Models for technology choice in a transit corridor with elastic demand

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Abstract

We present two optimization models for a transit line with elastic demand. These models can be used to strategically evaluate technology choices. We study the effect of demand elasticity on the technology choice by analytic and numerical comparison with the fixed demand models of **?**. We assume a range of objective functions having as two extrema the maximization of operator's profit and the maximization of social welfare. We show both analytically and numerically that accounting for demand elasticity does not change the conclusions that can be derived by an equivalent fixed demand model. For a given level of captured demand, the optimal configuration and the operating policies are invariant with respect to the assumptions on the demand model, fixed or elastic. This invariance holds for a broad range of objective functions in the elastic case. The significant difference between the two objective function extrema are in the fractions of captured demand.
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1. Introduction

We extend the transit technology choice models of **?** by assuming an elastic demand which depends on the passenger travel time and fare. Elastic demand models for the optimal supply and pricing of urban transit can frame several research questions. For example, **?** derived closed-form formulae for the optimal local area operating policies of a bus service under three objective functions, namely profit maximization, maximization of a combination of net user benefit and operator profit, and maximization of net user benefit under a deficit constraint. **?** studied the effects of supply and demand variations between service periods of bus feeder systems. **?** reassessed in an elastic demand framework the rationale for transit fare subsidization. Economies of scale on the passengers' waiting cost were found to motivate subsidization as in the fixed demand model of **?**. **?** analysed the effects of road congestion and bus crowding on optimal bus frequency, bus size, fare collection system, bus boarding policy, bus seating layout, fare, and road toll. **?** extended the model of **?** by including travel time elasticities. By so doing, they estimated environmental impact trade-off caused by mode shift from transit to autos.

The research question dealt in this paper is the effect of demand elasticity on the technology choice for a transit line. We assume a range of objective functions having as two extrema the maximization of operator's profit and the maximization of social welfare. We posit a general setting where passenger travel time components depend non-linearly on the demand level or other operating variables. For example, we incorporate in the in-vehicle time a crowding penalty which is a function of the vehicle load ratio.

Our contribution is an equivalence scheme between fixed and elastic demand optimization models. The significance is twofold. First, the equivalence scheme is instrumental to an effective solution method for the elastic demand optimization model. Second, we show both analytically and numerically that accounting for demand elasticity does not change the conclusions that can be derived by an equivalent fixed demand model. For a given level of captured demand, the optimal configuration and the operating policies are invariant with respect to the assumptions on the demand model, fixed or elastic. This invariance holds for a broad range of objective functions in the elastic case. The break-even points between technologies do not change significantly when demand elasticity is considered. These results do not rely on a specific assumption on the fare elasticity parameter. Henceforth, we reaffirm in a broader way the insight presented in **?** where an equivalence between elastic and fixed demand transit line optimization models was established in the case of passenger cost inversely proportional to the frequency of the transit line. We add the following insight. The difference between the two objective function extrema are in the fractions of captured demand. Profit maximization captures *circa* half of the demand captured by social welfare maximization.

The remainder of this paper is organized as follows. Section 2 presents mathematical models. Section 3 analyzes the models and introduces solution methods. Computational results are discussed in Section 4, and Section 5 reports some conclusions.

2. Models

As in **?**, we assume that a transit line of length L serves a bidirectional demand. The difference with respect to the previous paper is that we remove the assumption that the demand is fixed. Similarly to **?**, **?**, and **?**, we express the demand captured by transit as a linear function of average travel time components and fare. A linear demand model can be regarded as an approximation to logit and other demand models typically used in transportation planning. A caveat of this type of approximation is that it may be unreliable at the extrema of the demand range (**?**). However, this does not apply to our analysis since we do not deal with the question whether there it should be or not a transit line.

In the proposed model the fare is an optimized variable and the passenger travel time is composed of three parts: access and egress, waiting, and in-vehicle times. These travel time components depends on the following variables: the stop spacing, indicated by d, the frequency, f, and the length of the transit unit (TU), n. The frequency is expressed as the number of TU per hour. The concept of transit unit, see **?**, refers to a set of n physically linked vehicles traveling together. For single-vehicle operations, such as buses, n is equal to one, whereas for rail technology n can be larger than one and in that case is a variable to be optimized.

We introduce both a single and a two-period model. In a single period model the variables and the techno-economic coefficients are related to an average peak hour. This model allows to derive some analytic properties that are then deployed heuristically in a more realistic two-period model where variables and coefficients are related to both peak and off-peak average hours.

In the following we refer to the notation and the formulae of the fixed demand models of **?**, although the presented elastic models are of more general scope. We first introduce the single period model in Section 2.1, and in Section 2.2 we present the two-period model.

2.1. Single period model

In the single period model the maximum demand at peak hours is y, and the captured demand Y is a fraction of y. The average access and egress time, t_a, is a function of the stop spacing d. The average waiting time, t_w, is a function of the frequency f. This function can include the effect of schedule delay, see e.g. the equation (3) of **?**. The average in-vehicle time can be modeled as a fraction of the cycle time which depends on the frequency, the stop spacing, the TU length, and the captured demand Y which determines the boarding and alighting component of the cycle time (see e.g. equation (36) of **?**). Because of this dependence of the cycle time on the demand level, we indicate the average in-vehicle time as $t_{v,Y}$, and the average cycle time as $t_{c,Y}$. The average in-vehicle time includes a crowding multiplier which depends on the average vehicle load ratio. The average vehicle load ratio is inversely proportional to the frequency and to the consist length (see e.g. equation (37) of **?**). The crowding multiplier is equal to one, i.e. there is no crowding penalty, when the average vehicle load ratio is less or equal than a threshold related, for example, to the ratio of seats to total capacity. The crowding multiplier increases linearly for larger load ratios. Thus, the average in-vehicle time is here interpreted as an equivalent time which includes discomfort for crowding, if any.

We can now express the captured demand as a linear function of average travel time components and fare,

$$Y(f,d,n,P) \;=\; y\left(1 - e_a t_a(d) - e_w t_w(f) - e_v t_{v,Y}(f,d,n) - e_P P\right), \quad (1)$$

where e_a, e_w, and e_v are the elastic demand parameters for access, waiting, and in-vehicle times, P is the fare, and e_P is the elastic demand parameter for fare. This formula is recursive, since the in-vehicle travel time depends on Y.

