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Advantages of the mixed format in geometrically nonlinear analysis of beams and shells using solid finite elements

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SUMMARY

The paper deals with two main advantages in the analysis of slender elastic structures both achieved through the mixed (stress and displacement) format with respect to the more commonly used displacement one: i) the smaller error in the extrapolations usually employed in the solution strategies of nonlinear problems; ii) the lower polynomial dependence of the problem equations on the finite element degrees of freedom when solid finite elements are used. The smaller extrapolation error produces a lower number of iterations and larger step length in path-following analysis and a greater accuracy in Koiter asymptotic method. To focus on the origin of the phenomenon the two formats are derived for the same finite element interpolation. The reduced polynomial dependence improves the Koiter asymptotic strategy in terms of both computational efficiency, accuracy and simplicity.

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1. INTRODUCTION

In recent years an increasing amount of research has aimed at developing new efficient solid finite elements [**?**] for the linear and nonlinear analysis of thin structures. This is due to some advantages of solid elements in comparison to classical shell elements. In particular in the elastic nonlinear analysis of slender structures they allow the use of the 3D continuum strain and stress measures employing translational degrees of freedom only [**?**, **?**, **?**, **?**]. In this way it is possible to avoid the use of complicated and expensive rules for updating the rotations and, by using the Green-Lagrange strain measure, to coherently describe the structural behavior through a low order dependence on the displacement field without the need to employ complex, geometrically exact formulations, which are not always available or accurate [**?**, **?**, **?**]. In this way solid elements allow a simpler expression of the strain energy and its variations with a gain in computational efficiency.

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However, formulating robust solid-shell elements is more demanding than shell elements. To maintain an acceptable number of degrees of freedom, the elements proposed are usually based on a low order displacement interpolation. Consequently they have the disadvantages of *interpolation lockings*: the shear and membrane locking also present in classical shell elements and trapezoidal and thickness locking, typical of low order solid-shell elements [**?**]. Interpolation lockings are usually rectified by means of Assumed Natural Strain, Enhanced Assumed Strain [**?**, **?**, **?**, **?**] and mixed (stress-displacement) formulations [**?**, **?**, **?**, **?**]. In this way solid-shell elements have now reached a high level of efficiency and accuracy and have also been used to model composites or laminated beams [**?**, **?**, **?**, **?**] and shell structures in both the linear [**?**, **?**] and nonlinear [**?**, **?**, **?**] range. Among the most effective and interesting proposals we refer to are the mixed solid-shell elements of Sze and coauthors [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] which extend the initial PT18β hybrid element of Pian and Tong to thin shell.

When comparing mixed and displacement finite elements many authors (see for example [**?**] and [**?**]) observe that the mixed ones are more robust and allow larger steps in path–following geometrically nonlinear analyses. However the reasons for these better performances are, in our opinion, not clear, as they are often wrongly attributed to the properties of the finite element interpolation. One of the goals of this paper is therefore to clarify the true reason and origin of this phenomenon, extending the results presented some years ago [**?**, **?**] to the context of path-following and Koiter [**?**, **?**, **?**] asymptotic analyses of 2D framed structures.

Mixed and displacement descriptions, while completely equivalent at the continumm level, behave very differently when implemented in path-following and asymptotic solution strategies even when they are based on the same finite element interpolations, that is when they are equivalent also at the discrete level. This is an important, even if frequently misunderstood, point in developing numerical algorithms and it has been discussed in [**?**, **?**, **?**] to which we refer readers for more details.

Since the solution strategies of a nonlinear problem usually involve extrapolations or linearizations, a *smooth enough* description of the problem that makes the extrapolation error as small as possible is crucial. Mixed and displacement descriptions are characterized by a different extrapolation errors and so they behave very differently when used within a numerical solution process. For shells or beams, in the presence of large displacements (rigid rotations) and high membranal/flexural stiffness ratios, the extrapolation with the displacement description is affected by a large error that causes: i) a very slow convergence rate in path-following analysis; ii) an unreliable estimate of the bifurcation point along an extrapolated fundamental path and then a low accuracy of the Koiter method based on an asymptotic expansion in this point. As will be shown, the mixed description is unaffected by this phenomenon that we call *extrapolation locking*. Note that it is a locking of the nonlinear problem when described in a displacement format and it not related to the FE interpolation. We use the term locking in analogy to the interpolation locking of the finite element, because it produces an overestimated extrapolated stiffness which gets worse with the slenderness of the structure. On the contrary the mixed format of the nonlinear problem is unaffected by the extrapolation locking and this ensures: in path-following analysis, a fast convergence of the Newton (Riks) iterative process; in asymptotic analysis, which uses extrapolations which are not corrected by an iterative process [**?**, **?**], an accurate recovery of the equilibrium path.

