Homogenization of elliptic problems involving interfaces and singular data

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Abstract

We prove existence and homogenization results for a family of elliptic problems involving interfaces and a singular lower order term. These problems model heat or electrical conduction in composite media.

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1. Introduction

We consider a family of elliptic problems depending on a small parameter $\varepsilon > 0$, which represents the characteristic length of the microstructure underlying the model, and on a parameter $\alpha \geq -1$, taking into account different scalings (and, therefore, different physical properties) appearing in the model, as it will be better explained below. These problems involve also a singular lower order term and represent the Euler-Lagrange equations of energy functionals, which describe the equilibrium for the heat conduction in composite materials with two finely mixed phases having a microscopic periodic structure (for details on the related physical models see, for instance, [17, 18, 23] and the reference quoted therein). The same kind of energies can be also useful to study the electrical conduction in biological tissues (see, for instance, [6]–[9], where the related evolutive problems without singular source are considered). Similar models, in the framework of electrical or thermal conduction in composite materials, are treated in [5, 10, 11].

To be more precise, we assume that the domain $\Omega \subseteq \mathbb{R}^N$, which describes the region occupied by a material, has a periodic microstructure made by two phases Ω_1^ε, Ω_2^ε separated by an active interface Γ^ε; i.e., the physical properties of the interface are relevant in the model (for more details on the geometrical setting, see the next section). The mathematical description of our model in the microscopic setting is given by two non-homogeneous elliptic equations in each phase, complemented with the assumption that the flux of the solution u_ε is continuous across the interface and proportional to the jump of u_ε. Moreover, we assume that in both phases the rate of heat generation is given by a singular source, that is a function which blows up when the solution u_ε becomes small. We consider source terms of the form $\frac{f}{u_\varepsilon^\theta}$, with $0 < \theta < 1$ and $f \in L^{\frac{2}{1+\theta}}(\Omega)$. The restriction on θ is required in order to get suitable a priori estimates, although the source term is singular. In particular, the system we consider is the following

$$
\begin{array}{rll}
-\operatorname{div}(\lambda_1 \nabla u_\varepsilon) & = f/u_\varepsilon^\theta, & \text{in } \Omega_1^\varepsilon; \\
-\operatorname{div}(\lambda_2 \nabla u_\varepsilon) & = f/u_\varepsilon^\theta, & \text{in } \Omega_2^\varepsilon; \\
\lambda_1 \nabla u_\varepsilon \cdot \nu_\varepsilon & = \lambda_2 \nabla u_\varepsilon \cdot \nu_\varepsilon, & \text{on } \Gamma^\varepsilon; \\
\frac{\beta}{\varepsilon^\alpha}[u_\varepsilon] & = \lambda_2 \nabla u_\varepsilon \cdot \nu_\varepsilon, & \text{on } \Gamma^\varepsilon; \\
u_\varepsilon & > 0, & \text{in } \Omega; \\
u_\varepsilon & = 0, & \text{on } \partial\Omega,
\end{array}
$$

where $\lambda_1, \lambda_2, \beta$ are strictly positive constant, $[u_\varepsilon]$ denotes the jump of u_ε across the interface Γ^ε and ν_ε denotes the normal unit vector to Γ^ε pointing into Ω_2^ε.

Our main results concern the study of the homogenization limit (as $\varepsilon \to 0$) of the previous system, focusing our attention on the differences of the limit equations (characterizing the properties of the material from the macroscopic point of view) with respect to the parameter α (appearing in the interface condition). We confine our study to the case $\alpha \geq -1$, where a suitable Poincaré's inequality for general geometries is available. On the contrary, in the other scalings $\alpha < -1$, a uniform bound, with respect to ε, for the L^2-norm of the solution u_ε is missing without further assumptions on the geometry and, therefore, we cannot even assure that a limit does exist.

We point out that different choices of the scaling α keep different physical properties of the interface constant in the homogenization limit $\varepsilon \to 0$. For example, the cases $\alpha = 1$ or $\alpha = -1$ correspond to the cases where the permeability or the total thermal capacity per unit of volume, respectively, are preserved.

In order to get the homogenized problem, we use the *two-scale convergence* technique (see for instance [2, 3, 25]). In particular, we obtain four different behaviours:

$$\alpha > 1, \quad \alpha = 1, \quad \alpha \in (-1, 1), \quad \alpha = -1 \,.$$

In the first three cases, we get in the limit a second order elliptic equation with singular source, whose homogenized matrix is different in each case. Instead, for $\alpha = -1$, we get a bidomain governed by a system of two coupled elliptic equations. Moreover, we remark that, when $\alpha > 1$ or $\alpha \in (-1, 1)$, the homogenized problem loses memory of the physical properties of the interfaces, thus suggesting that the main models are those with $\alpha = \pm 1$.

In order to handle with the singular term in the homogenization procedure, we follow some ideas already present in [17] and in some previous papers (see, in particular, [19]), but our different geometrical setting, on one hand, produces different results (for instance, the resulting homogenized matrices keep memory of the jump across the interface at the microscopic level) and, on the other hand, gives rise to technical difficulties due to the interaction between jumps and singularities. This difficulties can be overcome by means of a new strategy. For instance, in the coupling between the singular source and the testing function, which jumps across the interface, we need a suitable factorization of such a test function, in order to pass to the limit, when ε goes to zero. Moreover, for some scalings, we loose the strong L^2-convergence of the sequence $\{u_\varepsilon\}$, so that we are forced to use some extension results.

Another crucial point is connected to the proof of the positivity of the solution, which is needed in order to guarantee that the singular term is well defined. The positivity of the solution, both for ε fixed and in the homogenization limit, is a non trivial result. In the first case, we need a careful analysis of the sign of the jump (of a suitable approximation) of the solution u_ε across the interface. In the second case, at least when $\alpha = -1$, our geometry does not allow to follow the arguments in [17], but it requires a new idea (see Lemma 5.7). To get this result, we follow an approach appealing to the *unfolding technique* introduced by Cioranescu, Damlamian and Griso in 2002 (see, for instance, [15, 16]).

The paper is organized as follows: in Section 2 we recall notations and preliminary results and we set our problems; in Section 3 we state the necessary estimates for the compactness results; in Section 4 we state and prove our main homogenization theorems. Finally, the paper contains an Appendix divided into two parts: in the first one, we prove the well-posedness of our microscopic problem (11), while in the second one we recall some tools from the unfolding technique and we prove the strict positivity of the homogenized solution for $\alpha = -1$.

2. Preliminaries

2.1. The geometrical setting

For $N \geq 3$, let $\Omega \subset \mathbb{R}^N$ be an open, connected and bounded set. Let E be a periodic open subset of $\mathbb{R}^N$, so that $E + z = E$ for all $z \in \mathbb{Z}^N$. For all $\varepsilon > 0$ we define the two open sets

$$\Omega_1^\varepsilon = \Omega \cap \varepsilon E, \qquad \Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}.$$

We assume that Ω and E have Lipschitz continuous boundary and that Ω_2^ε is connected. We set

$$\Gamma^\varepsilon = \partial\Omega_1^\varepsilon \cap \Omega = \partial\Omega_2^\varepsilon \cap \Omega,$$

so that we have $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. We also employ the notation $Y = (0,1)^N$, and $E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$ and we assume that $|\Gamma \cap \partial Y|_{N-1} = 0$ and that E_2 is connected.

In the following, we will consider two different situations.

- We will name the *connected/disconnected geometry* the case where $\Gamma \cap \partial Y = \emptyset$, and in this case we will assume that $\mathrm{dist}(\Gamma^\varepsilon, \partial\Omega) \geq \gamma_0 \varepsilon$, for a suitable $\gamma_0 > 0$. To this purpose, for each ε, we are ready to remove the inclusions in all the cells which are not completely contained in Ω. In this case, the sets Ω_1^ε and Ω_2^ε are usually called the *inner* and the *outer domain*, respectively (see Figure 1).

- We will name the *connected/connected geometry* the case where E_1, E_2, Ω_1^ε, Ω_2^ε are connected. In this case, we will assume that both ∂E_1 and ∂E_2 have Lipschitz regularity and, moreover, we will need that Ω, E_1 and E_2 are such that $\partial\Omega_1^\varepsilon$ and $\partial\Omega_2^\varepsilon$ are still Lipschitz regular at each

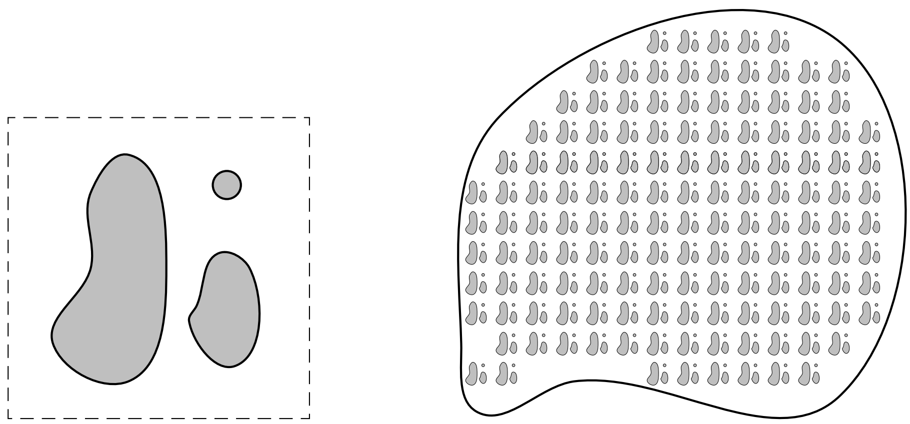

Figure 1: *Left*: the periodic cell Y. E_1 is the shaded region and E_2 is the white region. *Right*: the region Ω.

ε-step, at least for a suitable choice of a subsequence ε_n tending to zero. For instance, this is the case when Ω is a rectangular domain with $\varepsilon_n = |\Omega|/n$; indeed, this choice implies that Ω always contains an integer number of ε-cells. In the following, that regularity assumption will be always implicit; however, we will omit the subindex n, even in the case in which it should be necessary.

We denote by ν_ε the normal unit vector to Γ^ε pointing into Ω_2^ε and by ν the normal unit vector to Γ pointing into E_2.

For a function u defined on Ω, we denote by $u^{(1)}$ and $u^{(2)}$ the restriction of u to Ω_1^ε and Ω_2^ε, respectively. With abuse of notation, we denote by $u^{(1)}$ and $u^{(2)}$ also the trace on Γ^ε of these restrictions, so that we set

$$[u] := u^{(2)} - u^{(1)} \qquad \text{on } \Gamma^\varepsilon.$$

We use the same notation for functions defined in the unit cell Y, where $u^{(1)}$ and $u^{(2)}$ stands here for the restriction (or the trace of the restriction) of u to E_1 and E_2, respectively.

In the following x and y will denote the macro and micro-variable, respectively, so that, for a function $u(x,y)$ defined on $\Omega \times Y$, we denote by $\nabla_x u$, $\nabla_y u$ and $\operatorname{div}_x u$, $\operatorname{div}_y u$ the gradient and the divergence of u computed with respect to the variables x and y, respectively. When no confusion is possible, we write ∇u for $\nabla_x u$ and $\operatorname{div} u$ for $\operatorname{div}_x u$.

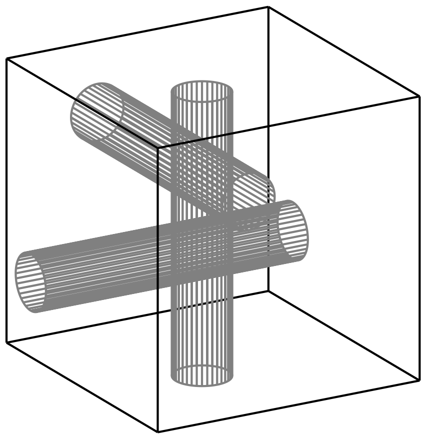

Figure 2: The periodic cell Y. E_1 is the shaded region and E_2 is the white region.

Given $\xi, \eta \in \mathbb{R}^N$, $\xi \otimes \eta$ will denote the matrix whose entries are $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_N$ the euclidian basis of $\mathbb{R}^N$. In the sequel, C will denote a positive constant, which may vary from line to line.

2.2. Functional spaces

We set

$$V_0^\varepsilon(\Omega) = \{u : \Omega \to \mathbb{R} \mid u^{(1)} \in H^1(\Omega_1^\varepsilon),\, u^{(2)} \in H^1(\Omega_2^\varepsilon),\, u = 0 \text{ on } \partial\Omega\},$$

and

$$\mathfrak{L}_0^\varepsilon(\Omega) = \{u : \Omega \to \mathbb{R} \mid u^{(1)} \in \mathrm{Lip}(\overline{\Omega_1^\varepsilon}), u^{(2)} \in \mathrm{Lip}(\overline{\Omega_2^\varepsilon}),\, u = 0 \text{ on } \partial\Omega\}.$$

Analogously, we define the following space

$$V_\#(Y) = \{v : Y \to \mathbb{R} \mid v \text{ is } Y\text{-periodic},\, v^{(1)} \in H^1_\#(E_1),\, v^{(2)} \in H^1_\#(E_2)\},$$

and

$$\mathfrak{L}_\#(Y) = \{v : Y \to \mathbb{R} \mid v \text{ is } Y\text{-periodic},\, v^{(1)} \in \mathrm{Lip}(\overline{E_1}), v^{(2)} \in \mathrm{Lip}(\overline{E_2})\}.$$

Here Y is identified with the flat torus in $\mathbb{R}^N$, so that, for every subset E of the flat torus Y, $H^1_\#(E)$ corresponds to the space of the H^1-functions $v : E \to \mathbb{R}$, such that v and ∇v coincide on opposite sides of $\partial E \cap \partial Y$.

Remark 2.1. *Notice that, if $u \in V_0^\varepsilon(\Omega)$, then $[u] \in L^2(\Gamma^\varepsilon)$ and, analogously, if $v \in V_\#(Y)$, then $[v] \in L^2(\Gamma)$.*

We recall the following Poincaré's inequality.

Theorem 2.2. *There exists $C > 0$, independent of ε, such that*

$$\int_\Omega v^2 \,\mathrm{d}x \le C \left\{ \int_\Omega |\nabla v|^2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \right\} \qquad \forall v \in V_0^\varepsilon(\Omega). \tag{1}$$

Remark 2.3. *Notice that, in presence of interfaces, the general Poincaré's inequality is given by* (1)*, where the factor ε in front of the integral over Γ^ε is replaced by ε^{-1} (see [6, Lemma 7.1]). However, since we have assumed that Ω_2^ε is connected, as stated in [9, Lemma 3.3], it is possible to prove the stronger version* (1) *of Poincaré's inequality. Indeed, in the connected/disconnected case it follows by [22, Lemma 6], complemented with [22, Lemma 4], while in the connected/connected case, since Ω_2^ε and Ω_1^ε are interchangeable, it follows by [4, Lemma A.4], applied to both sets.*

We also recall the following technical lemma proved in [2, Lemma 2.10], which will be useful in the sequel.