We denote by k the terms in (1) that are in a first approximation independent of P,

$$k \;=\; 1 - e_a t_a(d) - e_w t_w(f) - e_v t_{v,Y}(f,d,n). \quad (2)$$

We use the following compact equation for the elastic demand as a function of fare,

$$Y(P) \;=\; y(k - e_P P). \quad (3)$$

The net passenger welfare, W_p, is the difference between the total willingness to pay and what passengers actually pay. Similarly to **?**, W_p is computed as illustrated in Figure 1:

$$W_p(f,n,d,P) \;=\; \frac{Y(f,n,d,P)^2}{2 e_P y}. \quad (4)$$

The operator profit, W_o, is the difference between the revenue, the product of fare and captured demand, and the operator cost. The operator cost depends on the frequency, the stop spacing, the TU length, and the captured demand level Y and can be expressed by the equation (41) of **?**. We indicate the operator cost by $C_{o,Y}$. Thus, the operator profit is

$$W_o(f,n,d,P) \;=\; Y(f,n,d,P)P - C_{o,Y}(f,n,d). \quad (5)$$

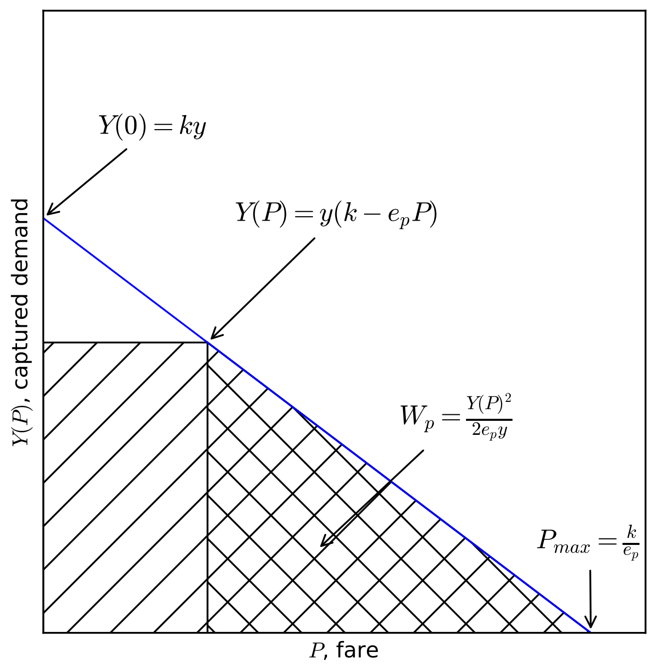

Figure 1: The net passenger welfare, W_p, is the double hatched area resulting from the difference between the total willingness to pay of captured passengers, and the fare paid by these passengers (single hatched area)

The single period model with elastic demand has the following objective

$$\text{maximize}\, W = \phi W_p + W_o, \tag{6}$$

where ϕ is parameter ranging from zero to one. Setting ϕ equal to zero maximizes the operator profit, setting ϕ equal to one maximizes the social welfare, sum of the net passenger welfare and the operator profit, and an intermediate value of ϕ maximizes the sum of the operator profit and a fraction ϕ of the net passenger welfare. The constraints of this optimization model are those that define feasible ranges for the frequency, stop spacing, and TU length, see for example (43)–(45) of **?**. Here we synthetically indicate the feasible range for the frequency as the interval between a minimal frequency f_{min} which depends on the TU length and the captured demand, and a maximum frequency which is technology dependent. Hence, the single period model can be stated as follows

$$\max_{f,n,d,P,Y} \quad \phi\frac{Y^2}{2e_P y} + YP - C_{o,Y}(f,n,d) \tag{7}$$

$$\text{s.t.} \quad d \geq d_{min} \tag{8}$$

$$n \in \{n_{min}, \ldots, n_{max}\} \tag{9}$$

$$f_{min}(n,Y) \leq f \leq f_{max} \tag{10}$$

$$P \geq 0 \tag{11}$$

$$Y = y\left(1 - e_a t_a(d) - e_w t_w(f) - e_v t_{v,Y}(f,n,d) - e_P P\right). \tag{12}$$

2.2. Two-period model

At peak and off-peak hours there are maximum demand levels y^p and y^o, respectively. The captured demand levels, Y^p and Y^o, are fractions of these maximum levels. We posit two frequencies for the two periods, namely f^p for the peak, and f^o for the off-peak period. The TU length for the peak period is n^p, and n^o denotes the TU length for the off-peak-period. The stop spacing, d, does not vary between periods. Analogously to the single period model, the captured demand levels are functions of the travel time components and fare:

$$Y^{(p,o)}(f^{(p,o)}, d, n^{(p,o)}, P) \quad = \quad y^{(p,o)}(1 - e_a t_a(d) - e_w t_w(f^{(p,o)}) - e_v t^{(p,o)}_{v,Y^{(p,o)}}(f^{(p,o)}, d, n^{(p,o)}) - e_P P). \tag{13}$$

The ratio of peak hours to total service hours is denoted by χ^p, and $\chi^o = 1 - \chi^p$ is the ratio of off-peak hours to total service hours. For example, if the typical weekday service is of 20 hours and the peak service is offered for five hours, then $\chi^p = 0.25$. Thanks to this notation, we can express the maximization of total welfare for the two-period model as follows:

$$\text{maximize}\, W = \phi \left(\chi^p \frac{Y^p(f^p, n^p, d, P)^2}{2e_P y^p} + \chi^o \frac{Y^o(f^o, n^o, d, P)^2}{2e_P y^o} \right) + \left(\chi^p Y^p(f^p, n^p, d, P) + \chi^o Y^o(f^o, n^o, d, P)\right) P - C_{o,Y^p,Y^o}(f^p, f^o, n^p, n^o, d), \tag{14}$$

where C_{o,Y^p,Y^o} is the operator cost function for the demand levels Y^p and Y^o at peak and off-peak periods, respectively. See equation (59) of **?** for an example of such two-period cost function. The constraints of this optimization model are the elastic demand constraint (13), and those that define feasible ranges for the frequencies, stop spacing, and TU lengths, see for example (61)–(64) of **?**.