In this paper a mixed and a displacement description are derived for the same finite element, so obtaining two completely equivalent discrete problems, in order to show that their different behavior

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is not due to the interpolation fields and that the extrapolation locking occurs for any displacement finite element. This allows us to thoroughly investigate this important phenomenon which has not been taken into account by the scientific community.

Another important advantage is related to the minimum strain energy dependence on the finite element (FE) discrete variables when solid elements based on the quadratic Green-Lagrange strain measure are employed: the fourth order dependence on displacement variables in the displacement formulation and the third order in stress and displacement variables in the mixed case. This has a significant effect on the efficiency, robustness and coherence of the asymptotic analysis when the mixed description is used. It allows, in fact, the zeroing of all the strain energy variations of an order greater than the third and, consequently, permits light numerical formulations and an improvement in accuracy. In this way it is possible to develop new asymptotic algorithms, which are more accurate and computationally efficient than those based on classical shell elements, well suited to the imperfection sensitivity analysis of structures presenting coincident or almost coincident buckling loads.

Finally it is worth mentioning that the use of both displacement and stress variables increases the dimension of the problem, but generally the computational extra–cost, with respect to a displacement analysis, is very low. This is because the global operations involve displacement dofs only by performing a static condensation of the stress variables defined at element level. This small computational extra-cost is largely compensated: in path-following analysis, by larger steps and fewer iterations with respect to the displacement case; in asymptotic analysis, by the zeroing of the computationally expensive fourth order strain energy variations. We will also show how the slow change of the Jacobian matrix when expressed in mixed variables allows, in path following analyses, an efficient use of the modified Newton method with a further significant reduction in the computational cost.

To summarize the paper deals with two important advantages, both achieved with the mixed format with respect to the commonly used displacement one: i) the smaller error in the extrapolations usually employed in the solution of nonlinear problems; ii) the lower polynomial dependence of the problem equations on the FE degrees of freedom when a solid finite element is used. The paper is organized as follows: Section 2 briefly reviews asymptotic and path-following methods; Section 3 presents the mixed and displacement descriptions based on the solid finite element and the advantages of mixed solid elements in Koiter analysis; Section 4 describes why extrapolation locking phenomenon occurs for slender structures and its effect on the two solution algorithms adopted; Section 5 presents some numerical tests; finally in Section 6 the conclusions are reported.

2. NUMERICAL STRATEGIES IN NONLINEAR FEM ANALYSIS

In this section we briefly summarize the path following and asymptotic methods. We refer readers to [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] for a complete review of both the approaches.

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2.1. The discrete nonlinear equations

We consider a slender hyperelastic structure subject to conservative loads $p[\lambda]$ proportionally increasing with the amplifier factor λ. The equilibrium is expressed by the virtual work equation

$$\Phi[u]'\,\delta u - \lambda\hat{p}\,\delta u = 0 \quad , \quad u \in \mathcal{U}\ ,\ \delta u \in \mathcal{T} \tag{1}$$

where $u \in \mathcal{U}$ is the field of configuration variables, $\Phi[u]$ denotes the strain energy, $\mathcal{T}$ is the tangent space of $\mathcal{U}$ at u and a prime is used to express the Frechèt derivative with respect to u. We assume that $\mathcal{U}$ will be a linear manifold so that its tangent space $\mathcal{T}$ will be independent of u. When a mixed format is adopted the configuration variables u collect both displacement and stress fields. Eq.(1) can be rewritten, using a FE discretization $u = \mathbf{N}\mathbf{u}$ as

$$\mathbf{r}[\mathbf{u},\,\lambda] \equiv \mathbf{s}[\mathbf{u}] - \lambda\,\hat{\mathbf{p}} = \mathbf{0}, \quad \text{with} \quad \begin{cases} \mathbf{s}^T\delta\mathbf{u} \equiv \Phi'[u]\delta u \\ \hat{\mathbf{p}}^T\delta\mathbf{u} \equiv \hat{p}\,\delta u \end{cases} \tag{2}$$