Lemma 2.4. *For any vector function $\Phi \in L^2(\Omega; \mathbb{R}^N)$, there exists a vector function $\Psi \in L^2(\Omega; H^1_\#(E_2; \mathbb{R}^N))$ such that*

$$\begin{aligned} &\operatorname{div}_y \Psi(x,y) = 0\,, && \text{in } E_2\,; \\ &\Psi(x,y) = 0\,, && \text{on } \Gamma\,; \\ &\int_{E_2} \Psi(x,y)\,\mathrm{d}y = \Phi(x)\,. && \end{aligned} \tag{2}$$

Moreover, $||\Psi||_{L^2(\Omega; H^1_\#(E_2;\mathbb{R}^N))} \le C ||\Phi||_{L^2(\Omega;\mathbb{R}^N)}$.

Clearly, in the connected/connected case, an analogous result holds with E_2 replaced by E_1.

2.3. Two-scale convergence

We recall some basic definitions and properties of the *two-scale convergence* technique. For more details see, for instance, [2, 3, 9, 25].

Definition 2.5. *A function $\varphi \in L^2(\Omega \times Y)$ is said an admissible test function if φ is Y-periodic with respect to the second variable and satisfies:*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi^2 \left(x, \frac{x}{\varepsilon} \right) \mathrm{d}x = \int_{\Omega \times Y} \varphi^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Remark 2.6. *If $\varphi \in \mathcal{C}^0(\overline{\Omega}; \mathcal{C}^0_\#(Y))$ or, more in general, if $\varphi \in L^2(\Omega; \mathcal{C}^0_\#(Y))$ or $\varphi \in L^2_\#(Y; \mathcal{C}^0(\overline{\Omega}))$, then φ is an admissible test function. Moreover, if $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$ with $\varphi_1 \in L^p(\Omega)$ and $\varphi_2 \in L^q_\#(Y)$, $1/p + 1/q = 1/2$, then φ is an admissible test function.*

Definition 2.7 (Two-scale convergence)**.** *For $\{u_\varepsilon\} \subset L^2(\Omega)$ and $u_0 \in L^2(\Omega \times Y)$, we say that $\{u_\varepsilon\}$ two-scale converges to u_0 in $L^2(\Omega \times Y)$ as $\varepsilon \to 0$ (and we write $u_\varepsilon \xrightarrow{2-sc} u_0$) if*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_{\Omega \times Y} u_0(x,y) \varphi(x,y) \, \mathrm{d}x \, \mathrm{d}y \,,$$

for every admissible test function φ.

Definition 2.8 (Two-scale convergence on surfaces)**.** *For $\{w_\varepsilon\} \subset L^2(\Gamma^\varepsilon)$ and $w_0 \in L^2(\Omega \times \Gamma)$, we say that $\{w_\varepsilon\}$ two-scale converges to w_0 in $L^2(\Omega \times \Gamma)$ as $\varepsilon \to 0$ (and, as above, we use the notation $w_\varepsilon \xrightarrow{2-sc} w_0$) if*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon} w_\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}\sigma = \int_{\Omega \times \Gamma} w_0(x,y) \varphi(x,y) \, \mathrm{d}x \ \mathrm{d}\sigma(y) \,,$$

for every $\varphi \in \mathcal{C}^0(\overline{\Omega}; \mathcal{C}^0_\#(Y))$.

Theorem 2.9. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then there exist a subsequence of $\{u_\varepsilon\}$ (still denoted by $\{u_\varepsilon\}$) and a function $u_0 \in L^2(\Omega \times Y)$ such that $u_\varepsilon \xrightarrow{2-sc} u_0$ in $L^2(\Omega \times Y)$.*

Proposition 2.10. *Let $\{u_\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$, which two-scale converges to a limit $u_0(x,y) \in L^2(\Omega \times Y)$. Then, u_ε converges weakly to $u(x) = \int_Y u_0(x,y) dy$ in $L^2(\Omega)$. Furthermore, we have*

$$\liminf_{\varepsilon \to 0} ||u_\varepsilon||_{L^2(\Omega)} \geq ||u_0||_{L^2(\Omega \times Y)} \geq ||u||_{L^2(\Omega)}.$$

Theorem 2.11. *Let $\{w_\varepsilon\} \subset L^2(\Gamma^\varepsilon)$. Assume that there exists $C > 0$, independent of ε, such that*

$$\varepsilon \int_{\Gamma^\varepsilon} |w_\varepsilon|^2 \, \mathrm{d}\sigma \leq C \,, \qquad \forall \varepsilon > 0.$$

Then, there exist a subsequence of $\{w_\varepsilon\}$ (still denoted by $\{w_\varepsilon\}$) and a function $w_0 \in L^2(\Omega \times \Gamma)$ such that $w_\varepsilon \xrightarrow{2-sc} w_0$ in $L^2(\Omega \times \Gamma)$.

Theorem 2.12. *Let* $\{u_\varepsilon\} \subset V_0^\varepsilon(\Omega)$*. Assume that there exists* $C > 0$ *(independent of* ε*) such that*

$$\int_\Omega |u_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x \le C\,, \qquad \forall \varepsilon > 0. \tag{3}$$

Then, there exists $u \in L^2(\Omega; V_\#(Y))$*, whose restrictions to* E_1 *and* E_2 *satisfy*

$$u(x,y) = u^{(1)}(x) \in H^1(\Omega)\,, \quad \textit{in } E_1\,, \qquad u(x,y) = u^{(2)}(x) \in H^1(\Omega)\,, \quad \textit{in } E_2\,;$$

and there exists $u_1 \in L^2(\Omega; V_\#(Y))$ *such that, up to subsequence, as* $\varepsilon \to 0$ *we have*

$$\chi_{\Omega_1^\varepsilon} u_\varepsilon^{(1)} \xrightarrow{2-sc} \chi_{E_1} u^{(1)} \quad \textit{and} \quad \chi_{\Omega_2^\varepsilon} u_\varepsilon^{(2)} \xrightarrow{2-sc} \chi_{E_2} u^{(2)}\,, \qquad \textit{in } L^2(\Omega \times Y)\,; \tag{4}$$

$$\chi_{\Omega_1^\varepsilon} \nabla u_\varepsilon^{(1)} \xrightarrow{2-sc} \chi_{E_1} \left(\nabla u^{(1)} + \nabla_y u_1^{(1)} \right)\,, \qquad \textit{in } L^2(\Omega \times Y)\,; \tag{5}$$

$$\chi_{\Omega_2^\varepsilon} \nabla u_\varepsilon^{(2)} \xrightarrow{2-sc} \chi_{E_2} \left(\nabla u^{(2)} + \nabla_y u_1^{(2)} \right)\,, \qquad \textit{in } L^2(\Omega \times Y)\,. \tag{6}$$

where, for $\mathcal{O} \subseteq \mathbb{R}^N$*,* $\chi_{\mathcal{O}}$ *denotes the characteristic function of* $\mathcal{O}$*. Moreover, we have also*

$$\varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma \le 2\varepsilon \int_{\Gamma^\varepsilon} \left(|u_\varepsilon^{(1)}|^2 + |u_\varepsilon^{(2)}|^2 \right) \,\mathrm{d}\sigma \le C\,, \qquad \forall \varepsilon > 0\,, \tag{7}$$

with C *independent of* ε*, and*

$$[u_\varepsilon] \xrightarrow{2-sc} [u]\,, \qquad \textit{in } L^2(\Omega \times \Gamma)\,. \tag{8}$$

Proof. The convergence stated in (4)–(6) can be obtained following the outlines of the first part of the proof in [2, Theorem 2.9], which can be applied both in Ω_1^ε and Ω_2^ε.
Inequality (7) follows by rescaling and summation over the ε-cells of Ω of the trace inequality in the unit cell Y and taking into account (3).
Finally, (8) can be obtained by (4)–(7) and Theorem 2.11, following the ideas in [3, Proposition 2.6], as done in [9, Proof of Theorem 4.12], where the time-dependent case is treated. □

2.4. Extension result

In this subsection, we recall an extension result which will be used in the proof of Theorems 4.7 and 4.10. This result permits to extend a function from the connected set Ω_2^ε to Ω, without any assumption on the connection of the set Ω_1^ε. Actually, when we are in the connected/disconnected geometry, we could apply a more classical extension theorem due to Tartar (see [14, 26]), but this is not the case in the connected/connected geometry.

We state below the version proposed in [24, Lemma 1] of the original extension result due to Acerbi-Chiadò Piat-Dal Maso-Percivale (see [1, Theorem 2.1]); to this purpose, let us define

$$V_{2,0}^\varepsilon = \{w \in H^1(\Omega_2^\varepsilon) \ : \ w\,|_{\partial\Omega\cap\partial\Omega_2^\varepsilon} = 0\}\,.$$

Theorem 2.13. *For every $\varepsilon > 0$, there exist a continuous linear operator $T_\varepsilon^2 : V_{2,0}^\varepsilon \to H^1(\Omega)$ and a constant $C > 0$ (independent on ε) such that $T_\varepsilon^2 w = w$ a.e. in Ω_2^ε and*

$$\|T_\varepsilon^2 w\|_{L^2(\Omega)} \le C\|w\|_{L^2(\Omega_2^\varepsilon)}\,, \tag{9}$$

$$\|\nabla T_\varepsilon^2 w\|_{L^2(\Omega)} \le C\|\nabla w\|_{L^2(\Omega_2^\varepsilon)}\,. \tag{10}$$

Notice that, in the connected/connected case, where the role of Ω_2^ε and Ω_1^ε can be interchanged, the previous theorem can be applied also to extend from Ω_1^ε into Ω, defining an operator T_ε^1, in an analogous way as done for T_ε^2.

2.5. Statement of the problem

Let $\lambda_1, \lambda_2, \beta$ be positive constants and $\theta \in (0,1)$. In the following, we will assume that $f \in L^{\frac{2}{1+\theta}}(\Omega)$ is a nonnegative function a.e. in Ω, not identically equal to zero both in Ω_1^ε and in Ω_2^ε, for every $\varepsilon > 0$. Let us define the functions $\lambda_\varepsilon : \Omega \to \mathbb{R}$ and $\lambda : Y \to \mathbb{R}$ as

$$\lambda_\varepsilon(x) = \begin{cases} \lambda_1, & \text{if } x \in \Omega_1^\varepsilon \\ \lambda_2, & \text{if } x \in \Omega_2^\varepsilon \end{cases} \qquad \text{and} \qquad \lambda(y) = \begin{cases} \lambda_1, & \text{if } y \in E_1 \\ \lambda_2, & \text{if } y \in E_2 \end{cases}$$

and set $\lambda_0 = \lambda_1|E_1| + \lambda_2|E_2|$. For $\alpha \ge -1$, we consider the problem

$$\begin{aligned} -\operatorname{div}(\lambda_\varepsilon \nabla u_\varepsilon) &= \frac{f}{u_\varepsilon^\theta}, && \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \\ [\lambda_\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] &= 0, && \text{on } \Gamma^\varepsilon; \\ \frac{\beta}{\varepsilon^\alpha}[u_\varepsilon] &= \lambda_2 \nabla u_\varepsilon^{(2)} \cdot \nu_\varepsilon, && \text{on } \Gamma^\varepsilon; \\ u_\varepsilon &> 0, && \text{in } \Omega; \\ u_\varepsilon &= 0, && \text{on } \partial\Omega\,. \end{aligned} \tag{11}$$

Definition 2.14. *We say that* $u \in V_0^\varepsilon(\Omega)$ *is a weak solution of* (11) *if* $u_\varepsilon > 0$ *a.e. in* Ω *and it satisfies*

$$\left| \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \right| < +\infty \,, \tag{12}$$

$$\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla \psi \, \mathrm{d}x + \frac{\beta}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon][\psi] \, \mathrm{d}\sigma = \int_\Omega \frac{f}{u_\varepsilon^\theta} \psi \, \mathrm{d}x \tag{13}$$

for every $\psi \in V_0^\varepsilon(\Omega)$.

Remark 2.15. *Note that the assumption* (12) *is indeed contained in* (13), *since it is a consequence of the finiteness of the left-hand side of* (13)*; nevertheless, we prefer to require it explicitly, being crucial in the proof of existence and homogenization results. Moreover, taking into account that* u_ε *and* f *are positive and recalling the decomposition* $\psi = \psi^+ - \psi^-$, (12) *can be rewritten for* $\psi > 0$ *and without the absolute value, or even in the apparently stronger form*

$$\int_\Omega \frac{f}{u_\varepsilon^\theta} |\psi| \, \mathrm{d}x < +\infty \,.$$

We will prove in the Appendix (Theorem 5.1) that, for every $\varepsilon > 0$ fixed, the problem (11) admits a unique solution $u_\varepsilon \in V_0^\varepsilon(\Omega)$.

Notice that, for the sake of simplicity, in the problem (11) we have considered only the model case, where the singular term is given by $\frac{f(x)}{s^\theta}$; however, all the proofs also work if we take into account a more general singularity of the form $f(x) \cdot g(s)$, with a non increasing function g such that $0 \leq g(s) \leq \frac{1}{s^\theta}$.

3. Estimates

The aim of this section is to prove that the solution u_ε satisfies some uniform (with respect to ε) estimates.

Proposition 3.1. *Let* u_ε *be the weak solution of problem* (11)*. Then there exists* $C > 0$*, independent of* ε *(and* α*), such that*

$$\int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \, \mathrm{d}\sigma \leq C \|f\|_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}} \,, \qquad \forall \varepsilon > 0 \,, \tag{14}$$

$$\int_\Omega u_\varepsilon^2 \, \mathrm{d}x \leq C ||f||_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{2}{1+\theta}} \,, \qquad \forall \varepsilon > 0 \,. \tag{15}$$

Proof. Taking $\psi = u_\varepsilon$ in (13), we get

$$\int_\Omega \lambda_\varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \frac{\beta}{\varepsilon^\alpha}\int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma = \int_\Omega f u_\varepsilon^{1-\theta} \,\mathrm{d}x \le ||f||_{L^{\frac{2}{1+\theta}}(\Omega)} ||u_\varepsilon||^{1-\theta}_{L^2(\Omega)}. \quad (16)$$

By Theorem 2.2, it follows

$$\begin{aligned} ||u_\varepsilon||^{1-\theta}_{L^2(\Omega)} &\le C\left[\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\right]^{\frac{1-\theta}{2}} \quad (17)\\ &\le C\left[\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x + \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\right]^{\frac{1-\theta}{2}}, \end{aligned}$$

where the last inequality is due to the fact that $\alpha \ge -1$. Hence, (14) follows by (16) and (17), and by (17) and (14), we get (15). □

Proposition 3.2. *Let u_ε be the weak solution of problem* (11). *Then, for every $\psi \in H_0^1(\Omega)$, we have*

$$\left|\int_\Omega \frac{f}{u_\varepsilon^\theta}\psi(x)dx\right| \le C||\nabla\psi||_{L^2(\Omega)}||\nabla u_\varepsilon||_{L^2(\Omega)} \quad (18)$$

with $C = \max(\lambda_1, \lambda_2)$.