3. Analysis and solution methods

We analyze the single period model and devise a solution method for it in Section 3.1. These analytic results are further discussed in Section 3.2.

The solution method for the single period model is heuristically extended to the two-period model in Section 3.3.

3.1. Solving the single period model

First, as observed by **?**, consistency requires that the ratios of travel times to fare sensitivity parameters are equal to the values of time, for example $e_a/e_P = P_a$, where P_a is the access and egress time value. We indicate by P_w and P_v the values of the waiting and in-vehicle times, respectively. The following equations holds

$$\begin{aligned} Y &= y(1 - e_a t_a - e_w t_w - e_v t_{v,Y} - e_P P) \\ &= y(1 - e_P \left(\frac{e_a}{e_P} t_a + \frac{e_w}{e_P} t_w + \frac{e_v}{e_P} t_{v,Y} \right) - e_P P) \\ &= y(1 - e_P(P_a t_a + P_w t_w + P_v t_{v,Y}) - e_P P). \end{aligned} \tag{15}$$

Second, we note that the term $P_a t_a + P_w t_w + P_v t_{v,Y}$ is the average passenger cost when the demand is equal to Y. We indicate by $C_{p,Y}$ the cost of Y passengers using a transit system with average access and egress, waiting, and in-vehicle times equal to t_a, t_w, and $t_{v,Y}$, respectively. Therefore, we can express the captured demand as

$$Y(f, n, d, P) = y \left(1 - e_P \frac{C_{p,Y}(f, n, d)}{Y} - e_P P \right). \tag{16}$$

Let $\lambda \in [0, 1]$ denote an auxiliary variable such that the elastic demand constraint (12) can be replaced by the following relations

$$\begin{cases} \lambda = 1 - e_P \frac{C_{p,y\lambda}(f,n,d)}{y\lambda} - e_P P, \\ Y = y\lambda, \\ 0 \leq \lambda \leq 1. \end{cases}$$

We can restate problem (7)-(12) as

$$W^* = \max_{f,n,d,P,\lambda} \quad \phi \frac{y}{2e_P}\lambda^2 + y\lambda P - C_{o,y\lambda}(f,n,d) \tag{17}$$

$$\text{s.t.} \quad d \geq d_{min} \tag{18}$$

$$n \in \{n_{min}, \dots, n_{max}\} \tag{19}$$

$$f_{min}(n, y\lambda) \leq f \leq f_{max} \tag{20}$$

$$0 \leq \lambda \leq 1 \tag{21}$$

$$P \geq 0 \tag{22}$$

$$\lambda = 1 - e_P \frac{C_{p,y\lambda}(f,n,d)}{y\lambda} - e_P P \tag{23}$$

where the dependence on Y has been replaced by $y\lambda$.

We observe that from equation (23) it follows that any feasible fare P is such that

$$P = \frac{1}{e_P} - \frac{C_{p,y\lambda}(f,n,d)}{y\lambda} - \frac{\lambda}{e_P}.$$

Hence, by getting rid of the dependence on P in problem (17)-(23), and taking into account that $P \geq 0$, we obtain the following equivalent formulation

$$\begin{aligned} W^* = \max_{f,n,d,\lambda} \quad & \phi \frac{y}{2e_P}\lambda^2 + y\lambda\left(\frac{1}{e_P} - \frac{C_{p,y\lambda}(f,n,d)}{y\lambda} - \frac{\lambda}{e_P}\right) \\ & - C_{o,y\lambda}(f,n,d) \qquad (24) \\ \text{s.t.} \quad & (18) - (21) \\ & \frac{1}{e_P} - \frac{C_{p,y\lambda}(f,n,d)}{y\lambda} - \frac{\lambda}{e_P} \geq 0. \qquad (25) \end{aligned}$$

Now observe that the objective function in (24) can be rewritten as

$$\frac{y\lambda}{e_P}\left(1 + \phi\frac{\lambda}{2} - \lambda\right) - C_{p,y\lambda}(f,n,d) - C_{o,y\lambda}(f,n,d)$$

where the first term depends on λ only. By adopting the following more compact notation

$$C_{tot,y\lambda}(f,n,d) \triangleq C_{p,y\lambda}(f,n,d) + C_{o,y\lambda}(f,n,d)$$

$$C(\lambda) \triangleq \frac{y\lambda}{e_P}\left(1 + \phi\frac{\lambda}{2} - \lambda\right)$$

where we indicate by $C_{tot,y\lambda}$ the total cost, sum of passenger and operator cost, at the demand level $y\lambda$, the problem can be reformulated as

$$\begin{aligned} W^* = \max_{f,n,d,\lambda} \quad & C(\lambda) - C_{tot,y\lambda}(f,n,d) \qquad (26) \\ \text{s.t.} \quad & (18) - (21) \\ & C_{p,y\lambda}(f,n,d) \leq \frac{y}{e_P}\lambda(1-\lambda). \qquad (27) \end{aligned}$$

In order to develop a decomposition-based method, we introduce another auxiliary variable $\eta \in [0,1]$ that will allow to break the circularity deriving from constraint (12). Hence, we will next focus on the following reformulation of the problem

$$W^* = \max_{f,n,d,\lambda,\eta} \quad W(f,n,d,\lambda,\eta) \triangleq C(\lambda) - C_{tot,y\eta}(f,n,d) \tag{28}$$

$$\text{s.t.} \quad (18) - (21)$$

$$C_{p,y\eta}(f,n,d) \leq \frac{y}{e_P}\lambda(1-\lambda), \tag{29}$$

$$0 \leq \eta \leq 1 \tag{30}$$

$$\lambda = \eta. \tag{31}$$

Now, we decompose the problem by successive separation on η and λ. In fact, we observe that problem (28)-(31) can be rewritten as

$$W^* = \max_{\eta} \{W^*(\eta) : (30)\} \tag{M^*}$$

where

$$W^*(\eta) = \max_{\lambda} \left\{\overline{W}_\eta(\lambda) : (21),\ (31)\right\} \tag{$M^*(\eta)$}$$

and, in turn

$$\overline{W}_\eta(\lambda) = \max_{f,n,d} \left\{\widehat{W}_{\eta,\lambda}(f,n,d) : (18) - (20), (29)\right\} \tag{$M^*(\eta,\lambda)$}$$

where

$$\widehat{W}_{\eta,\lambda}(f,n,d) = C(\lambda) - C_{tot,y\eta}(f,n,d). \tag{32}$$