where $\mathbf{r} : \mathbb{R}^{N+1} \to \mathbb{R}^N$ is a nonlinear vectorial function of the vector $\mathbf{z} \equiv \{\mathbf{u}, \lambda\} \in \mathbb{R}^{N+1}$, collecting the configuration $\mathbf{u} \in \mathbb{R}^N$ and the load multiplier $\lambda \in \mathbb{R}$, $\mathbf{s}[\mathbf{u}]$ is the *response vector* and $\hat{\mathbf{p}}$ the *reference load vector*. Eq.(2) represents a system of N-equations and $N+1$ unknowns and defines the *equilibrium path* as a curve in $\mathbb{R}^{N+1}$ from a known initial configuration $\mathbf{u}_0$, corresponding to $\lambda = 0$. In the following we also define the tangent stiffness matrix as

$$\delta\mathbf{u}_2^T\mathbf{K}[\mathbf{u}]\delta\mathbf{u}_1 = \Phi''[u]\delta u_1\delta u_2 \ \ , \ \ \forall\,\delta\mathbf{u}_1, \delta\mathbf{u}_2 \tag{3}$$

where δu_i are generic variations of the configuration field u and $\delta\mathbf{u}_i$ the corresponding FE vectors.

2.2. Path–following analysis

The Riks approach [**?**] completes the equilibrium equations (2) with the additional constraint $g[\mathbf{u}, \lambda] - \xi = 0$ which defines a surface in $\mathbb{R}^{N+1}$. Assigning successive values to the control parameter $\xi = \xi^{(k)}$ the solution of the nonlinear system

$$\mathbf{R}[\xi] \equiv \begin{bmatrix} \mathbf{r}[\mathbf{u}, \lambda] \\ g[\mathbf{u}, \lambda] - \xi \end{bmatrix} = \mathbf{0} \tag{4}$$

defines a sequence of points (steps) $\mathbf{z}^{(k)} \equiv \{\mathbf{u}^{(k)},\,\lambda^{(k)}\}$ belonging to the equilibrium path. Starting from a known equilibrium point $\mathbf{z}_0 \equiv \mathbf{z}^{(k)}$ the new one $\mathbf{z}^{(k+1)}$ is evaluated correcting a first *extrapolation* $\mathbf{z}_1 = \{\mathbf{u}_1,\,\lambda_1\}$ by a sequences of estimates $\mathbf{z}_j$ (loops) by a Newton–Raphson iteration

$$\begin{cases} \tilde{\mathbf{J}}\dot{\mathbf{z}} = -\mathbf{R}_j \\ \mathbf{z}_{j+1} = \mathbf{z}_j + \dot{\mathbf{z}} \end{cases} \tag{5a}$$

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where $\mathbf{R}_j \equiv \mathbf{R}[\mathbf{z}_j]$ and $\tilde{\mathbf{J}}$ is the Jacobian of the nonlinear system (4) at $\mathbf{z}_j$ or its suitable estimate. The simplest choice for $g[\mathbf{u}, \lambda]$ is the linear constraint corresponding to the orthogonal hyperplane

$$\mathbf{n}_u^T(\mathbf{u} - \mathbf{u}_1) + n_\lambda\,(\lambda - \lambda_1) = \Delta\xi \quad \text{where} \quad \begin{cases} \mathbf{n}_u \equiv \mathbf{M}\,(\mathbf{u}_1 - \mathbf{u}^{(k)}) \\ \mathbf{n}_\lambda \equiv \mu\,(\lambda_1 - \lambda^{(k)}) \end{cases} \tag{5b}$$

$\mathbf{M}$ and μ being some suitable metric factors [**?**, **?**], $\Delta\xi$ an assigned increment of ξ and

$$\tilde{\mathbf{J}} \approx \left[\frac{\partial \mathbf{R}[\mathbf{z}]}{\partial \mathbf{z}}\right]_{\mathbf{z}_j} = \begin{bmatrix} \tilde{\mathbf{K}} & -\hat{\mathbf{p}} \\ \mathbf{n}_u^T & n_\lambda \end{bmatrix} \tag{5c}$$

The standard load controlled scheme is obtained assuming $g[\mathbf{u}, \lambda] = \lambda$ (see [**?**] for further details) while keeping $\tilde{\mathbf{K}} = \mathbf{K}[\mathbf{u}_1]$ we have the modified Newton-Raphson scheme.