Proof. Taking in (13) a testing function $\psi \in H_0^1(\Omega)$ (i.e., $[\psi] = 0$), and applying Holder's inequality, we find that

$$\begin{aligned} \left|\int_\Omega \frac{f}{u_\varepsilon^\theta}\psi \,\mathrm{d}x\right| &= \left|\int_\Omega \lambda\nabla u_\varepsilon \cdot \nabla\psi \,\mathrm{d}x\right| \\ &\le \max(\lambda_1, \lambda_2)\left(\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla\psi|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned}$$

□

4. Homogenization

4.1. The case $\alpha = 1$

In this subsection we will assume to be in anyone of the geometrical settings described in Section 2. We will see that the homogenized problem will depend on the physical properties of the bulk regions (i.e., λ_1, λ_2) as well as the physical properties of the interfaces (i.e. β).

Theorem 4.1. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (11)*. Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; V_\#(Y))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ a.e. in Ω, such that, as $\varepsilon \to 0$, we have*

$$u_\varepsilon \to u\,, \qquad \text{strongly in } L^2(\Omega)\,; \tag{19}$$

$$u_\varepsilon \xrightarrow{2-sc} u\,, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{20}$$

$$\chi_{\Omega\setminus\Gamma^\varepsilon} \nabla u_\varepsilon \xrightarrow{2-sc} \nabla u + \nabla_y u_1\,, \qquad \text{in } L^2(\Omega \times Y)\ : \tag{21}$$

$$\frac{1}{\varepsilon}[u_\varepsilon] \xrightarrow{2-sc} [u_1]\,, \qquad \text{in } L^2(\Omega; L^2(\Gamma))\,. \tag{22}$$

Moreover,

$$\left| \int_\Omega \frac{f}{u^\theta} \varphi \,\mathrm{d}x \right| < +\infty\,, \qquad \forall \varphi \in H_0^1(\Omega)\,, \tag{23}$$

and the pair (u, u_1) solves

$$-\operatorname{div}\left(\lambda_0 \nabla u + \int_Y \lambda \nabla_y u_1 \,\mathrm{d}y \right) = \frac{f}{u^\theta}\,, \qquad \text{in } \Omega\,; \tag{24}$$

$$-\operatorname{div}_y \left(\lambda(\nabla u + \nabla_y u_1) \right) = 0\,, \qquad \text{in } \Omega \times (E_1 \cup E_2)\,; \tag{25}$$

$$[\lambda(\nabla u + \nabla_y u_1) \cdot \nu] = 0\,, \qquad \text{on } \Omega \times \Gamma\,; \tag{26}$$

$$\beta[u_1] = \lambda_2(\nabla u + \nabla_y u_1) \cdot \nu\,, \qquad \text{on } \Omega \times \Gamma\,; \tag{27}$$

$$u > 0\,, \qquad \text{in } \Omega\,; \tag{28}$$

$$u = 0\,, \qquad \text{on } \partial\Omega\,, \tag{29}$$

where λ_0 and λ are defined at the beginning of Subsection 2.5.

Remark 4.2. *Notice that the problem* (24)*–*(29) *admits at most one pair of solutions (u, u_1). Indeed, assume by contradiction that (u^i, u_1^i), for $i = 1, 2$ are two pairs of solutions and denote by $U = u^1 - u^2$ and $U_1 = u_1^1 - u_1^2$. Using U as test function in* (24) *written for u^1 and U_1 as test function in* (25) *written for u_1^1, adding the two equations, integrating by parts and using* (26)*–*(27)*, we get*

$$\int_\Omega\!\int_Y \lambda(\nabla u^1 + \nabla_y u_1^1) \cdot \nabla U \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\!\int_Y \lambda(\nabla u^1 + \nabla_y u_1^1) \cdot \nabla_y U_1 \,\mathrm{d}x\,\mathrm{d}y \\ + \beta \int_\Omega\!\int_\Gamma [u_1^1][U_1] \,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega \frac{f}{(u^1)^\theta} U \,\mathrm{d}x\,.$$

Repeating the same computation for (u^2, u_1^2) *and subtracting the equation for* (u^1, u_1^1) *from the equation for* (u^2, u_1^2)*, it follows*

$$\begin{aligned}&\int_\Omega\int_Y \lambda|\nabla U + \nabla_y U_1|^2 \,\mathrm{d}x\,\mathrm{d}y \\ + \;\; &\beta\int_\Omega\int_\Gamma [U_1]^2 \,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega f\left(\frac{1}{(u^1)^\theta} - \frac{1}{(u^2)^\theta}\right)(u^1 - u^2)\,\mathrm{d}x\,.\end{aligned}$$

Taking into account that the function $s \in (0, +\infty) \mapsto \frac{1}{s^\theta}$ *is decreasing, it follows that the right-hand side in the last equality is non positive, which implies* $[U_1] = 0$*. Moreover,*

$$\begin{aligned}\int_\Omega |\nabla U|^2 \,\mathrm{d}x + \int_\Omega\int_Y |\nabla_y U_1|^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega |\nabla U|^2 \,\mathrm{d}x + \int_\Omega\int_Y |\nabla_y U_1|^2 \,\mathrm{d}x\,\mathrm{d}y \\ + 2\int_\Omega \nabla u\cdot\left(\int_Y \nabla_y U_1 \,\mathrm{d}y\right)\mathrm{d}x &= \int_\Omega\int_Y |\nabla U + \nabla_y U_1|^2 \,\mathrm{d}x\,\mathrm{d}y = 0\,,\end{aligned}$$

where we have taken into account that $\int_Y \nabla_y U_1 \,\mathrm{d}y = 0$*, because of the* Y*-periodicity of* U_1 *and the fact that* $[U_1] = 0$*. Thus,* $\nabla U = \nabla_y U_1 = 0$*, which implies* $U = 0$ *in* Ω*, since it satisfies the homogeneous boundary condition, and* $U_1 = 0$*, since it has null mean average on* Y*.*

Finally, we also point out that, for any given $u \in H_0^1(\Omega)$*, reasoning in a similar (and even simpler) way as above, the problem* (25)–(27) *for the unknown* u_1*, admits a unique solution with null mean average on* Y *belonging to* $L^2(\Omega; V_\#(Y))$ *(clearly, depending on the given function* u*).*

Remark 4.3. *As usual in homogenization problems, when* (25)–(27) *perform a linear dependence of* u_1 *on* ∇u*, the function* u_1 *can be factorized in terms of* ∇u *(see, for instance, [12]). This leads to decouple the two-scale system resulting from the homogenization procedure and to characterize the limit function* u *by means of a single homogenized equation. In our case,* u_1 *can be written as*

$$u_1(x, y) = \chi(y)\cdot\nabla u(x), \tag{30}$$

where $\chi = (\chi_1, \dots, \chi_N)$ *and* $\chi_j \in V_\#(Y)$ *with* $\int_Y \chi_j \,\mathrm{d}y = 0$*, for each* $j = 1, \dots, N$*, satisfying*

$$\begin{aligned}&-\mathrm{div}_y(\lambda(\nabla_y\chi_j + \mathbf{e}_j)) = 0\,, &&\text{in } E_1\cup E_2\,;\\ &[\lambda(\nabla_y\chi_j + \mathbf{e}_j)\cdot\nu] = 0\,, &&\text{on } \Gamma\,;\\ &\beta\,[\chi_j] = \lambda_2(\nabla_y\chi_j + \mathbf{e}_j)\cdot\nu\,, &&\text{on } \Gamma\,.\end{aligned}\tag{31}$$

By [22, Theorem 2] the previous problem (31) *admits a unique solution with null mean average on* Y*. Replacing* (30) *in* (24)*, it follows that* u *solves*

$$\begin{aligned} & -\operatorname{div}(A_{hom}\nabla u) = \frac{f}{u^\theta}\,, && \text{in } \Omega\,; \\ & u > 0\,, && \text{in } \Omega\,, \\ & u = 0\,, && \text{on } \partial\Omega\,, \end{aligned} \tag{32}$$

where the matrix A_{hom} *is defined as*

$$A_{hom} = \lambda_0 I + \int_Y \lambda (\nabla_y \chi)^T \, \mathrm{d}y. \tag{33}$$

Here, $\nabla_y \chi$ *is the matrix whose entries are* $(\nabla_y \chi)_{ij} = \frac{\partial \chi_i}{\partial y_j}$ *and* $(\nabla_y \chi)^T$ *denotes its transposed. Therefore, we have*

$$\begin{aligned} \left(\int_Y \lambda (\nabla_y \chi)^T \, \mathrm{d}y \right)_{ij} &= \int_Y \lambda \frac{\partial \chi_j}{\partial y_i} \, \mathrm{d}y = \int_{E_1} \lambda_1 \frac{\partial \chi_j}{\partial y_i} \, \mathrm{d}y + \int_{E_2} \lambda_2 \frac{\partial \chi_j}{\partial y_i} \, \mathrm{d}y \\ &= \int_\Gamma \lambda_1 \chi_j \nu_i \, \mathrm{d}\sigma - \int_\Gamma \lambda_2 \chi_j \nu_i \, \mathrm{d}\sigma = - \int_\Gamma [\lambda \chi_j] \nu_i \, \mathrm{d}\sigma \end{aligned}$$

and hence we may write

$$A_{hom} = \lambda_0 I + \int_Y \lambda (\nabla_y \chi)^T \, \mathrm{d}y = \lambda_0 I - \int_\Gamma \nu \otimes [\lambda \chi] \, \mathrm{d}\sigma. \tag{34}$$

Notice that the factorization (30) *is unique. Indeed, as we have recalled above, there exists a unique solution* χ_j*,* $j = 1, \dots, N$*, with null mean average on* Y*, of the problem* (31)*. Moreover, the homogenized matrix* A_{hom} *is symmetric and positive definite, as proved in [22, end of Section 3.1]. Therefore, by [13, Theorem 5.2 and Remark 5.4] we obtain the existence and uniqueness of a solution of* (32)*.*

As a consequence of Remarks 4.2 and 4.3, we get that the homogenized problem (24)–(29) admits a unique solution and that such a solution can be uniquely factorized as in (30). More precisely, a pair (u, u_1) is the solution of (24)–(29) if and only if it coincides with the pair $(u, \chi \cdot \nabla u)$, where χ satisfies (31) and u is the unique solution of (32). Moreover, the function u_1 defined in (30) is also the unique solution of (25)–(27), once $u \in H_0^1(\Omega)$ is given.

Proof of Theorem 4.1. By Proposition 3.1 and [21, Proposition 5.5] we get that, up to a subsequence, (19)–(22) hold. Hence, taking into account (14) and (19) and passing to the limit in (18), when $\varepsilon \to 0$, by Fatou's Lemma we get (23) which also implies that u is not identically zero in Ω.

In order to get the limit problem, we have to pass to the two-scale limit in (13), with $\alpha = 1$. The main steps are to choose suitable testing functions to handle with the right-hand side of (13) and to provide the factorization of the function u_1. In particular, the factorization will be crucial, from one side, to obtain that the integral of the singular term close to the singular set $\{u = 0\}$ tends to zero in the limit and, from the other side, to apply the maximum principle to achieve that the limit function u is strictly positive a.e. in Ω.

Choosing $\psi(x) = \varphi(x) + \varepsilon\Phi\left(x, \frac{x}{\varepsilon}\right)$ with $\varphi \in \mathcal{C}^1_c(\Omega)$ and $\Phi \in \mathcal{C}^1_c(\Omega; \mathfrak{L}_\#(Y))$, we get

$$\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x \\ + \beta \int_{\Gamma^\varepsilon} [u_\varepsilon][\Phi] \,\mathrm{d}\sigma = \int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \,\mathrm{d}x + \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \Phi \,\mathrm{d}x. \quad (35)$$

By (21) and (22), as $\varepsilon \to 0$, the left-hand side of (35) converges to

$$\int_\Omega\int_Y \lambda(\nabla u + \nabla_y u_1) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega\int_Y \lambda(\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi \,\mathrm{d}x\,\mathrm{d}y \\ + \beta \int_\Omega\int_\Gamma [u_1][\Phi] \,\mathrm{d}x\,\mathrm{d}\sigma(y). \quad (36)$$

We now focus our attention on the right-hand side of (35) and we set

$$I_\varepsilon := \int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \,\mathrm{d}x\,, \qquad J_\varepsilon := \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \Phi \,\mathrm{d}x\,. \quad (37)$$

In order to deal with the term J_ε, we rewrite the test function $\Phi\left(x, \frac{x}{\epsilon}\right) = \varphi_1(x)\varphi_2\left(\frac{x}{\epsilon}\right)$; moreover, by the decomposition $\varphi_1 = \varphi_1^+ - \varphi_1^-$ and $\varphi_2 = \varphi_2^+ - \varphi_2^-$, we can assume $\varphi_1, \varphi_2 \geq 0$ (notice that the Lipschitz continuity of φ_1 is enough for our purposes). We have that

$$0 \leq J_\varepsilon = \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi_1(x)\varphi_2\left(\frac{x}{\varepsilon}\right) \,\mathrm{d}x \leq \varepsilon ||\varphi_2||_{L^\infty(Y)} \int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi_1(x) \,\mathrm{d}x \\ \leq C\varepsilon ||\varphi_2||_{L^\infty(Y)} ||\nabla\varphi_1||_{L^2(\Omega)} ||f||^{\frac{1}{1+\theta}}_{L^{\frac{2}{1+\theta}}(\Omega)}, \quad (38)$$

where we used (18) and (14). Since C is independent of ε, as $\varepsilon \to 0$, also $J_\varepsilon \to 0$. In order to study the limit of I_ε, having in mind the decomposition $\varphi = \varphi^+ - \varphi^-$ (notice again that the Lipschitz continuuity of φ is enough for our purposes), we may assume $\varphi \geq 0$. Moreover, we have to split the behaviour of the singular term into the part near to and far away from the singularity. To this purpose, we write

$$I_\varepsilon = \int_{\Omega\cap\{0<u_\varepsilon\leq\delta\}} \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x + \int_{\Omega\cap\{u_\varepsilon>\delta\}} \frac{f}{u_\varepsilon^\theta}\varphi\,\mathrm{d}x := I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta}\,. \tag{39}$$

where, by the Lebesgue dominated convergence theorem and taking into account that $0 \leq \frac{f}{u_\varepsilon^\theta}\varphi \leq \frac{f}{\delta^\theta}\varphi \in L^1(\Omega)$ in the set $\{u_\varepsilon > \delta\}$ (here it is crucial that φ is bounded), we get

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^2_{\varepsilon,\delta} = \int_{\Omega\cap\{u>0\}} \frac{f}{u^\theta}\varphi\,\mathrm{d}x\,, \tag{40}$$

once we have taken $\delta \notin \mathcal{C} = \{\delta > 0 : |\{u(x) = \delta\}| > 0\}$, which is at most countable (exactly as in [17, Proof of Theorem 4.6]).