The latter implies that $(M^*(\eta,\lambda))$ can be rewritten as

$$\overline{W}_\eta(\lambda) = C(\lambda) - \overline{C}(\eta,\lambda), \qquad (M^*(\eta,\lambda))$$

where

$$\overline{C}(\eta,\lambda) = \min_{f,n,d} \left\{ C_{tot,y\eta}(f,n,d) : (18) - (20), (29) \right\}. \qquad (33)$$

Summarizing, taking into account the whole decomposition process, namely, problems $(M^*(\eta))$ and $(M^*(\eta,\lambda))$, model (M^*) appears as

$$W^* = \max_{\eta\in[0,1]} \max_{\lambda\in[0,1],\lambda=\eta} C(\lambda) - \min_{f,n,d} C_{tot,y\eta}(f,n,d) \qquad (34)$$

$$\text{s.t. } (18) - (20), (29).$$

We are now ready to propose an algorithmic approach for the solution of problem (M^*) based on the decomposition presented in (34). We will make use of the following two relaxed problems

$$\widetilde{W}(\eta,\lambda) = C(\lambda) - \min_{f,n,d} \left\{ C_{tot,y\eta}(f,n,d) : (18) - (20) \right\} \qquad (\widetilde{M}(\eta,\lambda))$$

obtained from $(M^*(\eta,\lambda))$ by removing constraint (29), and

$$\widetilde{W}(\eta) = \max_{f,n,d,\lambda} \left\{ C(\lambda) - C_{tot,y\eta}(f,n,d) : (18) - (21), (29) \right\} \qquad (\widetilde{M}(\eta))$$

obtained from $(M^*(\eta))$ by relaxing constraint (31), i.e., $\lambda = \eta$.

Now, assume that at an iteration i one is given $\eta_i \in [0,1]$, and let $\tilde{\lambda}_i = \eta_i$. We observe that the inner minimization problem in $(\widetilde{M}(\eta,\lambda))$, for $\eta = \eta_i$ and $\lambda = \tilde{\lambda}_i$, has the same structure as the fixed-demand model introduced in **?**. Hence, by applying the lower convex envelope scheme of **?**, it is possible to approximately solve $(\widetilde{M}(\eta,\lambda))$, obtaining the maximizer $(\tilde{f}_i, \tilde{d}_i, \tilde{n}_i)$, and the corresponding optimal value $\widetilde{W}(\eta_i, \tilde{\lambda}_i)$. In case $(\tilde{f}_i, \tilde{d}_i, \tilde{n}_i)$ satisfies constraint (29), then an optimal solution for $(M^*(\eta_i, \tilde{\lambda}_i))$ has been found, which is feasible for (M^*), whose optimal value $\overline{W}_{\eta_i}(\tilde{\lambda}_i)$ possibly represents a new best optimal value for (M^*). Otherwise, if $(\tilde{f}_i, \tilde{d}_i, \tilde{n}_i)$ does not satisfy (29), then $\widetilde{W}(\eta_i, \tilde{\lambda}_i)$ is an upper bound on $\overline{W}_{\eta_i}(\tilde{\lambda}_i)$. In this case, consider the relaxed problem $(\widetilde{M}(\eta))$ for $\eta = \eta_i$, obtained from $(M^*(\eta))$ by relaxing constraint (30). Adopting $(\tilde{\lambda}_i, \tilde{f}_i, \tilde{n}_i, \tilde{d}_i)$ as a starting point, an appropriate nonlinear solver can be launched to get an optimal solution $(\lambda_i, f_i, n_i, d_i)$, and the corresponding optimal value $\widetilde{W}(\eta_i)$. If the solution $(\lambda_i, f_i, n_i, d_i)$ satisfies constraint (30), namely if $\lambda_i = \eta_i$, then an optimal solution for $(M^*(\eta_i))$ has been found, which is feasible for (M^*), with optimal value $W^*(\eta_i) = C(\lambda_i) - C_{tot,y\eta_i}(f_i, n_i, d_i)$, returning a possible new best optimal value for (M^*). Otherwise, $C(\lambda_i) - C_{tot,y\eta_i}(f_i, n_i, d_i)$ is an upper bound on $W^*(\eta_i)$, and λ_i, possibly combined with η_i, can be used to generate η_{i+1} and to iterate the procedure from the beginning. The whole procedure can then be embedded into an exploration scheme over the interval $[0, 1]$ for variable η, in order to generate new η every time a feasible solution is found. An algorithm scheme for such procedure is presented in Algorithm 1.

3.2. Analytic insights for the single period model

Observations on the optimal configuration, operating policies, break-even point between technologies, and differences in the captured demand levels for the single period model can be drawn thanks to the presented equivalence scheme. The equivalence (M^*) leads to expect frequencies under elastic demand similar to those under total cost minimization at the demand level equivalent to the captured demand. The same observation can be made for

Algorithm 1 Solve the elastic model

$Best \leftarrow$ a large negative number

$\eta_0 \leftarrow \bar{\eta} \in [0, 1]$

$i \leftarrow 0$

repeat

$\quad \tilde{\lambda}_i \leftarrow \eta_i$

$\quad$ Solve $(\widetilde{M}(\eta_i, \tilde{\lambda}_i))$ and compute $(\tilde{f}_i, \tilde{n}_i, \tilde{d}_i,)$ as in Model IV of **?**