2.2.1. Convergence of the path–following scheme. The convergence of the iterative process (5) has been widely discussed in [**?**] and can be expressed in the condition

$$\mathbf{R}_{j+1} = \left(\mathbf{I} - \mathbf{J}_s\tilde{\mathbf{J}}^{-1}\right)\mathbf{R}_j \tag{5d}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{J}_s \equiv \int_0^1 \mathbf{J}[\mathbf{z}_j + t(\mathbf{z}_{j+1} - \mathbf{z}_j)]\,dt$ the secant Jacobian matrix. The iteration converges if in some norm we have $\left\|\mathbf{I} - \mathbf{J}_s\tilde{\mathbf{J}}^{-1}\right\| < 1$ and it will be as fast as $\tilde{\mathbf{J}}$ is close to $\mathbf{J}_s$. Also note that the convergence condition for a load controlled scheme is obtained by replacing $\tilde{\mathbf{J}}$ and $\mathbf{J}_s$ with $\tilde{\mathbf{K}}$ and $\mathbf{K}_s$ respectively. For the displacement format in the case of positive definite $\tilde{\mathbf{K}}$ the convergence condition can be simplified as

$$0 < \mathbf{u}^T\mathbf{K}_s\mathbf{u} < 2\mathbf{u}^T\tilde{\mathbf{K}}\mathbf{u}, \quad \forall\mathbf{u} \tag{5e}$$

A convergence condition similar to Eq.(5e) but limited to the subspace of nonsingular values of $\tilde{\mathbf{K}}$ holds also for the arc-lenght scheme [**?**] that, like for the load controlled case, is as faster as

$$\mathbf{u}^T\mathbf{K}_s\mathbf{u} \approx \mathbf{u}^T\tilde{\mathbf{K}}\mathbf{u}, \quad \forall\mathbf{u} \tag{5f}$$

and it converges in a single iteration when $\mathbf{K}_s = \tilde{\mathbf{K}}$ because of the linearity of Eq.(5b).

The arc-length scheme provides a simple way to overcome limit points because $\tilde{\mathbf{J}}$ is not singular even when $\tilde{\mathbf{K}}$ is singular. The convergence is, however, strongly affected by the variables chosen to describe the problem since a smoother representation of the equilibrium path makes it easy to fulfill the condition (5f) allowing large steps and few loops. In the following we will show that this desirable behaviour occurs in the case of a mixed description while the displacement one, for any FE model, is affected by an *extrapolation locking* that could produce a pathological reduction in the step size (increase in iterations) and in some cases a loss of convergence.

2.2.2. Implementation details In the numerical tests of this paper the step length is evaluated in terms of the number of loops of the previous step (l) using the following expression $\Delta\xi^{(k)} = (1 - 0.7(l - dl)/(l + dl))\Delta\xi^{(k-1)}$ where dl is the number of desired loops set to 4. The constraints

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$0.5 \leq \Delta\xi^{(k)}/\Delta\xi^{(k-1)} \leq 2$ is are also added. Convergence is obtained when, in the Euclidean norm, $\|\mathbf{r}_j\| \leq 10^{-4}\|\Delta\lambda^{(0)}\mathbf{p}\|$ with $\Delta\lambda^{(0)}\mathbf{p}$ the first step load. When after 30 iterations the convergence is not achieved the step length is halved (false step). After 10 consecutive false steps a failure in convergence occurs.

2.3. *The asymptotic analysis*

The asymptotic approach, derived as a finite element implementation [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] of the Koiter theory of elastic stability [**?**] provides an effective and reliable strategy for predicting the initial post-critical behavior in both cases of limit or bifurcation points and makes the imperfection sensitivity analysis easy and affordable [**?**, **?**, **?**, **?**, **?**, **?**, **?**]. The solution process is based on a third order Taylor expansion of Eq.(1), in terms of load factor λ and modal amplitudes ξ_i. The steps of the algorithm are

1. The *fundamental path* is obtained as a linear extrapolation

$$\mathbf{u}^f[\lambda] := \mathbf{u}_0 + \lambda\hat{\mathbf{u}} \tag{6a}$$

where the initial path tangent $\hat{\mathbf{u}}$ is a solution of the linear system. Letting $\hat{u} := \mathbf{N}\hat{\mathbf{u}}$

$$(\Phi''[u_0]\hat{u} - p)\delta u = 0, \ \forall\delta u \quad \Rightarrow \quad \mathbf{K}_0\,\hat{\mathbf{u}} = \hat{\mathbf{p}} \ \ , \ \ \mathbf{K}_0 := \mathbf{K}[\mathbf{u}_0] \tag{6b}$$