Moreover, introducing the function $Z_\delta : \mathbb{R} \to [0, +\infty)$ defined by

$$Z_\delta(s) = \begin{cases} 1\,, & \text{if } 0 \leq s \leq \delta\,; \\ -\frac{s}{\delta} + 2\,, & \text{if } \delta \leq s \leq 2\delta\,; \\ 0\,, & \text{if } s \geq 2\delta\,, \end{cases} \tag{41}$$

using as test function in (13) (with $\alpha = 1$) the function $Z_\delta(u_\varepsilon)\varphi$, with φ as above, and recalling that $s \in [0, +\infty) \mapsto Z_\delta(s)$ is decreasing, we arrive at

$$\begin{aligned} I^1_{\varepsilon,\delta} &\leq \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi\, Z_\delta(u_\varepsilon)\,\mathrm{d}x \\ &= \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi [Z_\delta(u_\varepsilon) - Z_\delta(u)]\,\mathrm{d}x + \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi\, Z_\delta(u)\,\mathrm{d}x \end{aligned} \tag{42}$$

since

$$\frac{\beta}{\varepsilon}\int_{\Gamma^\varepsilon} [u_\varepsilon]\left(Z_\delta(u_\varepsilon^{(2)}) - Z_\delta(u_\varepsilon^{(1)})\right)\varphi\,\mathrm{d}x \leq 0$$

and

$$\int_{\Omega\cap\{\delta\leq u_\varepsilon\leq 2\delta\}} \frac{f}{u_\varepsilon^\theta} Z_\delta(u_\varepsilon)\varphi\,\mathrm{d}x \geq 0.$$

In order to pass to the two-scale limit in (42), we have to take into account that $\lambda_\varepsilon \nabla u_\varepsilon$ is bounded in $L^2(\Omega)$ and $Z_\delta(u_\varepsilon) - Z_\delta(u) \to 0$ strongly in $L^2(\Omega)$ (since $s \mapsto Z_\delta(s)$ is continuous and (19) holds), so that the first integral in (42) vanishes, while in the second integral, thanks to Remark 2.6, we can take $\lambda_\varepsilon \nabla\varphi Z_\delta(u)$ as admissible test function for the two-scale convergence. Therefore, we get

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^1_{\varepsilon,\delta} \le \int_{\Omega\cap\{u=0\}}\int_Y |\lambda(\nabla u + \nabla_y u_1)|\ |\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}y. \tag{43}$$

In order to prove that the right-hand side of (43) is zero, we notice that, choosing $\varphi \equiv 0$ in (35) and letting $\varepsilon \to 0$, we obtain

$$\int_\Omega\int_Y \lambda(\nabla u + \nabla_y u_1)\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y + \beta\int_\Omega\int_\Gamma [u_1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = 0\,,$$

which is the problem in the micro variable y (i.e. (25)–(27)); therefore, we get the factorization (30) for u_1. This implies that

$$\int_{(\Omega\cap\{u=0\})\times Y} |\lambda(\nabla u + \nabla_y u_1)|\,|\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}y = \int_{(\Omega\cap\{u=0\})\times Y} |\lambda(I+\nabla_y\chi)\nabla u|\,|\nabla\varphi|\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{44}$$

where, in the last equality, we used that u is a Sobolev function and hence its gradient vanishes on the level sets of u. Then, passing to the limit for $\varepsilon \to 0$ in (35), by (36), (40), (43), (44) and taking into account the density of our test functions in $H^1_0(\Omega)\times L^2(\Omega; V_\#(Y))$, we obtain

$$\int_{\Omega\times Y} \lambda(\nabla u + \nabla_y u_1)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y}\lambda(\nabla u + \nabla_y u_1)\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y \\ + \beta\int_\Omega\int_\Gamma [u_1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y) = \int_\Omega \frac{f}{u^\theta}\varphi\chi_{\{u>0\}}\,\mathrm{d}x\,. \tag{45}$$

Taking first $\varphi = 0$ and then $\Phi = 0$ in (45), it follows that the pair (u, u_1), with u nonnegative (being the limit of the sequence of positive solutions u_ε), is a weak solution of the problem (24)–(27) and (29), with $\frac{f}{u^\theta}$ replaced by $\frac{f}{u^\theta}\chi_{\{u>0\}}$. In order to conclude the proof, it remains to show that $u > 0$ a.e. in Ω. To this purpose, we recall again the factorization given in (30), where $u \in H^1_0(\Omega)$ solves the first equation in (32) with the new nonnegative source $\frac{f}{u^\theta}\chi_{\{u>0\}}$ and the matrix A_{hom} defined in (34) is positive definite. Therefore, taking into account (23), we can apply the strong maximum principle to deduce that $u > 0$ a.e. in Ω. Finally, by Remark 4.2, it follows that the whole sequence $\{u_\varepsilon\}$ converges and the thesis is accomplished. □

4.2. The case $\alpha > 1$

As in the previous subsection, we will assume to be in anyone of the geometrical setting described in Section 2. Moreover, we will see that, due to the particular scaling $\varepsilon^{-\alpha}$ in front of the interface term, the homogenized problem will not take memory of β, as pointed out in Remark 4.6.

Theorem 4.4. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem (11). Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\#(Y))$ with $\int_Y u_1(x,y)\,\mathrm{d}y = 0$ a.e. in Ω, such that, as $\varepsilon \to 0$, (19)–(23) hold. Moreover, the pair (u, u_1) solve*

$$-\operatorname{div}\left(\lambda_0 \nabla u + \int_Y \lambda \nabla_y u_1 \,\mathrm{d}y\right) = \frac{f}{u^\theta}, \qquad \text{in } \Omega; \tag{46}$$

$$-\operatorname{div}_y\left(\lambda(\nabla u + \nabla_y u_1)\right) = 0, \qquad \text{in } \Omega \times (E_1 \cup E_2); \tag{47}$$

$$[\lambda(\nabla u + \nabla_y u_1)\cdot \nu] = 0, \qquad \text{on } \Omega \times \Gamma; \tag{48}$$

$$u > 0, \qquad \text{in } \Omega; \tag{49}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{50}$$

where λ_0 and λ are defined at the beginning of Subsection 2.5.

Remark 4.5. *Notice that, similarly as in Remark 4.2, it is possible to prove that the problem (46)–(50) admits at most one pair of solutions (u, u_1). Moreover, in order to obtain, as before, a single homogenized equation for the limit function u, we can uniquely factorize u_1 as in (30), but with a different choice of cell functions $\chi = (\chi_1, \dots, \chi_N)$. In this case, we take $\chi_j \in H^1_\#(Y)$ such that $\int_Y \chi_j \,\mathrm{d}y = 0$, for each $j = 1, \dots, N$, and it satisfies the cell problem*

$$\begin{aligned} -\operatorname{div}_y(\lambda(\nabla_y \chi_j + \mathbf{e}_j)) &= 0, && \text{in } E_1 \cup E_2; \\ [\lambda(\nabla \chi_j + \mathbf{e}_j)\cdot \nu] &= 0, && \text{on } \Gamma. \end{aligned} \tag{51}$$

Replacing the factorization of u_1 in (46), it follows that u solves

$$\begin{aligned} -\operatorname{div}(A_{hom}\nabla u) &= \frac{f}{u^\theta}, && \text{in } \Omega; \\ u &> 0, && \text{in } \Omega; \\ u &= 0, && \text{on } \partial\Omega; \end{aligned} \tag{52}$$

where the matrix A_{hom} is defined as

$$A_{hom} = \lambda_0 I + \int_Y \lambda (\nabla_y \chi)^T \,\mathrm{d}y = \lambda_0 I - \int_\Gamma [\lambda] \nu \otimes \chi \,\mathrm{d}\sigma. \tag{53}$$

We recall that by standard arguments equation (51) *admits a unique solution with null mean average on Y, for $j = 1, \dots, N$. Moreover, by [6, Proposition 4.1] we know that A_{hom} is symmetric and positive definite and therefore, by [13, Theorem 5.2 and Remark 5.4], also the solution of equation* (52) *is unique.*

Remark 4.6. *Notice that, from the definition* (51)*, the cell functions do not depend on the coefficient β. Therefore, the homogenized matrix and, hence, the macroscopic function u lose any memory of the physical properties of the interfaces.*

Proof of Theorem 4.4. By Proposition 3.1, [21, Proposition 5.5] and Fatou's Lemma, we get that, up to a subsequence, (19)–(22) and (23) hold (as in the proof of Theorem 4.1).

Moreover, by (14) we also know that

$$\varepsilon \int_{\Gamma^\varepsilon} \left(\frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} \right)^2 \mathrm{d}x = \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \, \mathrm{d}x \le C \,, \tag{54}$$

uniformly with respect to ε. Hence, as ε tends to 0, by Theorem 2.11 it follows that there exists $v \in L^2(\Omega \times \Gamma)$ such that, up to a subsequence, $v_\varepsilon := \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} \xrightarrow{2-sc} v$ in $L^2(\Omega^\varepsilon)$. However, by (22) we already know that $\frac{[u_\varepsilon]}{\varepsilon} \xrightarrow{2-sc} [u_1]$; therefore, taking into account that $\frac{[u_\varepsilon]}{\varepsilon} = \varepsilon^{\frac{\alpha-1}{2}} v_\varepsilon$, with $\frac{\alpha-1}{2} > 0$, we infer that $[u_1] = 0$, so that $u_1 \in L^2(\Omega; H^1_\#(Y))$.

In order to pass to the two-scale limit in (13), with $\alpha > 1$, we choose as test function $\psi(x) = \varphi(x) + \varepsilon \Phi\left(x, \frac{x}{\varepsilon}\right)$ with $\varphi \in \mathcal{C}^1_c(\Omega)$ and $\Phi \in \mathcal{C}^1_c(\Omega; \mathcal{C}^1_\#(\overline{Y}))$ (i.e., we can take $[\Phi] = 0$, since $[u_1] = 0$) and we get

$$\begin{aligned} &\int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla \varphi \, \mathrm{d}x + \varepsilon \int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla_x \Phi \, \mathrm{d}x + \int_\Omega \lambda \nabla u_\varepsilon \cdot \nabla_y \Phi \, \mathrm{d}x = \\ = \; &\int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \, \mathrm{d}x + \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \Phi \, \mathrm{d}x. \end{aligned} \tag{55}$$

By (21), we obtain that the left-hand side of (55) converges to

$$\int_{\Omega \times Y} \lambda (\nabla u + \nabla_y u_1) \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}y + \int_{\Omega \times Y} \lambda (\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi \, \mathrm{d}x \, \mathrm{d}y \,.$$

Moreover, by (18) and reasoning as in (38), the second term in the right-hand side tends to 0. Finally, arguing as in the proof of Theorem 4.1 for the study of the first integral in the right-hand side of (55), as ε goes to 0, we have

$$\int_\Omega \frac{f}{u_\varepsilon^\theta}\varphi \,\mathrm{d}x \to \int_\Omega \frac{f}{u^\theta}\varphi\chi_{\{u>0\}} \,\mathrm{d}x. \tag{56}$$

The proof that $u > 0$ a.e. in Ω follows, as usual, from the strong maximum principle, taking into account (23), so that we can replace the source $\frac{f}{u^\theta}\varphi\chi_{\{u>0\}}$ with $\frac{f}{u^\theta}\varphi$. Finally, recalling the density of our test functions in $H_0^1(\Omega) \times L^2(\Omega; H^1_\#(Y))$, taking alternatively $\varphi = 0$ and $\Phi = 0$ in (56) and integrating by parts, we deduce (46)-(47). Therefore, by the uniqueness of the problem (46)–(50) (see Remark 4.5), it follows that the whole sequence $\{u_\varepsilon\}$ converges and the thesis is accomplished.

□

4.3. The case $\alpha \in (-1, 1)$

As in the previous subsections, we will assume to be in anyone of the geometrical settings described in Section 2. Moreover, analogously to the case $\alpha > 1$, we will see that also in this case, due to the particular scaling $\varepsilon^{-\alpha}$ in front of the interface term, the homogenized problem will not take memory of β (see the end of Remark 4.8).

Theorem 4.7. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (11). *Then, there exist $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; V_\#(Y))$ with $u_1 = (u_1^{(1)}, u_1^{(2)})$, $\int_{E_1} u_1^{(1)}(x,y)\,\mathrm{d}y = 0 = \int_{E_2} u_1^{(2)}(x,y)\,\mathrm{d}y$ a.e. in Y, such that, as $\varepsilon \to 0$, we have*

$$u_\varepsilon \xrightarrow{2-sc} u\,, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{57}$$

$$\chi_{\Omega\setminus\Gamma^\varepsilon}\nabla u_\varepsilon \xrightarrow{2-sc} \nabla u + \nabla_y u_1\,, \qquad \text{in } L^2(\Omega \times Y)\,; \tag{58}$$

$$[u_\varepsilon] \xrightarrow{2-sc} 0\,, \qquad \text{in } L^2(\Omega; L^2(\Gamma))\,. \tag{59}$$

Moreover, (19) *and* (23) *hold and the pair* (u, u_1) *solves*

$$-\operatorname{div}\left(\lambda_0\nabla u + \int_Y \lambda\nabla_y u_1\,\mathrm{d}y\right) = \frac{f}{u^\theta}\,, \quad \text{in } \Omega\,; \tag{60}$$

$$-\operatorname{div}_y(\lambda(\nabla u + \nabla_y u_1)) = 0\,, \qquad \text{in } \Omega \times (E_1 \cup E_2)\,; \tag{61}$$

$$[\lambda(\nabla u + \nabla_y u_1)\cdot\nu] = 0\,, \qquad \text{on } \Omega \times \Gamma\,; \tag{62}$$

$$\lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)})\cdot\nu = 0 \qquad \text{on } \Omega \times \Gamma\,; \tag{63}$$

$$u > 0\,, \qquad \text{in } \Omega\,; \tag{64}$$

$$u = 0\,, \qquad \text{on } \partial\Omega\,, \tag{65}$$

where λ_0 and λ_1 are defined in Subsection 2.5.