$\quad$ **if** $(\tilde{f}_i, \tilde{n}_i, \tilde{d}_i,)$ fulfills (29) **then**

$\quad\quad$ **if** $Best < C(\tilde{\lambda}_i) - C_{tot,y\eta_i}(\tilde{f}_i, \tilde{n}_i, \tilde{d}_i)$ **then**

$\quad\quad\quad Best \leftarrow C(\tilde{\lambda}_i) - C_{tot,y\eta_i}(\tilde{f}_i, \tilde{n}_i, \tilde{d}_i)$

$\quad\quad\quad \eta_{i+1} \leftarrow \text{explore}(0, 1)$

$\quad\quad$ **end if**

$\quad$ **else**

$\quad\quad$ Solve $(\widetilde{M}(\eta_i))$ and find $(\lambda_i, f_i, n_i, d_i)$ starting from $(\tilde{\lambda}_i, \tilde{f}_i, \tilde{n}_i, \tilde{d}_i)$

$\quad\quad$ **if** $\lambda_i = \eta_i$ **then**

$\quad\quad\quad$ **if** $Best < C(\lambda_i) - C_{tot,y\eta_i}(f_i, n_i, d_i)$ **then**

$\quad\quad\quad\quad Best \leftarrow C(\lambda_i) - C_{tot,y\eta_i}(f_i, n_i, d_i)$

$\quad\quad\quad$ **end if**

$\quad\quad$ **else**

$\quad\quad\quad \eta_{i+1} \leftarrow \text{combine}(\lambda_i, \eta_i)$

$\quad\quad$ **end if**

$\quad$ **end if**

until an iteration limit is reached

return $Best$

the stop spacing, and hence we conclude that both the optimal configuration and operating policies are similar in the fixed and in the elastic models at the demand level equivalent to the captured demand.

For the break-even point we note that when we compare two technologies for a given potential demand y and a given weight ϕ in the objective function, the term $C(\lambda)$ may vary between them only because of different values of λ, the fraction of captured demand. We can reasonably assume that when two technologies break-even their captured demand levels are similar. Thus, the term $C(\lambda)$ can be assumed to be similar between technologies at their break-even point. Hence, the location of the break-even point depends mainly on the other component of (M^*) which reflects the sum of the passenger and operator costs. As a result, the location of the break-even point in the elastic demand model is expected to be close to that of the fixed demand model with respect to the same level of captured demand.

We now derive an approximation of the captured demand levels as a function of ϕ. First observe that the objective function W can be rewritten as

$$W = \phi \frac{y}{2e_p}(1 - e_P \frac{C_{p,Y}}{Y} - e_p P)^2 + y(1 - e_P \frac{C_{p,Y}}{Y} - e_p P)P - C_{o,Y}. \quad (35)$$

For an approximation, we assume that the terms $C_{p,Y}$ and $C_{o,Y}$ do not depend significantly on P, namely, that

$$\partial C_{p,Y}/\partial P = \partial C_{o,Y}/\partial P \approx 0. \quad (36)$$

Thus, the objective function W is twice continuously differentiable with respect to P, and we have

$$\frac{\partial W}{\partial P} = Pye_p(\phi-2)+y(1-e_P\frac{C_{p,Y}}{Y})(1-\phi), \tag{37}$$

$$\frac{\partial^2 W}{\partial P^2} = ye_p(\phi-2)<0. \tag{38}$$

We indicate by P^* the unique value of P which satisfies the necessary conditions for an unconstrained local maximum:

$$P^* = \frac{(1-e_P\frac{C_{p,Y}}{Y})(1-\phi)}{e_p(2-\phi)}. \tag{39}$$

In case P^* is feasible, by observing that $(1-e_P\frac{C_{p,Y}}{Y})$ is equal to $\lambda+e_pP$, we can express P^* as a function of λ:

$$P^* = \frac{\lambda}{e_p}(1-\phi). \tag{40}$$

Hence, P^* is the optimal fare under the assumption (36), and when it satisfies the other constraints. For $\phi=1$, the previous formula returns $P^*=0$, a classic result in the case of social welfare maximization, see e.g. **?**, when the capacity constraint of the vehicles is not binding, and therefore the marginal operator cost of a passenger is zero.

We indicate by $\hat{\lambda}$ the fraction of captured demand when the optimal fare is equal to P^*. By combining equations (39) and (40) it follows that

$$\hat{\lambda} = \frac{1-e_P\frac{C_{p,Y}}{Y}}{2-\phi}. \tag{41}$$

The term $C_{p,Y}/Y$ is the unit passenger cost for a fixed demand Y. The unit passenger cost as a function of the demand level is “U” shaped. At first, the passenger cost decreases because of the lower waiting time due to the higher frequencies induced by higher demand levels. Then, as the maximum frequency is approached, the optimal vehicle load factor increases, and thus the crowding cost penalty prevails, see for example Figure 10 of **?**. The focus of this paper is on technology choice, and the demand range where a technology is competitive is, typically, where the passenger cost is non-increasing. Hence, we can conservatively assume it to be a constant and the following relation can be drawn for the two cases of profit maximization, $\phi = 0$, and social welfare maximization, $\phi = 1$,

$$\hat{\lambda}\Big|_{\phi=0} = \frac{1}{2}\hat{\lambda}\Big|_{\phi=1}. \tag{42}$$

As a result, under the above assumptions, the fraction of captured demand under profit maximization is half of that under social welfare maximization.

3.3. Solving the two-period model

The solution method for the single period model can be generalized to the two-period model. A similar treatment leads to single out an auxiliary problem with fixed demand. The auxiliary problem with fixed demand has the same constraints as in Model IV of **?**, and the following objective function:

$$\text{minimize}\, \chi^p C_{p,Y^p} + \chi^o C_{p,Y^o} + C_{o,Y^p,Y^o} \tag{43}$$

The solution method is hence similar to that of the single period model with the main difference being that the decomposition relies on two variables, namely λ^p for the peak period, and λ^o for the off-peak period.

Computational experiments presented in the following section show the effectiveness of the auxiliary problem (43) in solving the problem. Moreover, we numerically confirm for the two-period model the observations drawn for the single period model with respect to the optimal configuration, operating policies, break-even point between technologies, and differences in the captured demand levels when ϕ varies.