2. A cluster of *buckling loads* λ_i, $i = 1\cdots m$ and associated *buckling modes* $\dot{v}_i := \mathbf{N}\dot{\mathbf{v}}_i$, are obtained along the extrapolated $\mathbf{u}^f[\lambda]$ from the critical condition

$$\Phi''[u_f[\lambda]]\dot{v}_i\delta u \equiv \delta\mathbf{u}^T\mathbf{K}[\lambda]\dot{\mathbf{v}}_i = 0 \quad i = 1\cdots m \quad \forall\delta\mathbf{u} \tag{6c}$$

Eq.(6c) defines the following nonlinear eigenvalue problem

$$\mathbf{K}[\lambda_i]\,\dot{\mathbf{v}}_i = \mathbf{0} \ \ , \ \ \mathbf{K}[\lambda] \equiv \mathbf{K}[\mathbf{u}_0 + \lambda\,\hat{\mathbf{u}}] \tag{6d}$$

that, in the multi-modal buckling case, is usually simplified by means of a linearization near $(\mathbf{u}_b, \lambda_b)$ [**?**],

$$\Phi''[u_f[\lambda]]\dot{v}_i\delta u \approx (\Phi_b'' + (\lambda - \lambda_b)\Phi_b'''\hat{u})\dot{v}_i\delta u \tag{6e}$$

where λ_b is a reference value for the cluster, the subscript "b" denotes quantities evaluated at $\mathbf{u}_b \equiv \mathbf{u}^f[\lambda_b]$ and, letting δ_{ij} the Kronecker symbol, the following normalization is used

$$\Phi_b'''\hat{u}\dot{v}_i\dot{v}_j = -\delta_{ij}. \tag{6f}$$

We will denote with $\mathcal{V} := \{\dot{\mathbf{v}} = \sum_{i=1}^m \xi_i\dot{\mathbf{v}}_i\}$ the subspace spanned by the buckling modes and $\mathcal{W} := \{w := \mathbf{N}\mathbf{w} : \Phi_b'''\hat{u}\dot{v}_i w = 0\,,\ i = 1\cdots m\}$ its orthogonal complement.

3. The asymptotic approximation for the required path is defined by the expansion

$$\mathbf{u}[\lambda, \xi_k] := \lambda\hat{\mathbf{u}} + \sum_{i=1}^m \xi_i\dot{\mathbf{v}}_i + \frac{1}{2}\sum_{i,j=1}^m \xi_i\xi_j\mathbf{w}_{ij} + \frac{1}{2}\lambda^2\hat{\hat{\mathbf{w}}} \tag{6g}$$

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where $\mathbf{w}_{ij}, \hat{\hat{\mathbf{w}}} \in \mathcal{W}$ are quadratic corrections introduced to satisfy the projection of the equilibrium equation (1) into $\mathcal{W}$ and obtained by the linear systems

$$\begin{aligned} &\delta\mathbf{w}^T(\mathbf{K}_b\mathbf{w}_{ij} + \mathbf{p}_{ij}) = 0 \\ &\delta\mathbf{w}^T(\mathbf{K}_b\hat{\hat{\mathbf{w}}} + \hat{\hat{\mathbf{p}}}) = 0 \end{aligned} \quad , \quad \forall\mathbf{w} \in \mathcal{W} \tag{6h}$$

where $\mathbf{K}_b \equiv \mathbf{K}[\lambda_b]$ and vectors $\mathbf{p}_{ij}$ and $\hat{\hat{\mathbf{p}}}$ are defined by the energy equivalence

$$\delta\mathbf{w}^T\mathbf{p}_{ij} = \Phi_b'''\dot{v}_j\dot{v}_j\,\delta w, \quad \delta\mathbf{w}^T\hat{\hat{\mathbf{p}}} = \Phi_b'''\hat{u}^2\,\delta w$$

4. The following energy terms are computed for $i, j, k = 1 \cdots m$:

$$\begin{aligned} \mathcal{A}_{ijk} &= \Phi_b'''\dot{v}_i\dot{v}_j\dot{v}_k \\ \mathcal{B}_{00ik} &= \Phi_b''''\hat{u}^2\dot{v}_i\dot{v}_k - \Phi_b''\hat{\hat{w}}w_{ik} \\ \mathcal{B}_{0ijk} &= \Phi_b''''\hat{u}\dot{v}_i\dot{v}_j\dot{v}_k \\ \mathcal{B}_{ijhk} &= \Phi_b''''\dot{v}_i\dot{v}_j\dot{v}_h\dot{v}_k - \Phi_b''(w_{ij}w_{hk} + w_{ih}w_{jk} + w_{ik}w_{jh}) \\ \mathcal{C}_{ik} &= \Phi_b''\hat{\hat{w}}w_{ik} \\ \mu_k[\lambda] &= \frac{1}{2}\lambda^2\Phi_b'''\hat{u}^2\dot{v}_k + \frac{1}{6}\lambda^2(\lambda_b - 3\lambda)\Phi_b''''\hat{u}^3\dot{v}_k \end{aligned} \tag{6i}$$

