3.23	2.71×10^{-4}	1.62
400	2.87	0.78	7.33×10^{-4}	0.25
800	0.71	0.19	7.71×10^{-4}	0.001
1000	0.45	0.12	7.89×10^{-4}	0.001
12000	0.05	7.86×10^{-4}	1.41×10^{-4}	5.06×10^{-4}

TABLE II
RELATIVE ESTIMATION ERROR $E(k)[\%]$ AT TIME $0.5\ s$. $\left(k = \frac{0.5}{T_s}\right)$

$1/T_s[Hz]$	Forward Euler	Tustin pre-warping	Proposed FLL	Runge Kutta
200	89.25	45.76	2.41×10^{-10}	42.62
400	26.17	8.07	1.27×10^{-7}	8.04
800	6.47	1.88	1.88×10^{-6}	0.18
1000	4.41	1.19	2.33×10^{-6}	0.04
12000	0.02	0.008	1.57×10^{-6}	0.0011

TABLE III
EXAMPLE 1: RELATIVE ESTIMATION ERROR $E(k)[\%]$ AT TIME $3.5\ s$. $\left(k = \frac{3.5}{T_s}\right)$

SNR_{dB}	30	20	10	5	0	-5	-10
Proposed FLL	2.23	2.31	2.38	2.54	3.04	3.41	4.78
Runge-Kutta	2.26	2.52	2.6	2.61	3.23	3.56	4.82

TABLE IV
EXAMPLE 1: RELATIVE ESTIMATION ERROR $E_N[\%]$ IN THE PRESENCE OF NOISE

in (48) is corrupted by a zero-mean white gaussian noise $\eta(t) \sim WN(0,\sigma)$, i.e. $\tilde{v}(t) = v(t)+\eta(t)$, with the variance σ characterizing the *Signal-to-Noise Ratio* expressed as $\text{SNR}_{dB} = 20\log(A/\sigma)$.

Figure 4 shows some frequency estimations performed by the proposed discrete-time FLL filter with a sampling period of $T_s = 1.25\times10^{-3}$ (800 [Hz]). The results seem satisfactory for $\text{SNR}_{dB} \geq -5$. Notice however that an increment of the sampling frequency can lead to lower estimation errors. Comparisons have been undertaken between the filter discretized by means of Runge-Kutta method. In Table IV, for each discretization policy, the values of E_N have been reported for several values of SNR_{dB}. In all cases, the filter discretized by means of the proposed method achieves the best performance.

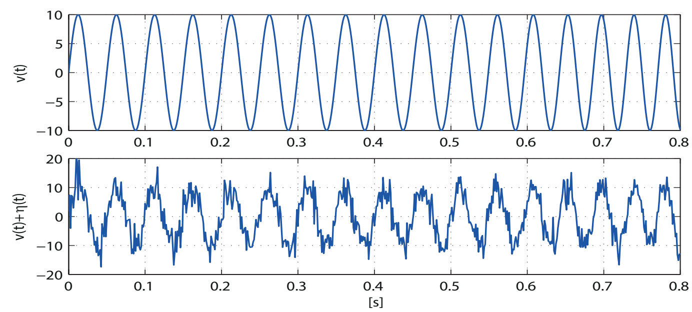

Fig. 3. (Up) Original signal $v(t)$ in the time interval $[0, 0.8]$ secs. (Down) Corrupted signal $\tilde{v}(t)$ with $\text{SNR}_{db} = 0$ in the interval $[0, 0.8]$ secs.

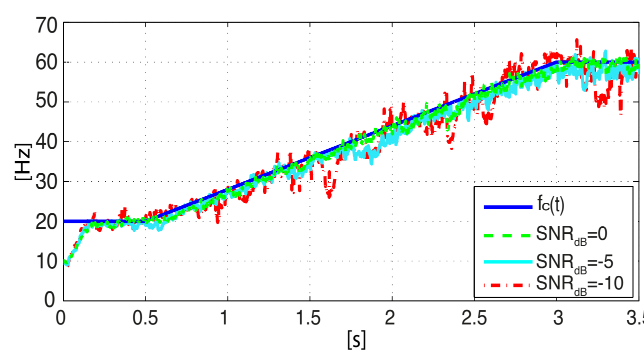

Fig. 4. Example 1: Estimated signal frequency profile for different values of SNR_{dB}

B. Example 2: Comparison with an alternative adaptive PLL filter approach

In order to further investigate the effectiveness of the proposed discrete-time FLL filter design approach, comparisons with the discrete-time adaptive Phase-Look Loop (PLL) based estimation technique introduced in [28], namely PLL-TD, are presented. In this case, the goal of the experiment is to estimate the amplitude and frequency of the signal

$$v(t) = A\sin(2\pi f_c(t)t) \tag{51}$$

where $A = 1$ and

$$f_c(t) = \begin{cases} 2 \text{ Hz}, & 0s \leq t \leq 30s, \\ 10 \text{ Hz}, & 30s < t \leq 60s \end{cases}$$

The above signal has been sampled with a period of $T_s = 0.03$s (33.33 [Hz]) and processed by the proposed discrete-time FLL filter, namely FLL-TD, with parameters $K_s = 5$, $\gamma = 29$ and $\varepsilon = 10^{-5}$. The PLL-TD has been tuned with parameters $g_\omega = 0.0396$, $z_\alpha = 0.9005$, $k_\alpha = 10.05$ and $g_m = 0.01$. In Figure 5 the estimated frequency and amplitude are depicted for both schemes. From that figure it clearly results that both methods behave similarly during the first transient while, after the frequency step change, PLL-TD presents a slower response.

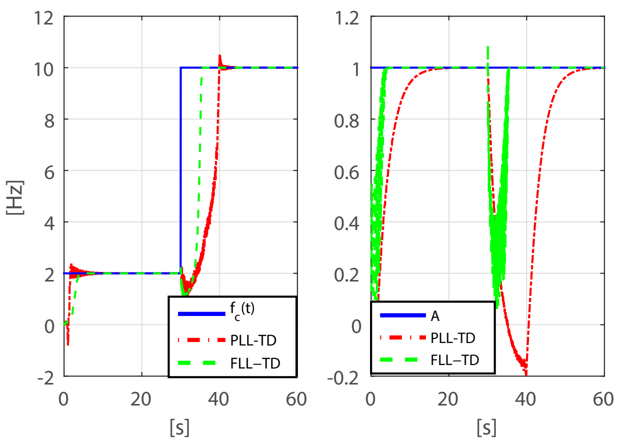

Fig. 5. Example 2: (left) frequency estimates, (right) amplitude estimates

V. Conclusions

A novel effective discrete-time FLL nonlinear filter has been derived starting from their continuous-time counterparts. The latter class of filters enjoy suitable features in terms of semi-global stability, convergence speed and estimation accuracy and have been used in several applications involving the instantaneous estimation of the frequency, phase and amplitude of a time-varying sinusoidal signal.

The proposed discrete-time FLL filter results semi-globally exponentially stable and unbiased. In particular, it has been proved that such a filter is able to converge to the true values of the signal parameters regardless of its initial conditions when both the initial estimates and the true values of them are contained in a prefixed but arbitrarily large compact set. On the contrary, it has been demonstrated that standard discretization techniques, although giving rise to stable filters, usually lead to biased estimates.

Final examples are provided where the proposed discrete-time FLL filter is used as a frequency tracker of a time-varying sinusoidal signal even in presence of noise. The responses of several discrete-time FLL filters synthesized via other discretization techniques are also reported for allowing comparisons. Further comparisons with a discrete-time PLL alternative scheme have confirmed the effectiveness of the approach.

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