4. Computational results

We use the same case study of **?** and **?** where a bidirectional line of 20 km is patronized by passengers traveling on average 10 km per trip, and accessing the line by walking at a speed of 4 km/h. The critical load is equal to 35% of the total demand. Four technologies are studied, namely buses in mixed traffic lanes (Bus), bus rapid transit (BRT), light rail transit (LRT), and heavy rail (HR). The demand is studied from 3000 to 60000 pax/h with a step of 500 pax/h. The algorithm is implemented in Python 2.7 with the L-BFGS-B solver, a quasi-Newton code for bound-constrained optimization, see **?**. We refer to Section 4.1 of **?** for further details. The introduced models require new parameters for the elastic demand function. We assume e_P equal to 0.07, as in **?**, and Table 1 lists this set of parameters. We report break-even points between technologies, but we emphasize, as discussed in **?**, that there are plausible scenarios and additional issues that could lead to different conclusions. We focus on the introduced two-period model and we compare the results with the fixed demand model of **?**, in the following referred to as Model IV. In Model IV break-even point between BRT and LRT occurs at *circa* 13000 pax/h, and that between LRT and HR occurs at *circa* 20000 pax/h. We first present results obtained with the maximization of social welfare, i.e. with $\phi = 1$, and then those with the maximization of profit, $\phi = 0$. In all cases the algorithm reaches a feasible solution. The starting point $(\tilde{f}^{(p,o)}, \tilde{d})$ derived from the auxiliary fixed demand model is instrumental to this. We report that a random starting point would have failed at obtaining convergence.

Figure 2 illustrates the optimal social welfare averaged with respect to the maximum peak demand. Break-even points appear larger than those of Model IV. However, when the optimal social welfare is scaled to the captured demand, see Figure 3, the break-even points between technologies are very close to those of Model IV. Optimal peak frequency is depicted in Figure 4 and these results are very close to those of Model IV, see Figure 17 of of **?**. Optimal off-peak frequency and stop spacing are similarly close to those of Model IV and then these figures are not presented here for brevity. The ratios of the captured demands to the maximum peak and off-peak demands are reported in Figure 5 and 6, respectively. The optimal fare is always equal to zero.

Figure 7 illustrates the optimal profit averaged with respect to the maximum peak demand. Break-even points appear much larger than those of Model IV. This stems from the significantly lower captured demand under profit maximization. However, as in the case of social welfare maximization, scaling the optimal profit to the captured demand, see Figure 8, leads to break-even points that are very close to those of Model IV. Optimal peak frequency is reported in Figure 9. Comparing Figure 9 with Figure 4 we observe that the optimal frequency is similar at the same demand level, the difference lies in the larger demand levels captured by social welfare maximization. The ratios of the captured demands to the maximum peak and off-peak demands are reported in Figure 10 and 11, respectively. As analytically observed in Section 3.2 for the single period model, these ratios are *circa* half of those yielded by social welfare maximization. Figure 12 reports the optimal fare.

We now test the sensitivity to the fare elasticity parameter. We reduce this parameter of an order of magnitude, i.e. we set it equal to 0.007. The other elastic demand parameters are modified accordingly. In the profit maximization case, this very low sensitivity to fare yields higher fractions of captured demand than in the previous experiment. These fractions are close to 0.5 for all technologies. However, the same effect occur in the social welfare maximization case where the fractions of captured demand are close to one for all technologies. The break-even points between technologies, the optimal configuration, and the operating policies do not change. Thus, we have confirmed numerically for the two-period model the analytic observation derived for the single period model that the model results do not depend on the fare elasticity parameter.

Parameter	Definition	Value
e_a	Elastic demand parameter for access and egress time	0.87
e_P	Elastic demand parameter for fare	0.07
e_v	Elastic demand parameter for in-vehicle time	0.70
e_w	Elastic demand parameter for waiting time	1.05

Table 1: Parameters specific to the two-period elastic model

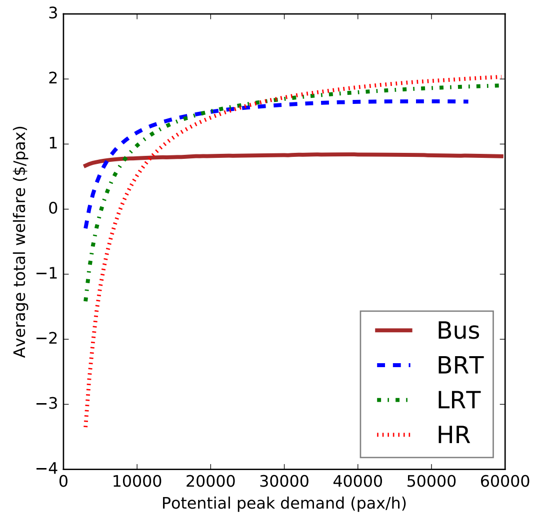

Figure 2: Two-period elastic model with maximization of social welfare, social welfare averaged to the maximum demand at peak hour

5. Conclusions

We have devised an equivalence scheme between fixed and elastic demand models for the single period optimization of a transit line. The maximization of a weighted sum of net passenger welfare and operator profit is equivalent to the minimization of the total cost with fixed demand. The break-even points between technologies under elastic demand are similar to those under fixed demand when accounting for the effective demand captured. The approximated fraction of captured demand under profit maximization is half of that under social welfare maximization. These results do not depend on a specific assumption on the fare elasticity parameter, and are derived for a general setting where the passenger cost is not necessarily decreasing with

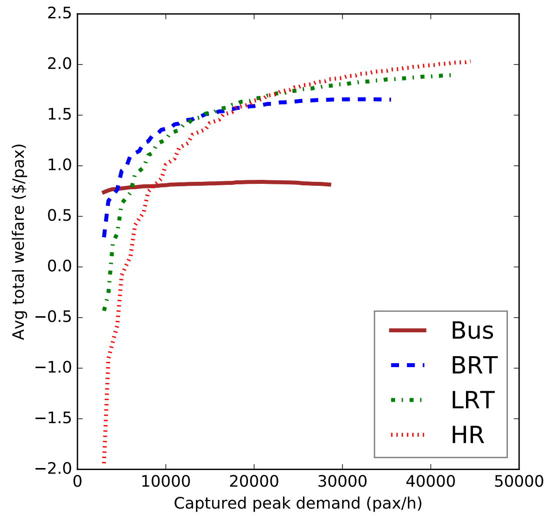

Figure 3: Two-period elastic model with maximization of social welfare, social welfare averaged to the captured demand at peak hour

higher demand levels.