5. The equilibrium path is obtained by projecting the equilibrium equation (1) into $\mathcal{V}$ and assuming a coherent Taylor expansion in λ and ξ_i

$$\begin{aligned} \mu_k[\lambda] + (\lambda_k - \lambda)\xi_k - \lambda_b(\lambda - \frac{\lambda_b}{2})\sum_{i=1}^{m}\xi_i\mathcal{C}_{ik} + \frac{1}{2}\sum_{i,j=1}^{m}\xi_i\xi_j\mathcal{A}_{ijk} + \frac{1}{2}(\lambda - \lambda_b)^2\sum_{i=1}^{m}\xi_i\mathcal{B}_{00ik} \\ + \frac{1}{2}(\lambda - \lambda_b)\sum_{i,j=1}^{m}\xi_i\xi_j\mathcal{B}_{0ijk} + \frac{1}{6}\sum_{i,j,h=1}^{m}\xi_i\xi_j\xi_h\mathcal{B}_{ijhk} = 0, \quad k = 1 \cdots m \end{aligned} \tag{6j}$$

The equations (6j) are an algebraic nonlinear system of m equations in the $m+1$ variables λ, $\xi_1 \cdots \xi_m$, with known coefficients and solved using a path–following algorithm.

2.3.1. Remarks on Koiter analysis. Asymptotic analysis uses fourth order variations of the strain energy in an extrapolated bifurcation point and requires fourth order accuracy to be guaranteed. In the past, much effort has been devoted to developing geometrically exact structural models [**?**, **?**, **?**]. These models however use 3D finite rotations and consequently they have complex and expensive strain energy variations. Also, being based on the fundamental path extrapolation, the accuracy of the method is very sensitive to the format used in the problem description. Both these problems are solved naturally when using a mixed solid element which, furthermore, improves the computational efficiency and accuracy as shown in the next sections.

3. THE SOLID FINITE ELEMENT

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In this section, two equivalent descriptions, one in stresses and displacements called *mixed description* based on the Hellinger-Reissner functional and another, in displacement variables only, called *displacement description*, are derived for the mixed Pian and Tong finite element [?]. The use of the same interpolations makes it possible to directly compare the different formats maintaining the same discrete approximation. The framework proposed is easy to extend to other mixed solid or solid-shell elements [?, ?, ?]. Obviously the displacement description is natural when displacement based finite elements are employed. The dramatic improvement in efficiency due to the joint use of a solid element and mixed description in Koiter analysis is also discussed.

3.1. *Solid element equations in convective coordinates*

We consider a solid finite element and denote with $\boldsymbol{\zeta} = \{\zeta^1, \zeta^2, \zeta^3\}$ the convective coordinates used to express the FE interpolation. The initial configuration, assumed as reference, is described by the position vector $\mathbf{X}[\boldsymbol{\zeta}]$ while $\mathbf{x}[\boldsymbol{\zeta}]$ represents the same position in the current configuration. They are related by the transformation

$$\mathbf{x}[\boldsymbol{\zeta}] = \mathbf{X}[\boldsymbol{\zeta}] + \mathbf{d}[\boldsymbol{\zeta}] \tag{7}$$

where $\mathbf{d}[\boldsymbol{\zeta}]$ is the displacement field. The covariant (or convected) base vectors are obtained by partial derivatives of the position vectors with respect to the convective coordinates as $\mathbf{G}_i = \mathbf{X}_{,i}$ where the comma followed by i denotes differentiation with respect to ζ^i. The contravariant base vectors are defined by the orthonormality conditions $\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j$ where δ_i^j is the Kronecker symbol and a dot denotes the scalar product. Adopting the convention of summing on repeated indexes the Green-Lagrange strain measure in covariant components becomes

$$\boldsymbol{\varepsilon} = \bar{\varepsilon}_{ij}\left(\mathbf{G}^i \otimes \mathbf{G}^j\right) \quad \text{with} \quad \bar{\varepsilon}_{ij} = \frac{1}{2}\left(\mathbf{X}_{,i} \cdot \mathbf{d}_{,j} + \mathbf{d}_{,i} \cdot \mathbf{X}_{,j} + \mathbf{d}_{,i} \cdot \mathbf{d}_{,j}\right) \tag{8}$$