Remark 4.8. *Following the same ideas as in Remarks 4.2, 4.3 and 4.5, we have that the problem (60)–(65) admits at most one pair of solutions (u, u_1) and that u_1 can be factorized as in (30) where, in this case, the cell function $\chi = (\chi_1, \dots, \chi_N)$ is such that $\chi_j \in V_\#(Y)$ with $\int_{E_1} \chi_j^{(1)} \, \mathrm{d}y = 0 = \int_{E_2} \chi_j^{(2)} \, \mathrm{d}y$ for each $j = 1, \dots, N$ and it satisfies the cell problem*

$$
\begin{aligned}
&-\operatorname{div}_y(\lambda(\nabla_y \chi_j + \mathbf{e}_j)) = 0\,, && \text{in } E_1 \cup E_2\,;\\
&[\lambda(\nabla_y \chi_j + \mathbf{e}_j) \cdot \nu] = 0\,, && \text{on } \Gamma\,;\\
&\lambda_2(\nabla_y \chi_j^{(2)} + \mathbf{e}_j) \cdot \nu = 0\,, && \text{on } \Gamma\,,
\end{aligned}
\tag{66}
$$

which admits a unique solution. Replacing the factorization of u_1 in (60), we still obtain that u solves an elliptic problem analogous to (32), where the new matrix A_{hom} is defined as in (33) and (34) in terms of the cell functions given in (66). Notice that, following the same ideas as in [12, Remark 2.6], we obtain

$$
A^{ij}_{hom} = \int_Y \lambda \nabla(\chi_i + y_i) \cdot \nabla(\chi_j + y_j) \, \mathrm{d}y\,.
$$

Moreover, for every $\xi \in \mathbb{R}^N$, we get

$$
\begin{aligned}
A_{hom}\xi \cdot \xi &= \int_Y \lambda \nabla\big((\chi_i + y_i)\xi_i\big) \cdot \nabla\big((\chi_j + y_j)\xi_j\big) \, \mathrm{d}y = \int_Y \lambda |\nabla\big((\chi + y) \cdot \xi\big)|^2 \, \mathrm{d}y \\
&= \int_{E_1} \lambda_1 |\nabla\big((\chi + y) \cdot \xi\big)|^2 \, \mathrm{d}y + \int_{E_2} \lambda_2 |\nabla\big((\chi + y) \cdot \xi\big)|^2 \, \mathrm{d}y \\
&\geq \int_{E_2} \lambda_2 |\nabla\big((\chi + y) \cdot \xi\big)|^2 \, \mathrm{d}y \geq \ell |\xi|^2\,,
\end{aligned}
$$

for a suitable constant $\ell > 0$. The last inequality is a consequence of the Y-periodicity of the cell function χ, as in [4, Section 1] (see, also, [11, Proof of Lemma 4.7], for the same idea applied in a different framework). Indeed, if by contradiction $|\nabla\big((\chi + y) \cdot \xi\big)| = 0$, we obtain that, up to an additive constant, $\chi(y) \cdot \xi = -y \cdot \xi$, a.e. in E_2. However, this is not possible, since $\partial E_2 \cap \partial Y \neq \emptyset$ and, while χ is Y-periodic, y is not. Hence, A_{hom} is a symmetric and positive definite matrix.

As in the case $\alpha > 1$, from the definition (66), we see that the cell functions do not depend on the coefficient β.

Proof of Theorem 4.7. As a consequence of Theorem 2.2 and Proposition 3.1, we can apply Theorem 2.12, obtaining that (4)–(6) and (8) hold. Moreover, as in the proof of Theorem 4.4, we get that (14) implies (54) with C independent of ε. Hence, after setting $v_\varepsilon := \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}}$, we obtain that, up to a subsequence, v_ε two-scale converges in $L^2(\Omega \times \Gamma)$ to some $v \in L^2(\Omega \times \Gamma)$, so that

$$0 \overset{2-sc}{\longleftarrow} v_\varepsilon \varepsilon^{\frac{\alpha+1}{2}} = [u_\varepsilon] \overset{2-sc}{\longrightarrow} [u] \,, \tag{67}$$

where we have taken into account that $\alpha + 1 > 0$. Therefore, (59) holds and $[u] = 0$. Taking into account (67), (4)–(6) become (57)–(58).

Now, let us choose $\psi(x) = \varphi(x) + \varepsilon\Phi\left(x, \frac{x}{\varepsilon}\right)$, with $\varphi \in \mathcal{C}^1_c(\Omega)$ and $\Phi \in \mathcal{C}^1_c(\Omega; \mathfrak{L}_\#(Y))$, as test function in (13). Then, we get

$$\begin{aligned} &\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \\ + &\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x + \beta\varepsilon^{1-\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon][\Phi] \,\mathrm{d}\sigma = \\ = &\int_\Omega \frac{f}{u_\varepsilon^\theta} \varphi \,\mathrm{d}x + \varepsilon \int_\Omega \frac{f}{u_\varepsilon^\theta} \Phi \,\mathrm{d}x := I_\varepsilon + J_\varepsilon. \end{aligned} \tag{68}$$

By (58), as $\varepsilon \to 0$, we obtain

$$\begin{aligned} &\int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x + \varepsilon \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x + \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla_y \Phi \,\mathrm{d}x \\ &\to \int_{\Omega \times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla\varphi \,\mathrm{d}x \,\mathrm{d}y + \int_{\Omega \times Y} \lambda(\nabla u + \nabla_y u_1) \cdot \nabla_y \Phi \,\mathrm{d}x \,\mathrm{d}y. \end{aligned} \tag{69}$$

Moreover, we can write

$$\beta\varepsilon^{1-\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon][\Phi] \,\mathrm{d}\sigma = \beta\varepsilon^{\frac{1-\alpha}{2}} \varepsilon \int_{\Gamma^\varepsilon} \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} [\Phi] \,\mathrm{d}\sigma \to 0 \,, \tag{70}$$

as a consequence of the fact that $1 - \alpha > 0$ and that $v_\varepsilon = \frac{[u_\varepsilon]}{\varepsilon^{\frac{\alpha+1}{2}}} \overset{2-sc}{\longrightarrow} v$.

In order to pass to the limit in the right-hand side of (68), i.e. to deal with the singular term, we consider the extension of u_ε from Ω_2^ε to Ω_1^ε as in Theorem 2.13, and for the sake of simplicity, let us denote by $T(u_\varepsilon)$ this extension, i.e. $T(u_\varepsilon) \in H^1(\Omega)$, $T(u_\varepsilon) = u_\varepsilon$ in Ω_2^ε, $\|T(u_\varepsilon)\|_{L^2(\Omega)} \le C\|u_\varepsilon\|_{L^2(\Omega_2^\varepsilon)}$ and $\|\nabla T(u_\varepsilon)\|_{L^2(\Omega)} \le C\|\nabla u_\varepsilon\|_{L^2(\Omega_2^\varepsilon)}$, with C independent of ε. Then, by

(14), it follows that there exists $\upsilon \in H^1(\Omega)$ such that, up to a subsequence, $T(u_\varepsilon) \rightharpoonup \upsilon$ weakly in $H^1(\Omega)$ and $T(u_\varepsilon) \to \upsilon$ strongly in $L^2(\Omega)$. Moreover, recalling [2, Proposition 1.14 (i)] we have also $T(u_\varepsilon) \stackrel{2-sc}{\longrightarrow} \upsilon$ in $L^2(\Omega \times Y)$. By Lemma 6 of [22] applied to $\|u_\varepsilon - T(u_\varepsilon)\|_{L^2(\Omega_1^\varepsilon)}$, we have that

$$
\begin{aligned}
\|u_\varepsilon - \upsilon\|^2_{L^2(\Omega)} &= \|(u_\varepsilon - T(u_\varepsilon)) + (T(u_\varepsilon) - \upsilon)\|^2_{L^2(\Omega)} \\
&\leq 2\left(\|u_\varepsilon - T(u_\varepsilon)\|^2_{L^2(\Omega)} + \|T(u_\varepsilon) - \upsilon\|^2_{L^2(\Omega)}\right) \\
&\leq 2\left(\|u_\varepsilon - T(u_\varepsilon)\|^2_{L^2(\Omega_2^\varepsilon)} + \|u_\varepsilon - T(u_\varepsilon)\|^2_{L^2(\Omega_1^\varepsilon)} + \|T(u_\varepsilon) - \upsilon\|^2_{L^2(\Omega)}\right) \\
&\leq C\Big(\|u_\varepsilon - T(u_\varepsilon)\|^2_{L^2(\Omega_2^\varepsilon)} + \varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon - T(u_\varepsilon)]^2 \,\mathrm{d}\sigma \\
&\qquad\quad + \varepsilon^2 \|\nabla u_\varepsilon - \nabla T(u_\varepsilon)\|^2_{L^2(\Omega)} + \|T(u_\varepsilon) - \upsilon\|^2_{L^2(\Omega)}\Big) \\
&\leq C\Big(\varepsilon^{1+\alpha} \frac{1}{\varepsilon^\alpha} \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma + \varepsilon^2 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \\
&\qquad\quad + \varepsilon^2 \|\nabla T(u_\varepsilon)\|^2_{L^2(\Omega)} + \|T(u_\varepsilon) - \upsilon\|^2_{L^2(\Omega)}\Big) \to 0\,,
\end{aligned}
$$

where we have taken into account again that $\alpha + 1 > 0$. It remains to prove that $\upsilon = u$, but this is a direct consequence of the fact that $T(u_\varepsilon) = u_\varepsilon$ in Ω_2^ε, indeed taking a test function $\phi\left(x, \frac{x}{\varepsilon}\right) = \phi_1(x)\phi_2\left(\frac{x}{\varepsilon}\right)$, with $\phi_1 \in \mathcal{C}^0_c(\Omega)$ and $\phi_2 \in \mathcal{C}^0_\#(\overline{Y})$ with compact support in E_2, it follows

$$
\begin{aligned}
\left(\int_\Omega u(x)\phi_1(x)\,\mathrm{d}x\right)\left(\int_{E_2} \phi_2(y)\,\mathrm{d}y\right) &\leftarrow \int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x \\
= \int_\Omega T(u_\varepsilon)(x)\phi\left(x, \frac{x}{\varepsilon}\right)\,\mathrm{d}x &\to \left(\int_\Omega \upsilon(x)\phi_1(x)\,\mathrm{d}x\right)\left(\int_{E_2} \phi_2(y)\,\mathrm{d}y\right).
\end{aligned}
$$

Therefore, $u_\varepsilon \to u$ strongly in $L^2(\Omega)$, i.e. (19) holds. In order to get the homogenous boundary condition (65), we proceed as in [9, Proof of Theorem 2.2]. Let $\Phi \in L^2(\Omega;\mathbb{R}^N)$ and let Ψ be the function associated to Φ on E_2 by

Lemma 2.4. Integrating by parts and passing to the two-scale limit, we have

$$\int_\Omega \int_{E_2} (\nabla u(x) + \nabla_y u_1(x,y))\, \Psi(x,y)\, \mathrm{d}y\, \mathrm{d}x \leftarrow \int_{\Omega_2^\varepsilon} \nabla u_\varepsilon(x) \cdot \Psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x$$
$$= -\int_{\Omega_2^\varepsilon} u_\varepsilon(x)\, \mathrm{div}_x\, \Psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \to -\int_\Omega \int_{E_2} u(x)\, \mathrm{div}_x\, \Psi(x,y)\, \mathrm{d}y\, \mathrm{d}x$$
$$= -\int_\Omega u(x)\, \mathrm{div}\, \Phi(x)\, \mathrm{d}x\,. \quad (71)$$

Moreover, by (2) there holds

$$\int_{E_2} \nabla_y u_1(x,y) \Psi(x,y)\, \mathrm{d}y = -\int_\Gamma u_1(x,y) \Psi \cdot \nu\, \mathrm{d}\sigma - \int_{E_2} u_1(x,y)\, \mathrm{div}_y\, \Psi(x,y)\, \mathrm{d}y = 0. \quad (72)$$

By (71) and (72), we conclude

$$\int_\Omega \nabla u(x) \Phi(x)\, \mathrm{d}x = \int_\Omega \nabla u(x) \left(\int_{E_2} \Psi(x,y)\, \mathrm{d}y \right) \mathrm{d}x = -\int_\Omega u(x)\, \mathrm{div}\, \Phi(x)\, \mathrm{d}x\,, \quad (73)$$

and hence $u = 0$ on $\partial\Omega$. Then, we can repeat the argument in the proof of Theorem 4.1 in order to obtain (23) and

$$I_\varepsilon \to \int_\Omega \frac{f}{u^\theta} \varphi \chi_{\{u>0\}}\, \mathrm{d}x\,, \qquad J_\varepsilon \to 0\,, \qquad \text{for } \varepsilon \to 0. \quad (74)$$

Moreover, using the strong maximum principle as in Theorem 4.1, we obtain $u > 0$ a.e. in Ω, so that we can drop the characteristic function $\chi_{\{u>0\}}$ in (74). Finally, taking first $\varphi = 0$ and then $\Phi = 0$, we deduce the strong formulation (60)–(65). □

Remark 4.9. *Notice that, when we are in the connected/disconnected case, as already pointed out in Subsection 2.4, we can refer to the more classical extension theorem in [14, 26], where the extension is found directly in $H_0^1(\Omega)$. Thus the proof of Theorem 4.7 can be achieved in a simpler way, avoiding steps* (71)*–*(73)*.*

4.4. The case $\alpha = -1$

In this subsection we will assume to be in the connected/connected geometry. Moreover, we stipulate that the source $f \in L^{\frac{2}{1+\theta}}(\Omega)$ is strictly positive a.e. in Ω. We will see that the homogenized problem will take into account the physical properties of the bulk regions (i.e., λ_1, λ_2) as well as the physical properties of the interfaces (i.e. β).

Theorem 4.10. *For $\varepsilon > 0$, let $u_\varepsilon \in V_0^\varepsilon(\Omega)$ be the weak solution of the problem* (11). *Then, there exist $u = \left(u^{(1)}, u^{(2)}\right) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $u_1 = \left(u_1^{(1)}, u_1^{(2)}\right) \in L^2(\Omega; V_\#(Y))$ with $\int_{E_1} u_1^{(1)}\,\mathrm{d}y = 0 = \int_{E_2} u_1^{(2)}\,\mathrm{d}y$, such that*

$$\chi_{\Omega_1^\varepsilon} u_\varepsilon^{(1)} \overset{2-sc}{\longrightarrow} \chi_{E_1} u^{(1)}, \quad \chi_{\Omega_2^\varepsilon} u_\varepsilon^{(2)} \overset{2-sc}{\longrightarrow} \chi_{E_2} u^{(2)}, \qquad \text{in } L^2(\Omega \times Y); \tag{75}$$

$$\chi_{\Omega_1^\varepsilon} \nabla u_\varepsilon^{(1)} \overset{2-sc}{\longrightarrow} \chi_{E_1} \left(\nabla u^{(1)} + \nabla_y u_1^{(1)}\right), \qquad \text{in } L^2(\Omega \times Y); \tag{76}$$

$$\chi_{\Omega_2^\varepsilon} \nabla u_\varepsilon^{(2)} \overset{2-sc}{\longrightarrow} \chi_{E_2} \left(\nabla u^{(2)} + \nabla_y u_1^{(2)}\right), \qquad \text{in } L^2(\Omega \times Y); \tag{77}$$

$$[u_\varepsilon] \overset{2-sc}{\longrightarrow} [u], \qquad \text{in } L^2(\Omega; L^2(\Gamma)). \tag{78}$$

Moreover,

$$\left| \int_\Omega \frac{f}{(u^{(i)})^\theta} \varphi \,\mathrm{d}x \right| < +\infty, \qquad \forall \varphi \in H_0^1(\Omega), \quad i = 1, 2, \tag{79}$$

and the pair (u, u_1) solve the bidomain system

$$-\operatorname{div}\left(\lambda_1 |E_1| \nabla u^{(1)} + \int_{E_1} \lambda_1 \nabla_y u_1^{(1)}\,\mathrm{d}y\right) = |E_1| \frac{f}{(u^{(1)})^\theta} + |\Gamma| \beta [u], \ \text{in } \Omega; \tag{80}$$

$$-\operatorname{div}\left(\lambda_2 |E_2| \nabla u^{(2)} + \int_{E_2} \lambda_2 \nabla_y u_1^{(2)}\,\mathrm{d}y\right) = |E_2| \frac{f}{(u^{(2)})^\theta} - |\Gamma| \beta [u], \ \text{in } \Omega; \tag{81}$$

$$-\operatorname{div}_y(\lambda(\nabla u + \nabla_y u_1)) = 0, \ \text{in } \Omega \times (E_1 \cup E_2); \tag{82}$$

$$\lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)}) \cdot \nu = 0, \ \text{on } \Omega \times \Gamma; \tag{83}$$

$$\lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)}) \cdot \nu = 0, \ \text{on } \Omega \times \Gamma; \tag{84}$$

$$u^{(1)}, u^{(2)} > 0, \ \text{in } \Omega, \tag{85}$$

$$u^{(1)} = u^{(2)} = 0, \ \text{on } \partial\Omega, \tag{86}$$

where, with a slight abuse of notation, we set $[u] = u^{(2)} - u^{(1)}$.