The equivalence scheme derived for the single period model has been extended to the two-period model to obtain a solution method. This method always obtain feasible solutions in our computational experiments, whereas a straightforward implementation would have failed. Finally, we have confirmed numerically for the two-period model the analytic observations drawn for the single period model.

Acknowledgements

This work was partly supported by —- under project —-. This support is gratefully acknowledged.

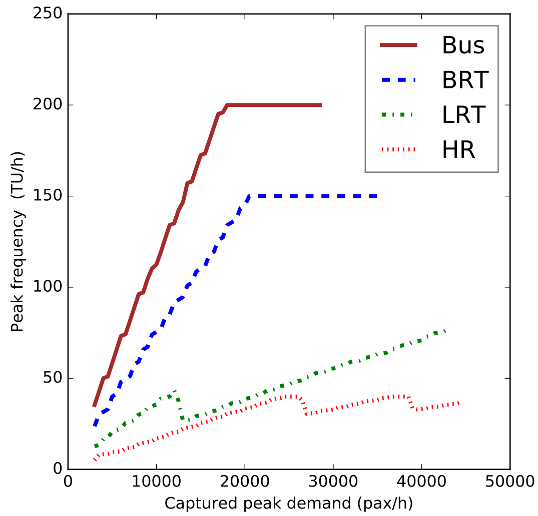

Figure 4: Two-period elastic model with maximization of social welfare, optimal peak frequency

Appendix A. Notational glossary

Table A.2 reports the primary symbols used in this paper. Some other symbols are derived from those listed in this table as explained in the following. The subscripts *min* and *max* specify bounds of a parameter or of a variable. A variable with the tilde symbol is related to an approximation scheme. The superscripts (p, o) refer to the peak or off-peak periods.

Table A.2: List of primary symbols, and units of measure used in the formulae

Symbol	Definition	Unit
$C_{o,Y}$	Operator cost	\$/h

Continued on next page

Table A.2 – *Continued from previous page*

Symbol	Definition	Unit
$C_{p,Y}$	Passenger cost	\$/h
$C_{tot,Y}$	Total cost, sum of passenger and operator costs	\$/h
d	Distance between stops	km
e_a	Elastic demand parameter for access and egress time	-
e_P	Elastic demand parameter for fare	-
e_v	Elastic demand parameter for in-vehicle time	-
e_w	Elastic demand parameter for waiting time	-
f	Frequency	TU/h
L	Transit line length	km
n	Number of vehicles per TU	veh
P	Fare	\$
P_a	Value of the access and egress time	\$/h
P_v	Value of the in-vehicle time	\$/h
P_w	Value of the waiting time	\$/h
t_a	Average access and egress time of a passenger	h
$t_{c,Y}$	Cycle time	h
TU	Transit unit	-
$t_{v,Y}$	Average in-vehicle time of a passenger	h
t_w	Average waiting time of a passenger	h
W	Weighted sum of the net passenger welfare and profit	\$/h
W_o	Net operator welfare, i.e. the profit	\$/h
W_p	Net passenger welfare	\$/h

Continued on next page

Table A.2 – *Continued from previous page*

Symbol	Definition	Unit
y	Maximum bidirectional demand	pax/h
Y	Captured bidirectional demand	pax/h
λ	Ratio of the captured demand to the maximum demand	-
ϕ	Weight of the net passenger welfare	-

References

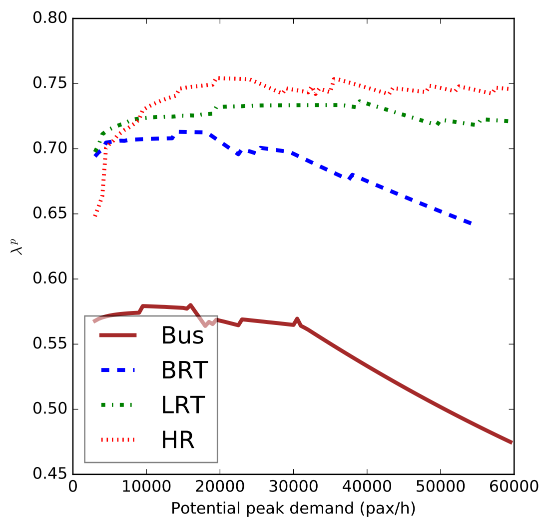

Figure 5: Two-period elastic model with maximization of social welfare, ratio of the captured demand to the maximum peak demand

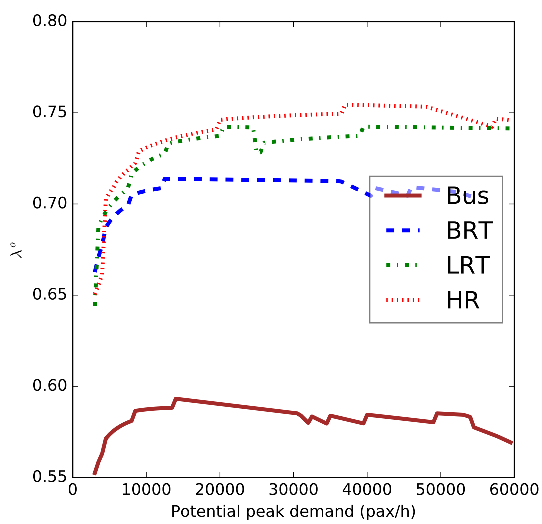

Figure 6: Two-period elastic model with maximization of social welfare, ratio of the captured demand to the maximum off-peak demand