The second Piola-Kirchoff stress tensor in contravariant components is

$$\boldsymbol{\sigma} = \bar{\sigma}^{ij}\left(\mathbf{G}_i \otimes \mathbf{G}_j\right) \tag{9}$$

Finally the constitutive law is assumed to be linear as $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ with the elastic tensor $\mathbf{C}$ that, assuming an isotropic and homogeneous material, is expressed in the fixed global orthonormal reference frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ can be expressed in the fixed system in terms of the so-called *physical components*. For the stress we have

$$\boldsymbol{\sigma} = \sigma^{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \bar{\sigma}^{ij}\left(\mathbf{G}_i \otimes \mathbf{G}_j\right) \tag{10}$$

that exploiting the relation $\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j$ furnish

$$\sigma^{rs} = \bar{\sigma}^{ij} t_i^r t_j^s \quad \text{with} \quad t_i^r = \left(\mathbf{e}^r \cdot \mathbf{G}_i\right) = \frac{\partial X^r}{\partial \zeta^i} \tag{11}$$

or in matrix format by collecting the components of t_i^r in the Jacobiam matrix $\mathbf{J}_e$ we obtain $\boldsymbol{\sigma} = \mathbf{J}_e \bar{\boldsymbol{\sigma}} \mathbf{J}_e^T$. By adopting as usual a Voigt notation we can express stress and strain tensors in a vector form

$$\boldsymbol{\varepsilon} = \left[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{23}, 2\varepsilon_{13}, 2\varepsilon_{12}\right]^T, \quad \boldsymbol{\sigma} = \left[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\right]^T \tag{12}$$

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where the same symbol is used to denote both quantities in Voight or tensorial notation and we have

$$\boldsymbol{\sigma} = \mathbf{T}_\sigma \bar{\boldsymbol{\sigma}}, \qquad \boldsymbol{\varepsilon} = \mathbf{T}_\sigma^{-T} \bar{\boldsymbol{\varepsilon}} \tag{13}$$

where $\mathbf{T}_\sigma$ is defined by Eq.(11).

3.2. *The finite element interpolation*

The position vector of a point inside the element and its displacement are interpolated, using a trilinear 8 nodes hexahedron, as

$$\mathbf{X}[\boldsymbol{\zeta}] = \mathbf{N}_d[\boldsymbol{\zeta}]\mathbf{X}_e \quad , \quad \mathbf{d}[\boldsymbol{\zeta}] = \mathbf{N}_d[\boldsymbol{\zeta}]\mathbf{d}_e \tag{14a}$$

where vectors $\mathbf{d}_e$ and $\mathbf{X}_e$ collect the element nodal displacements and coordinates and matrix $\mathbf{N}_d[\boldsymbol{\zeta}]$ the trilinear interpolation functions. The Green-Lagrange strain components are obtained from Eq.(8) as

$$\bar{\boldsymbol{\varepsilon}} = \left(\boldsymbol{\mathcal{L}}[\boldsymbol{\zeta}] + \frac{1}{2} \boldsymbol{\mathcal{Q}}[\boldsymbol{\zeta}, \mathbf{d}_e] \right) \mathbf{d}_e, \tag{14b}$$

where matrices $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{Q}}$ are so defined