Remark 4.11. *Following the same ideas as in Remark 4.2, we obtain that problem* (80)–(86) *admits at most one pair of solutions (u, u_1). Moreover, we can factorize u_1 as*

$$u_1^{(1)}(x, y) = \chi^{(1)}(y) \nabla u^{(1)}(x), \qquad u_1^{(2)}(x, y) = \chi^{(2)}(y) \nabla u^{(2)}(x), \tag{87}$$

where $\chi^{(k)} = (\chi_1^{(k)}, \dots, \chi_N^{(k)})$*, for* $k = 1, 2$*,* $\int_{E_1} \chi_j^{(1)} \,\mathrm{d}y = 0 = \int_{E_2} \chi_j^{(2)} \,\mathrm{d}y$*, for each* $j = 1, \dots, N$*, and, recalling the usual notation, we set* $\chi = (\chi^{(1)}, \chi^{(2)}) \in \big(V_\#(Y)\big)^N$*. Then by* (82)*–*(84) *we obtain that, for each* $j = 1, \dots, N$*,* χ_j *satisfies* (66) *and* $u^{(1)}, u^{(2)}$ *solve the following bidomain system*

$$\begin{aligned} & -\operatorname{div}(A^{(1)}_{hom} \nabla u^{(1)}) = |E_1| \tfrac{f}{(u^{(1)})^\theta} + |\Gamma| \beta \left(u^{(2)} - u^{(1)}\right) , && \text{in } \Omega\,; \\ & -\operatorname{div}(A^{(2)}_{hom} \nabla u^{(2)}) = |E_2| \tfrac{f}{(u^{(2)})^\theta} - |\Gamma| \beta \left(u^{(2)} - u^{(1)}\right) , && \text{in } \Omega\,; \\ & u^{(1)} = u^{(2)} = 0\,, && \text{on } \partial\Omega\,, \end{aligned} \tag{88}$$

where, for $k = 1, 2$*, the matrix* $A^{(k)}_{hom}$ *is defined as*

$$A^{(k)}_{hom} = \lambda_k |E_k| I + \lambda_k \int_{E_k} (\nabla_y \chi^{(k)})^T \,\mathrm{d}y.$$

Since

$$\left(\int_{E_k} (\nabla \chi^{(k)})^T \,\mathrm{d}y \right)_{ij} = \int_{E_k} \frac{\partial \chi_j^{(k)}}{\partial y_i} \,\mathrm{d}y = -(-1)^k \int_\Gamma \chi_j^{(k)} \nu_i \,\mathrm{d}y,$$

we also have

$$A^{(k)}_{hom} = \lambda_k |E_k| I - (-1)^k \lambda_k \int_\Gamma \nu \otimes \chi^{(k)} \,\mathrm{d}\sigma. \tag{89}$$

As in Remark 4.8, we obtain that the matrices $A^{(k)}_{hom}$ *are symmetric and positive definite. Therefore, the solution* $u = (u^{(1)}, u^{(2)})$ *of* (88) *is unique. In fact, if* $\hat{u} = (\hat{u}^{(1)}, \hat{u}^{(2)})$ *and* $\bar{u} = (\bar{u}^{(1)}, \bar{u}^{(2)})$ *are two different solutions, then for* $\varphi \in H^1_0(\Omega)$ *and* $k = 1, 2$*, we have*

$$\int_\Omega A^{(k)}_{hom} \nabla \hat{u}^{(k)} \cdot \nabla \varphi \,\mathrm{d}x = |E_k| \int_\Omega \frac{f}{(\hat{u}^{(k)})^\theta} \varphi \,\mathrm{d}x - (-1)^k |\Gamma| \beta \int_\Omega [\hat{u}] \varphi \,\mathrm{d}x\,, \tag{90}$$

$$\int_\Omega A^{(k)}_{hom} \nabla \bar{u}^{(k)} \cdot \nabla \varphi \,\mathrm{d}x = |E_k| \int_\Omega \frac{f}{(\bar{u}^{(k)})^\theta} \varphi \,\mathrm{d}x - (-1)^k |\Gamma| \beta \int_\Omega [\bar{u}] \varphi \,\mathrm{d}x\,. \tag{91}$$

Subtracting (91) *from* (90) *and taking* $\varphi = \hat{u}^{(k)} - \bar{u}^{(k)}$*, separately for* $k = 1, 2$*, we have*

$$\begin{aligned} & \int_\Omega A^{(k)}_{hom} \nabla \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right) \cdot \nabla \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right) \,\mathrm{d}x = \\ & |E_k| \int_\Omega \left(\frac{f}{(\hat{u}^{(k)})^\theta} - \frac{f}{(\bar{u}^{(k)})^\theta} \right) \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right) \,\mathrm{d}x - |\Gamma| \beta \int_\Omega \left(\hat{u}^{(k)} - \bar{u}^{(k)}\right)^2 \,\mathrm{d}x \\ & \qquad + |\Gamma| \beta \int_\Omega \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \,\mathrm{d}x. \end{aligned} \tag{92}$$

Summing (92) *for* $k = 1, 2$, *we get*

$$
\begin{aligned}
&\int_\Omega A^{(1)}_{hom} \nabla \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \cdot \nabla \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \, \mathrm{d}x \\
+&\int_\Omega A^{(2)}_{hom} \nabla \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \cdot \nabla \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \, \mathrm{d}x \\
=&|E_1| \int_\Omega \left(\frac{f}{(\hat{u}^{(1)})^\theta} - \frac{f}{(\bar{u}^{(1)})^\theta} \right) \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \, \mathrm{d}x \\
+&|E_2| \int_\Omega \left(\frac{f}{(\hat{u}^{(2)})^\theta} - \frac{f}{(\bar{u}^{(2)})^\theta} \right) \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \, \mathrm{d}x \\
-&|\Gamma| \beta \int_\Omega \left((\hat{u}^{(1)} - \bar{u}^{(1)}) - (\hat{u}^{(2)} - \bar{u}^{(2)}) \right)^2 \mathrm{d}x.
\end{aligned}
\tag{93}
$$

Recalling that $A^{(1)}_{hom}$ *and* $A^{(2)}_{hom}$ *are positive definite and taking into account that the function* $s \in (0, +\infty) \mapsto \frac{1}{s^\theta}$ *is decreasing, by* (93) *we infer*

$$
\int_\Omega \left| \nabla \left(\hat{u}^{(1)} - \bar{u}^{(1)}\right) \right|^2 + \int_\Omega \left| \nabla \left(\hat{u}^{(2)} - \bar{u}^{(2)}\right) \right|^2 \leq 0 \,,
$$

which implies $\hat{u}^{(1)} = \bar{u}^{(1)}$ *and* $\hat{u}^{(2)} = \bar{u}^{(2)}$.

Proof. First we note that (75)–(78) follow by Proposition 3.1 and Theorem 2.12. In order to proceed with the homogenization, we choose $\psi = (\psi^{(1)}, \psi^{(2)})$, $\psi^{(i)}(x) = \varphi_i(x) + \varepsilon \Phi_i \left(x, \frac{x}{\varepsilon}\right)$ in $\Omega^\varepsilon_i \times E_i$, with $\varphi_i \in \mathcal{C}^1_c(\Omega)$ and $\Phi_i \in \mathcal{C}^1_c(\Omega; \mathfrak{L}_\#(Y))$, for $i = 1, 2$, as test function in (13), with $\alpha = -1$. We get

$$
\begin{aligned}
& \int_{\Omega^\varepsilon_1} \lambda_1 \nabla u_\varepsilon \cdot \nabla \varphi_1 \, \mathrm{d}x + \int_{\Omega^\varepsilon_2} \lambda_2 \nabla u_\varepsilon \cdot \nabla \varphi_2 \, \mathrm{d}x + \varepsilon \int_{\Omega^\varepsilon_1} \lambda_1 \nabla u_\varepsilon \cdot \nabla_x \Phi_1 \, \mathrm{d}x \\
+ \quad & \varepsilon \int_{\Omega^\varepsilon_2} \lambda_2 \nabla u_\varepsilon \cdot \nabla_x \Phi_2 \, \mathrm{d}x + \int_{\Omega^\varepsilon_1} \lambda_1 \nabla u_\varepsilon \nabla_y \Phi_1 \, \mathrm{d}x + \int_{\Omega^\varepsilon_2} \lambda_2 \nabla u_\varepsilon \nabla_y \Phi_2 \, \mathrm{d}x \\
+ \quad & \beta \varepsilon \int_{\Gamma^\varepsilon} [u_\varepsilon][\psi] \, \mathrm{d}\sigma \\
= \quad & \int_{\Omega^\varepsilon_1} \frac{f}{u^\theta_\varepsilon} \varphi_1 \, \mathrm{d}x + \int_{\Omega^\varepsilon_2} \frac{f}{u^\theta_\varepsilon} \varphi_2 \, \mathrm{d}x + \varepsilon \int_{\Omega^\varepsilon_1} \frac{f}{u^\theta_\varepsilon} \Phi_1 \, \mathrm{d}x + \varepsilon \int_{\Omega^\varepsilon_2} \frac{f}{u^\theta_\varepsilon} \Phi_2 \, \mathrm{d}x \\
=: \quad & I^1_\varepsilon + I^2_\varepsilon + J^1_\varepsilon + J^2_\varepsilon.
\end{aligned}
\tag{94}
$$

Hence, taking into account (75)–(78), as $\varepsilon \to 0$, the left-hand side converges to

$$\int_{\Omega\times E_1} \lambda_1(\nabla u^{(1)}+\nabla_y u_1^{(1)})\cdot\nabla\varphi_1 \,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega\times E_2} \lambda_2(\nabla u^{(2)}+\nabla_y u_1^{(2)})\cdot\nabla\varphi_2 \,\mathrm{d}x\,\mathrm{d}y$$
$$+\int_{\Omega\times E_1} \lambda_1(\nabla u^{(1)}+\nabla_y u_1^{(1)})\cdot\nabla_y\Phi_1 \,\mathrm{d}x\,\mathrm{d}y+\int_{\Omega\times E_2} \lambda_2(\nabla u^{(2)}+\nabla_y u_1^{(2)})\cdot\nabla_y\Phi_2 \,\mathrm{d}x\,\mathrm{d}y$$
$$+\beta\int_{\Omega\times\Gamma}[u][\varphi]\,\mathrm{d}x\,\mathrm{d}\sigma(y). \quad (95)$$

In order to treat the right-hand side of (94), we will need to making use of the extension operator introduced in Subsection 2.4. More precisely, we use the extensions of $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$, which can be obtained applying Theorem 2.13 both in Ω_1^ε and Ω_2^ε. In the sequel, for the sake of simplicity, we set $T(u_\varepsilon^{(i)}) = T_\varepsilon^i u_\varepsilon^{(i)}$, $i = 1, 2$. We recall that $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ are positive and, without loss of generality, we can assume that also $T(u_\varepsilon^{(1)})$ and $T(u_\varepsilon^{(2)})$ are positive (in fact, if the extension given by Theorem 2.13 would not be positive, we could replace it with its positive part). Moreover, by Theorem 2.13 and (75) we get

$$T(u_\varepsilon^{(1)})\chi_{\Omega_1^\varepsilon} = u_\varepsilon^{(1)}\chi_{\Omega_1^\varepsilon}, \qquad T(u_\varepsilon^{(2)})\chi_{\Omega_2^\varepsilon} = u_\varepsilon^{(2)}\chi_{\Omega_2^\varepsilon}, \quad (96)$$
$$u_\varepsilon^{(1)}\chi_{\Omega_1^\varepsilon} \xrightarrow{2-sc} u^{(1)}\chi_{E_1}, \qquad u_\varepsilon^{(2)}\chi_{\Omega_2^\varepsilon} \xrightarrow{2-sc} u^{(2)}\chi_{E_2}, \quad (97)$$

and, by (9), (10), (14) and (15), it follows that there exist υ_1, υ_2 such that

$$T(u_\varepsilon^{(1)}) \to \upsilon_1\,, \qquad T(u_\varepsilon^{(2)}) \to \upsilon_2 \quad \text{strongly in } L^2(\Omega). \quad (98)$$

Finally, we obtain

$$\upsilon_1(x) = u^{(1)}(x), \qquad \upsilon_2(x) = u^{(2)}(x), \qquad \text{for a.e. } x \in \Omega. \quad (99)$$

In fact, for $i = 1, 2$, we have that $T(u_\varepsilon^{(i)})\chi_{\Omega_i^\varepsilon} \xrightarrow{2-sc} \upsilon_i\chi_{E_i}$, since $T(u_\varepsilon^{(i)}) \to \upsilon_i$ strongly in $L^2(\Omega)$. Hence, by (97), it follows

$$\int_\Omega u^{(i)}|E_i|\varphi\,\mathrm{d}x \leftarrow \int_\Omega u_\varepsilon^{(i)}\chi_{\Omega_i^\varepsilon}\varphi\,\mathrm{d}x = \int_\Omega T(u_\varepsilon^{(i)})\chi_{\Omega_i^\varepsilon}\varphi\,\mathrm{d}x \to \int_\Omega \upsilon_i|E_i|\varphi\,\mathrm{d}x\,,$$

for every $\varphi \in \mathcal{C}_c^1(\Omega)$. Therefore, we have proved that $\upsilon_i = u^{(i)}$ a.e. in Ω; i.e.,

$$T(u_\varepsilon^{(1)}) \to u^{(1)}\,, \qquad T(u_\varepsilon^{(2)}) \to u^{(2)} \quad \text{strongly in } L^2(\Omega). \quad (100)$$

We remark also that, arguing as in (71)–(73), both for $u^{(1)}$ and $u^{(2)}$, we get $u^{(1)} = u^{(2)} = 0$ on $\partial\Omega$.