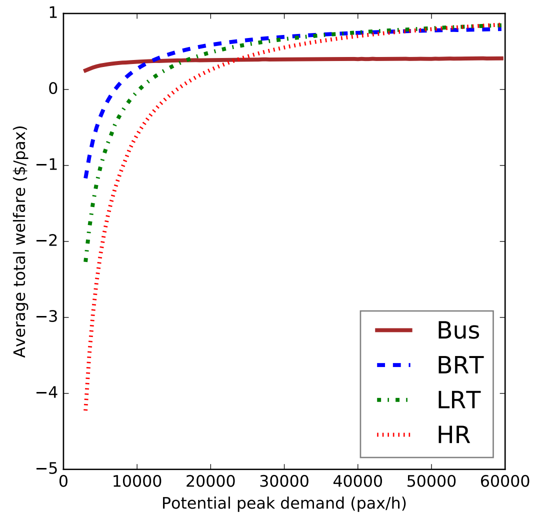

Figure 7: Two-period elastic model with maximization of profit, profit averaged to the maximum demand at peak hour

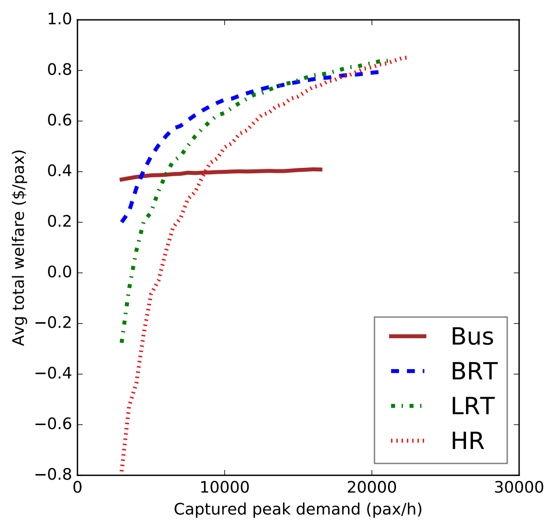

Figure 8: Two-period elastic model with maximization of profit, profit averaged to the captured demand at peak hour

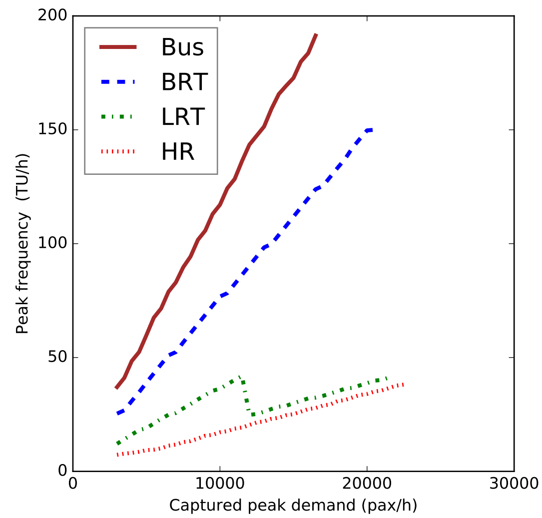

Figure 9: Two-period elastic model with maximization of profit, optimal peak frequency

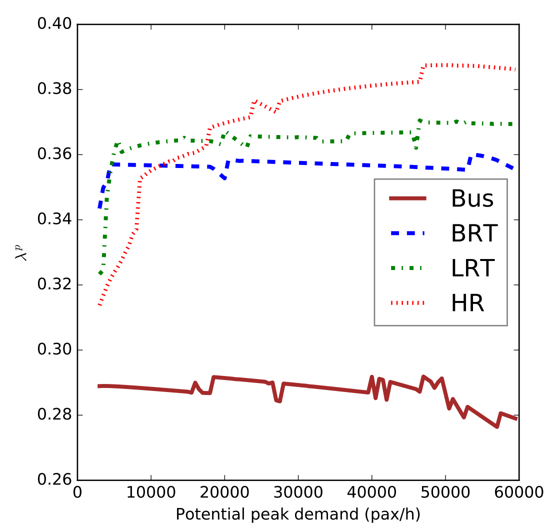

Figure 10: Two-period elastic model with maximization of profit, ratio of the captured demand to the maximum peak demand

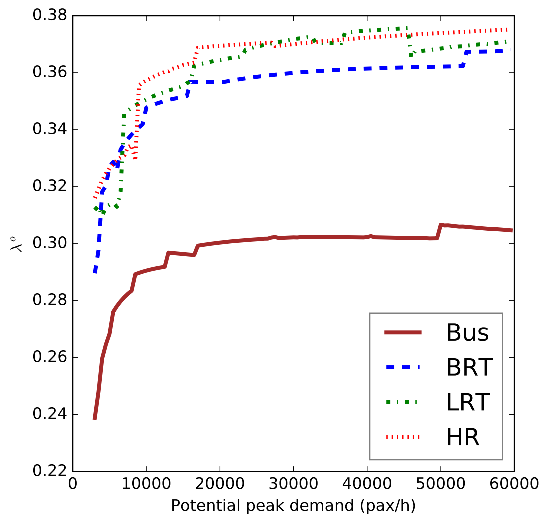

Figure 11: Two-period elastic model with maximization of profit, ratio of the captured demand to the maximum off-peak demand

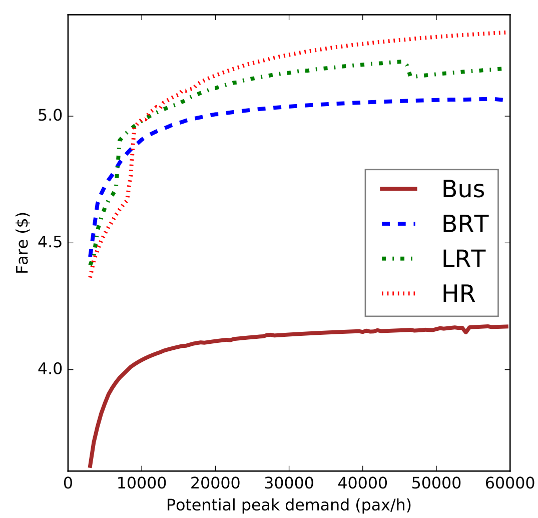

Figure 12: Two-period elastic model with maximization of profit, optimal fare