We are now ready to deal with the right hand side of (94). Taking into account that the integrands in J^1_ε and J^2_ε can be assumed positive, we can estimate from above each J^i_ε, $i = 1, 2$, with the integral over the whole Ω. Therefore, reasoning as in (38), we obtain that, as $\varepsilon \to 0$,

$$J^1_\varepsilon \to 0 \qquad \text{and} \qquad J^2_\varepsilon \to 0.$$

On the other hand, we rewrite I^i_ε, $i = 1, 2$, in the following way

$$I^i_\varepsilon = \int_{\Omega^\varepsilon_i \cap \{0 \le u_\varepsilon < \delta\}} \frac{f}{u^\theta_\varepsilon} \varphi_i \,\mathrm{d}x + \int_{\Omega^\varepsilon_i \cap \{u_\varepsilon \ge \delta\}} \frac{f}{u^\theta_\varepsilon} \varphi_i \,\mathrm{d}x := I^{i,1}_{\varepsilon,\delta} + I^{i,2}_{\varepsilon,\delta}\,.$$

We can adapt the same argument used in the case $\alpha = 1$ for the term $I^{i,1}_{\varepsilon,\delta}$. In particular, as in the proof of Theorem 4.1, we take $Z_\delta(u_\varepsilon)\varphi_i$ as test function in (13) with Z_δ defined in (41) and we assume $\varphi_i \ge 0$, obtaining

$$\begin{aligned}
I^{i,1}_{\varepsilon,\delta} \;&\le\; \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi_i Z_\delta(u_\varepsilon)\,\mathrm{d}x = \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi_i Z_\delta(u_\varepsilon)\big(\chi_{\Omega^\varepsilon_1} + \chi_{\Omega^\varepsilon_2}\big)\,\mathrm{d}x \\
&= \sum_{k=1}^2 \int_\Omega \lambda^k_\varepsilon \nabla u^{(k)}_\varepsilon \cdot \nabla\varphi_i Z_\delta(u^{(k)}_\varepsilon)\chi_{\Omega^\varepsilon_k}\,\mathrm{d}x \\
&= \sum_{k=1}^2 \int_\Omega \lambda^k_\varepsilon \nabla T(u^{(k)}_\varepsilon) \cdot \nabla\varphi_i Z_\delta(T(u^{(k)}_\varepsilon))\chi_{\Omega^\varepsilon_k}\,\mathrm{d}x \\
&= \sum_{k=1}^2 \int_\Omega \lambda^k_\varepsilon \nabla T(u^{(k)}_\varepsilon) \cdot \nabla\varphi_i \big(Z_\delta(T(u^{(k)}_\varepsilon)) - Z_\delta(u^{(k)})\big)\chi_{\Omega^\varepsilon_k}\,\mathrm{d}x \\
&+ \sum_{k=1}^2 \int_\Omega \lambda^k_\varepsilon \nabla T(u^{(k)}_\varepsilon) \cdot \nabla\varphi_i Z_\delta(u^{(k)})\chi_{\Omega^\varepsilon_k}\,\mathrm{d}x\,.
\end{aligned} \tag{101}$$

Recalling that $\lambda_\varepsilon \nabla T(u^{(k)}_\varepsilon)\chi_{\Omega^\varepsilon_k}$ is equi-bounded in $L^2(\Omega)$, using (100) in order to obtain that $Z_\delta(T(u^{(k)}_\varepsilon)) \to Z_\delta(u^{(k)})$ strongly in $L^2(\Omega)$, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \lambda_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi_i Z_\delta(u_\varepsilon)\,\mathrm{d}x = \\
&= \sum_{k=1}^2 \int_{\Omega\times E^k} \lambda^k \left(\nabla u^{(k)} + \nabla_y u^{(k)}_1\right) \nabla\varphi_i Z_\delta(u^{(k)})\,\mathrm{d}x\,\mathrm{d}y\,,
\end{aligned}$$

where we have taken into account (76), (77) and (96). Hence,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{i,1}_{\varepsilon,\delta} \le \sum_{k=1}^{2}\int_{(\Omega\cap\{u^{(k)}=0\})\times E^k} \left|\lambda^k(\nabla u^{(k)} + \nabla_y u_1^{(k)})\right| |\nabla\varphi_i|. \tag{102}$$

By Remark 4.11, for $k = 1, 2$, we may rewrite

$$\begin{aligned}&\int_{(\Omega\cap\{u^{(k)}=0\})\times E^k} \left|\lambda^k(\nabla u^{(k)} + \nabla_y u_1^{(k)})\right| |\nabla\varphi_i| = \\ = &\int_{(\Omega\cap\{u^{(k)}=0\})\times E^k} \left|\lambda^k(I + \nabla_y \chi^{(k)})\nabla u^{(k)}\right| |\nabla\varphi_i| = 0\,,\end{aligned}$$

because $\nabla u^{(k)}$ vanishes on $\{u^{(k)} = 0\}$. Therefore, we conclude

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{i,1}_{\varepsilon,\delta} = 0\,. \tag{103}$$

We now focus our attention on the term $I^{i,2}_{\varepsilon,\delta}$. We have

$$I^{i,2}_{\varepsilon,\delta} = \int_{\Omega_i^\varepsilon\cap\{u_\varepsilon^{(i)}\ge\delta\}} \frac{f}{(u_\varepsilon^{(i)})^\theta}\varphi_i\,\mathrm{d}x = \int_\Omega \frac{f}{(T(u_\varepsilon^{(i)}))^\theta}\chi_{\Omega_i^\varepsilon}\chi_{\{T(u_\varepsilon^{(i)})\ge\delta\}}\varphi_i\,\mathrm{d}x\,. \tag{104}$$

Since $0 \le \frac{f}{(T(u_\varepsilon^{(i)}))^\theta}\varphi_i \le \frac{f}{\delta^\theta}\varphi_i \in L^1(\Omega)$ in the set $\{T(u_\varepsilon^{(i)}) \ge \delta\}$ and $\chi_{\Omega_i^\varepsilon} \rightharpoonup |E^i|$ weakly* in $L^\infty(\Omega)$, we can argue as in (40), once we have taken $\delta \notin \mathcal{C} = \bigcup_{k=1}^2\{\delta > 0\ :\ |\{u^{(k)}(x) = \delta\}| > 0\}$, which is at most countable. Thus we obtain

$$\lim_{\varepsilon\to 0} I^{i,2}_{\varepsilon,\delta} = |E_i|\int_\Omega \frac{f}{(u^{(i)})^\theta}\chi_{\{u^{(i)}>\delta\}}\varphi_i\,\mathrm{d}x\,. \tag{105}$$

Finally, by (94), (95), (103) and (105), we arrive at

$$\begin{aligned}&\int_{\Omega\times E_1}\lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)})\cdot\nabla\varphi_1\,\mathrm{d}x\,\mathrm{d}y \\ +&\int_{\Omega\times E_2}\lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)})\cdot\nabla\varphi_2\,\mathrm{d}x\,\mathrm{d}y \\ +&\int_{\Omega\times E_1}\lambda_1(\nabla u^{(1)} + \nabla_y u_1^{(1)})\cdot\nabla_y\Phi_1\,\mathrm{d}x\,\mathrm{d}y \\ +&\int_{\Omega\times E_2}\lambda_2(\nabla u^{(2)} + \nabla_y u_1^{(2)})\cdot\nabla_y\Phi_2\,\mathrm{d}x\,\mathrm{d}y + \beta\int_{\Omega\times\Gamma}[u][\varphi]\,\mathrm{d}x\,\mathrm{d}\sigma(y) \\ =\ & |E_1|\int_\Omega\frac{f}{(u^{(1)})^\theta}\chi_{\{u^{(1)}>0\}}\varphi_1\,\mathrm{d}x + |E_2|\int_\Omega\frac{f}{(u^{(2)})^\theta}\chi_{\{u^{(2)}>0\}}\varphi_2\,\mathrm{d}x\,.\end{aligned} \tag{106}$$

Choosing $\varphi_1, \varphi_2, \Phi_1, \Phi_2$ respectively equal to 0 in (106), we obtain (80)–(84) and (86) with $\frac{f}{(u^{(i)})^\theta}$ replaced by $\frac{f}{(u^{(i)})^\theta}\chi_{\{u^{(i)}>0\}}$, $i = 1, 2$. Moreover, using the factorization of $u_1^{(1)}$ and $u_1^{(2)}$ given in Remark 4.11, we obtain that $(u^{(1)}, u^{(2)})$ solve the system (88), with the new sources $\frac{f}{(u^{(i)})^\theta}\chi_{\{u^{(i)}>0\}}$, $i = 1, 2$. In order to conclude the proof, we have to show that (79) and (85) hold, so that we can drop $\chi_{\{u^{(i)}>0\}}$ in (106). These properties will be proved in Subsection 5.2, Lemma 5.7. □

5. Appendix

5.1. Existence and uniqueness for the ε-problem

We devote this subsection to prove the existence and uniqueness for problem (11), following the ideas in [13] as done in [17, Theorem 4.1]. The main difference in the present case is the underline geometrical setting, which requires different a-priori estimates and a more careful approach to obtain that the weak solution of problem (11) is nonnegative a.e. in Ω, which is the essential step to achieve its strict positivity. For this reason and for convenience of the reader, we will give a sketch of the proof.

Since here ε is fixed, we will omit it so that, similarly to Section 2, we rewrite $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ and

$$V_0(\Omega) = \{u : \Omega \to \mathbb{R} \,|\, u^{(1)} \in H^1(\Omega_1),\, u^{(2)} \in H^1(\Omega_2),\, u = 0 \text{ on } \partial\Omega\},$$

endowed with the norm defined by

$$\|u\|_{V_0(\Omega)} := \|\nabla u\|_{L^2(\Omega_1\cup\Omega_2)} + \|[u]\|_{L^2(\Gamma)} .$$

Moreover, we denote by

$$\mathfrak{L}_0(\Omega) = \{\varphi : \Omega \to \mathbb{R} \,|\, \varphi^{(1)} \in \mathrm{Lip}(\overline{\Omega}_1), \varphi^{(2)} \in \mathrm{Lip}(\overline{\Omega}_2), \varphi = 0 \text{ on } \partial\Omega\}.$$

Finally, we set $\lambda(x) = \lambda_1$ a.e. in Ω_1 and $\lambda(x) = \lambda_2$ a.e. in Ω_2.

Theorem 5.1. *Assume that $f \in L^{\frac{2}{1+\theta}}(\Omega)$, $\theta \in (0, 1)$, and $f \geq 0$ a.e. in Ω, with f not identically zero both in Ω_1 and in Ω_2. Then, the problem*

$$\begin{aligned} &\left|\int_\Omega \frac{f}{u^\theta}\psi \,\mathrm{d}x\right| < +\infty\,, \\ &\int_\Omega \lambda\nabla u\cdot\nabla\psi \,\mathrm{d}x + \beta\int_\Gamma [u][\psi]\,\mathrm{d}\sigma = \int_\Omega \frac{f}{u^\theta}\psi\,\mathrm{d}x\,, \qquad \forall\psi\in V_0(\Omega)\,, \end{aligned} \tag{107}$$

admits a unique solution $u \in V_0(\Omega)$, with $u > 0$ a.e. in Ω.

In order to prove the previous result, we first need a preliminary existence result for a sequence of approximating problems. To this purpose, for $n \in \mathbb{N}$, we set

$$f_n(x) = \min\{f(x), n\} \tag{108}$$

and we consider the problem to find $u_n \in V_0(\Omega)$ satisfying the system

$$\begin{aligned} -\operatorname{div}(\lambda \nabla u_n) &= \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta}, && \text{in } \Omega_1 \cup \Omega_2; \\ [\lambda \nabla u_n \cdot \nu] &= 0, && \text{on } \Gamma; \\ \beta[u_n] &= \lambda \nabla u_n^{(2)} \cdot \nu, && \text{on } \Gamma; \\ u_n &\geq 0, && \text{in } \Omega; \\ u_n &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{109}$$

which corresponds to find a nonnegative solution of the weak formulation

$$\int_\Omega \lambda \nabla u_n \cdot \nabla \psi \, \mathrm{d}x + \beta \int_\Gamma [u_n][\psi] \, \mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta} \psi \, \mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega). \tag{110}$$

Theorem 5.2. *The problem* (109) *admits a unique nonnegative solution* $u_n \in V_0(\Omega)$.

Proof. Let $w \in L^2(\Omega)$ be fixed. For any $n \in \mathbb{N}$ we consider the following nonsingular linear problem

$$\begin{aligned} -\operatorname{div}(\lambda \nabla u_n) &= \frac{f_n}{\left(|w| + \frac{1}{n}\right)^\theta}\,, && \text{in } \Omega_1 \cup \Omega_2\,; \\ [\lambda \nabla u_n \cdot \nu] &= 0\,, && \text{on } \Gamma\,; \\ \beta[u_n] &= \lambda_2 \nabla u_n^{(2)} \cdot \nu\,, && \text{on } \Gamma\,; \\ u_n &= 0\,, && \text{on } \partial\Omega, \end{aligned} \tag{111}$$

whose weak formulation is

$$\int_\Omega \lambda \nabla u_n \cdot \nabla \psi \, \mathrm{d}x + \beta \int_\Gamma [u_n][\psi] \, \mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(|w| + \frac{1}{n}\right)^\theta} \psi \, \mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega). \tag{112}$$

Since the datum $(|w| + \frac{1}{n})^{-\theta} f_n$ is bounded by $n^{1+\theta}$, there exists a unique solution $u_n \in V_0(\Omega)$, as a consequence of the well-known Lax-Milgram Lemma. Moreover, by standard energy estimates and by Poincaré's inequality (2.2), there exists a positive constant C, depending on n but not on w, such that

$$||u_n||_{L^2(\Omega)} \leq C ||u_n||_{V_0(\Omega)} \leq C. \tag{113}$$

In order to prove the existence of a solution to problem (109), we will use Schauder's Theorem. To this purpose we introduce the map $F : L^2(\Omega) \to L^2(\Omega)$ defined by $F(w) = u_n$, where u_n is the solution of (111). Let B be the ball in $L^2(\Omega)$ of radius C, where C is the constant appearing in (113). Clearly $F(B) \subseteq B$. In order to apply the Schauder's Theorem, we need to prove that F is continuous and compact on B. The compactness of F follows by the fact that the inclusion of V_0 in $L^2(\Omega)$ is compact. In order to prove that F is continuous we proceed as follows. Let $\{w_r\} \subset B$ be a sequence in $L^2(\Omega)$ strongly converging to a function $w \in L^2(\Omega)$. We want to prove that $u_{n,r} := F(w_r)$ strongly converges in $L^2(\Omega)$ to $u_n = F(w)$, for $r \to +\infty$. Since w_r is strongly convergent in $L^2(\Omega)$ to w, we have also that, up to a subsequence, $w_r(x) \to w(x)$ for a.e. $x \in \Omega$ and therefore also $\left(|w_r| + \frac{1}{n}\right)^{-\theta} f_n$ converges to $\left(|w| + \frac{1}{n}\right)^{-\theta} f_n$ a.e. in Ω, which implies the strong convergence in $L^q(\Omega)$ for every $q \geq 1$. By (113) with u_n replaced by $u_{n,r}$ and the compactness of the inclusion of V_0 in $L^2(\Omega)$, it follows that there exists $u_n \in V_0$ such that, up to a subsequence,

$$
\begin{aligned}
&u_{n,r} \to u_n\,, && \text{strongly in } L^2(\Omega),\\
&\nabla u_{n,r} \rightharpoonup \nabla u_n\,, && \text{weakly in } L^2(\Omega),\\
&[u_{n,r}] \rightharpoonup [u_n]\,, && \text{weakly in } L^2(\Gamma).
\end{aligned}
$$

Notice that the last convergence in indeed strong. Passing to the limit in (112) written for $u_{n,r}$ and w_r, it follows that $u_n = F(w)$ and, by the uniqueness of the solution of problem (111)–(112), we have that the whole sequence $F(w_{n,r}) = u_{n,r} \to u_n = F(w)$, strongly in $L^2(\Omega)$, for $r \to +\infty$. Hence F is continuous and therefore there exists a fixed point u_n which is a solution of the problem

$$
\int_\Omega \lambda \nabla u_n \cdot \nabla \psi \,\mathrm{d}x + \beta \int_\Gamma [u_n][\psi] \,\mathrm{d}\sigma = \int_\Omega \frac{f_n}{\left(|u_n| + \frac{1}{n}\right)^\theta} \psi \,\mathrm{d}x\,, \qquad \forall \psi \in V_0(\Omega).
$$

The proof that u_n is nonnegative can be obtained following the same idea in [17, page 4062 - Proof of Theorem 4.1], where $-u_n^-$ is chosen as test function in the previous equation. A simple computations shows that $[u_n][-u_n^-] \geq 0$ a.e. on Γ, so that, taking into account that $f_n \geq 0$, we obtain

$$
\int_\Omega \lambda |\nabla u_n^-|^2 \,\mathrm{d}x \leq -\int_\Omega \lambda \nabla u_n \cdot \nabla u_n^- \,\mathrm{d}x + \beta \int_\Gamma [u_n][-u_n^-] \,\mathrm{d}\sigma \leq 0\,. \tag{114}
$$

In particular, (114) implies that $\nabla u_n^- = 0$ a.e. in Ω. Thus, in the connected/connected geometry, we get that $u_n^- = 0$ a.e. in Ω, because of the homogenous boundary condition and the fact that $\partial\Omega_1 \cap \partial\Omega \neq \emptyset$ as well as $\partial\Omega_2 \cap \partial\Omega \neq \emptyset$. On the contrary, in the connected/disconnected geometry, we have to pay more attention to the role of the jump part. Indeed, we can still assure that $u_n^- = 0$ a.e. in Ω_2 (i.e. $u_n \geq 0$ a.e. in Ω_2), since the boundary of Ω_2 still intersects $\partial\Omega$, where the homogenous condition is assigned. However, since in this case $\partial\Omega_1 \cap \partial\Omega = \emptyset$, from $\nabla u_n^- = 0$ a.e. in Ω, we can only deduce that u_n^- equals a nonnegative constant (let us write $u_n^- = \gamma^2$) a.e. in Ω_1. In order to assure that $\gamma^2 = 0$, assume, by contradiction, that $\gamma^2 > 0$. This implies that $u_n = -\gamma^2$ a.e. in Ω_1, being $u_n \in H^1(\Omega_1)$. Moreover, a.e. on Γ, we have $[u_n] = u_n^{(2)} - u_n^{(1)} = u_n^{(2)} + \gamma^2 > 0$ and $[-u_n^-] = -(u_n^{(2)})^- + (u_n^{(1)})^- = \gamma^2 > 0$. Hence, by (114), we get

$$\gamma^2 \int_\Gamma [u_n] \, \mathrm{d}\sigma = \int_\Gamma [u_n][-u_n^-] \, \mathrm{d}\sigma \leq 0 \, ,$$

and this is a contradiction, if we take into account that $[u_n] > 0$ a.e. on Γ and, thus, $\int_\Gamma [u_n] \, \mathrm{d}\sigma > 0$. Therefore, $\gamma^2 = 0$, so that $u_n^- = 0$ a.e. in Ω_1 and we have proved that $u_n \geq 0$ a.e. in Ω. Thus, u_n is a solution of problem (109).

Finally, the proof that the solution u_n is unique follows by [17, Proof of Theorem 4.5], taken into account that the function $s \in (0, +\infty) \mapsto \frac{1}{(s+1/n)^\theta}$ is decreasing. □

Proof of Theorem 5.1. Taking u_n as test function in (110) and using Poincaré's inequality (1), we obtain

$$\begin{aligned} \int_\Omega u_n^2 \, \mathrm{d}x \leq C \left(\int_\Omega |\nabla u_n|^2 \, \mathrm{d}x + \int_\Gamma [u_n]^2 \, \mathrm{d}\sigma \right) &\leq C \int_\Omega \frac{f_n}{\left(u_n + \frac{1}{n}\right)^\theta} u_n \, \mathrm{d}x \\ &\leq C ||f||_{L^{\frac{2}{1+\theta}}(\Omega)} ||u_n||_{L^2(\Omega)}^{1-\theta} \end{aligned} \quad (115)$$

and hence

$$||u_n||_{L^2(\Omega)} \leq C ||u_n||_{V_0} \leq C ||f||_{L^{\frac{2}{1+\theta}}(\Omega)}^{\frac{1}{1+\theta}} , \quad (116)$$

where C is independent of n. By (116) and the compactness of the inclusion of V_0 in $L^2(\Omega)$, we infer that there exists $u \in V_0$, $u \geq 0$ a.e. in Ω, such that, up to a subsequence,

$$\begin{array}{ll} u_n \to u\,, & \text{strongly in } L^2(\Omega); \\ \nabla u_n \rightharpoonup \nabla u\,, & \text{weakly in } L^2(\Omega); \\ [u_n] \rightharpoonup [u]\,, & \text{weakly in } L^2(\Gamma). \end{array} \tag{117}$$

Notice that the last convergence in actually strong. Moreover, by (110), with $\psi \in V_0(\Omega)$, and (115)-(116), we obtain

$$\left| \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\theta} \psi \,\mathrm{d}x \right| \le C\,,$$

so that, when $n \to +\infty$, by Fatou's Lemma it follows

$$\left| \int_\Omega \frac{f}{u^\theta} \psi \,\mathrm{d}x \right| \le C\,, \tag{118}$$

which also implies that u is not identically zero in Ω (neither in Ω_1 nor in Ω_2). Now, we can pass to the limit in the weak formulation (110). Clearly, the left-hand side converges to the left-hand side of (107). In order to pass to the limit in the right-hand side, we proceed again as in [17, Proof of Theorem 4.1 and Proposition 5.4] obtaining that u satisfies

$$\int_\Omega \lambda \nabla u \cdot \nabla \psi \,\mathrm{d}x + \beta \int_\Gamma [u][\psi] \,\mathrm{d}\sigma = \int_\Omega \frac{f}{u^\theta} \chi_{\{u>0\}} \psi \,\mathrm{d}x\,,$$

for every $\psi \in V_0(\Omega)$. It remains to prove that $u > 0$ a.e. in Ω, in order to replace $\frac{f}{u^\theta}\chi_{\{u>0\}}$ with $\frac{f}{u^\theta}$. This is a direct consequence of the maximum principle (see [20, Theorem 8.19] and also [19, Proposition 3.5]) applied to u in Ω_1 and Ω_2, separately, recalling that (118) implies that u is not identically zero both in Ω_1 and in Ω_2. Indeed, in the connected/connected geometry the maximum principle can be applied since $\inf u = 0$ in each Ω_i, $i = 1, 2$ (being $u = 0$ in $\partial\Omega \cap \partial\Omega_i \neq \emptyset$, $i = 1, 2$). The same approach can be followed in the connected/disconnected geometry for the outer domain Ω_2, where we have $u = 0$ on $\partial\Omega \cap \partial\Omega_2 \neq \emptyset$. On the contrary in Ω_1, taking into account that u is nonnegative (being the strong L^2-limit of the sequence of nonnegative functions u_n) we should distinguish two different situations: or $\inf u > 0$ in Ω_1 and, therefore, there is nothing to prove, or $\inf u = 0$ in Ω_1 and in this case we can appeal again to the maximum principle. Finally, in a similar way as in Remark 4.2, the uniqueness of the solution can be proved taking into account that the function $s \in (0, +\infty) \mapsto \frac{1}{s^\theta}$ is decreasing.

□

5.2. Positivity of the bidomain homogenized solution

We devote this subsection to the proof of the strict positivity a.e. in Ω of the solution of the bidomain problem (80)–(86) obtained from the homogenization of the system (11) in the case $\alpha = -1$ (Lemma 5.7 below). This result can be obtained from (18), by following the approach used in [28, Section 1], which exploits the notion of *two-scale decomposition* introduced in [27], in order to prove the lower semicontinuity of a suitable functional, which implies as a by-product, the requested positivity of the limit solution.

However, due to the special factorized form of the integral in the left-hand side of (18), we prefer to give a direct proof based on the unfolding homogenization technique which, in this case, essentially corresponds to the *two-scale decomposition.*

To this purpose, we recall the definition and those properties of the unfolding operator which are necessary in order to achieve our result (see [15, 16]). Let us set

$$\Xi_\varepsilon = \left\{\xi \in \mathbb{Z}^N , \quad \varepsilon(\xi + Y) \subset \Omega\right\} , \quad \widehat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\} .$$

Denoting by $[r]$ the integer part of $r \in \mathbb{R}$, we define for $x \in \mathbb{R}^N$

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right], \dots, \left[\frac{x_N}{\varepsilon}\right] \right), \quad \text{so that} \qquad x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y \right) .$$

Definition 5.3. *For w Lebesgue-measurable on Ω the periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined as*

$$\mathcal{T}_\varepsilon(w)(x, y) = \begin{cases} w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) , & (x, y) \in \widehat{\Omega}_\varepsilon \times Y , \\ 0 , & \textit{otherwise.} \end{cases}$$

Clearly, $\mathcal{T}_\varepsilon$ is linear and for w_1, w_2 as in Definition 5.3

$$\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1)\mathcal{T}_\varepsilon(w_2) . \tag{119}$$

Proposition 5.4. *Let $w \in L^1(\Omega)$, then*

$$\int_{\Omega\times Y} |\mathcal{T}_\varepsilon(w)| \,\mathrm{d}x \,\mathrm{d}y \le \int_\Omega |w| \,\mathrm{d}x . \tag{120}$$

Proposition 5.5. *Let $\{w_\varepsilon\}$ be a sequence of functions in $L^p(\Omega)$, $p > 1$. If $w_\varepsilon \to w$ strongly in $L^p(\Omega)$ as $\varepsilon \to 0$, then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w\,, \quad \textit{strongly in}\, L^p(\Omega \times Y)\,. \tag{121}$$

Proposition 5.6. *Let $\phi : Y \to \mathbb{R}$ be a function extended by Y-periodicity to the whole of $\mathbb{R}^N$ and define the sequence*

$$\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)\,, \qquad x \in \mathbb{R}^N\,. \tag{122}$$

If ϕ is measurable on Y, then

$$\mathcal{T}_\varepsilon(\phi^\varepsilon)(x,y) = \begin{cases} \phi(y)\,, & (x,y) \in \widehat{\Omega}_\varepsilon \times Y\,, \\ 0\,, & \textit{otherwise.} \end{cases} \tag{123}$$

Moreover, if $\phi \in L^p(Y)$, $p > 1$, as $\varepsilon \to 0$

$$\mathcal{T}_\varepsilon(\phi^\varepsilon) \to \phi\,, \qquad \textit{strongly in}\, L^p(\Omega \times Y)\,. \tag{124}$$

Lemma 5.7. *Under the assumption of Theorem 4.10,*

$$\left|\int_\Omega \frac{f}{(u^{(i)})^\theta}\varphi\,\mathrm{d}x\right| < +\infty\,, \qquad \forall \varphi \in H_0^1(\Omega)\,, \quad i = 1,2\,, \tag{125}$$

holds and the functions $u^{(1)}$ and $u^{(2)}$ are strictly positive a.e. in Ω.

Proof. As in the proof of Theorem 4.10, let T denotes the extension operator. Recalling that, for a.e. $x \in \Omega$, $\chi_{\Omega_1^\varepsilon}(x) = \chi_{E_1}(\varepsilon^{-1}x)$ and $\chi_{\Omega_2^\varepsilon}(x) = \chi_{E_2}(\varepsilon^{-1}x)$, extended by periodicity from Y to the whole of $\mathbb{R}^N$, and taking into account (100) and the properties of the unfolding operator (119), (121) and (124), we have that

$$\begin{aligned}\mathcal{T}_\varepsilon(u_\varepsilon) &= \mathcal{T}_\varepsilon(u_\varepsilon\chi_{\Omega_1^\varepsilon} + u_\varepsilon\chi_{\Omega_2^\varepsilon}) = \mathcal{T}_\varepsilon\left(T(u_\varepsilon^{(1)})\chi_{\Omega_1^\varepsilon} + T(u_\varepsilon^{(2)})\chi_{\Omega_2^\varepsilon}\right) \\ &= \mathcal{T}_\varepsilon\left(T(u_\varepsilon^{(1)})\right)\mathcal{T}_\varepsilon\left(\chi_{\Omega_1^\varepsilon}\right) + \mathcal{T}_\varepsilon\left(T(u_\varepsilon^{(2)})\right)\mathcal{T}_\varepsilon\left(\chi_{\Omega_2^\varepsilon}\right) \\ &\longrightarrow u^{(1)}\chi_{E_1} + u^{(2)}\chi_{E_2}\,, \qquad \text{strongly in } L^1(\Omega \times Y).\end{aligned}$$

Therefore, there exists a set $\mathcal{N} \subset \Omega \times Y$, with $|\mathcal{N}| = 0$, such that

$$\mathcal{T}_\varepsilon(u_\varepsilon)(x,y) \to u^{(1)}(x)\chi_{E_1}(y) + u^{(2)}(x)\chi_{E_2}(y)$$

for every $(x, y) \in (\Omega \times Y) \setminus \mathcal{N}$. Then, by (18) with $\psi \in \mathcal{C}_c^1(\Omega)$, $\psi \geq 0$, (14) and applying Fatou's Lemma, we get

$$\begin{aligned}\int_{\Omega\times Y} \frac{f}{(u^{(1)}\chi_{E_1} + u^{(2)}\chi_{E_2})^\theta}\psi \,\mathrm{d}x \,\mathrm{d}y &\leq \liminf_{\varepsilon\to 0} \int_{\Omega\times Y} \frac{\mathcal{T}_\varepsilon(f)}{\mathcal{T}_\varepsilon(u_\varepsilon)^\theta}\mathcal{T}_\varepsilon(\psi) \,\mathrm{d}x \,\mathrm{d}y \\ = \liminf_{\varepsilon\to 0} \int_{\Omega\times Y} \mathcal{T}_\varepsilon\left(\frac{f}{u_\varepsilon^\theta}\psi\right) \,\mathrm{d}x \,\mathrm{d}y &\leq \liminf_{\varepsilon\to 0} \int_\Omega \frac{f}{u_\varepsilon^\theta}\psi \,\mathrm{d}x \leq C\,, \end{aligned} \tag{126}$$

where we used also (120). Inequality (126) implies, in particular,

$$|E_i| \int_\Omega \frac{f}{(u^{(i)})^\theta}\psi \,\mathrm{d}x = \int_{\Omega\times E_i} \frac{f}{(u^{(i)}\chi_{E_i})^\theta}\psi \,\mathrm{d}x \,\mathrm{d}y \leq C\,, \qquad i = 1, 2\,; \tag{127}$$

thus, (125) is proved and hence, taking into account that $f > 0$ a.e. in Ω, (127) implies $u^{(i)} > 0$ a.e. in Ω, $i = 1, 2$. □

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