$$\boldsymbol{\mathcal{L}}[\boldsymbol{\zeta}] \equiv \begin{bmatrix} \mathbf{G}_1^T \mathbf{N}_{d,1} \\ \mathbf{G}_2^T \mathbf{N}_{d,2} \\ \mathbf{G}_3^T \mathbf{N}_{d,3} \\ \mathbf{G}_3^T \mathbf{N}_{d,2} + \mathbf{G}_2^T \mathbf{N}_{d,3} \\ \mathbf{G}_1^T \mathbf{N}_{d,3} + \mathbf{G}_3^T \mathbf{N}_{d,1} \\ \mathbf{G}_1^T \mathbf{N}_{d,2} + \mathbf{G}_2^T \mathbf{N}_{d,1} \end{bmatrix}, \quad \boldsymbol{\mathcal{Q}}[\boldsymbol{\zeta}, \mathbf{d}_e] \equiv \begin{bmatrix} \mathbf{d}_e^T \mathbf{N}_{d,1}^T \mathbf{N}_{d,1} \\ \mathbf{d}_e^T \mathbf{N}_{d,2}^T \mathbf{N}_{d,2} \\ \mathbf{d}_e^T \mathbf{N}_{d,3}^T \mathbf{N}_{d,3} \\ \mathbf{d}_e^T (\mathbf{N}_{d,3}^T \mathbf{N}_{d,2} + \mathbf{N}_{d,2}^T \mathbf{N}_{d,3}) \\ \mathbf{d}_e^T (\mathbf{N}_{d,1}^T \mathbf{N}_{d,3} + \mathbf{N}_{d,3}^T \mathbf{N}_{d,1}) \\ \mathbf{d}_e^T (\mathbf{N}_{d,1}^T \mathbf{N}_{d,2} + \mathbf{N}_{d,2}^T \mathbf{N}_{d,1}) \end{bmatrix} \tag{14c}$$

and, from now on, we omit the dependence on $\boldsymbol{\zeta}$ to simplify the notation. Note that the Assumed Natural Strain techniques can be applied to Eq.(14) in order to improve the element performance for curved shells (see [**?**, **?**]) simply changing the definition of matrices $\boldsymbol{\mathcal{L}}[\boldsymbol{\zeta}]$ and $\boldsymbol{\mathcal{Q}}[\boldsymbol{\zeta}, \mathbf{d}_e]$ without affecting the format of the equations.

For the contravariant stress components we use the "optimal" interpolation proposed by Pian and Tong [**?**, **?**] defined as

$$\bar{\boldsymbol{\sigma}}[\boldsymbol{\zeta}] = \mathbf{N}_\sigma[\boldsymbol{\zeta}]\boldsymbol{\beta}_e \tag{14d}$$

where $\boldsymbol{\beta}_e$ collects the 18 stress parameters and the interpolation functions $\mathbf{N}_\sigma[\boldsymbol{\zeta}]$ are given in [**?**, **?**]. The FE defined by Eqs (14) is called PT18β. Finally we use Eq.(13) to obtain the Cartesian components with $\mathbf{T}_\sigma$ and its inverse evaluated for $\boldsymbol{\zeta} = \mathbf{0}$.

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3.3. The Mixed finite element strain energy

The strain energy is expressed as a sum of element contributions $\Phi[u] \equiv \sum_e \Phi_e[u]$. Making V_e the finite element volume and using the interpolations defined above, we obtain

$$\begin{aligned}\Phi_e[u] &\equiv \int_{\Omega_e} \left(\boldsymbol{\sigma}^T \boldsymbol{\varepsilon} - \frac{1}{2}\boldsymbol{\sigma}^T \mathbf{C}^{-1}\boldsymbol{\sigma}\right) dV_e \\ &= \boldsymbol{\beta}_e^T (\mathbf{L}_e + \frac{1}{2}\mathbf{Q}_e[\mathbf{d}_e])\mathbf{d}_e - \frac{1}{2}\boldsymbol{\beta}_e^T \mathbf{H}_e \boldsymbol{\beta}_e\end{aligned} \quad \text{with} \quad \begin{cases} \mathbf{H}_e \equiv \int_{\Omega_e} \mathbf{N}_\sigma^T \mathbf{T}_\sigma^T \mathbf{C}^{-1} \mathbf{T}_\sigma \mathbf{N}_\sigma dV_e \\ \mathbf{L}_e \equiv \int_{\Omega_e} \mathbf{N}_\sigma^T \boldsymbol{\mathcal{L}}[\boldsymbol{\zeta}] dV_e \\ \mathbf{Q}_e \equiv \int_{\Omega_e} \mathbf{N}_\sigma^T \boldsymbol{\mathcal{Q}}[\boldsymbol{\zeta}, \mathbf{d}[\boldsymbol{\zeta}]] dV_e \end{cases} \tag{15}$$

where $dV_e = \det[\mathbf{J}_e[\mathbf{0}]] d\zeta_1 d\zeta_2 d\zeta_3$ and the integrals are evaluated with $2 \times 2 \times 2$ Gauss points. Note that exploiting the linear dependence of $\mathbf{Q}_e[\mathbf{d}_e]$ from $\mathbf{d}_e$ and its symmetry we have

$$\begin{aligned}&\mathbf{Q}_e[\mathbf{d}_{e1}]\mathbf{d}_{e2} = \mathbf{Q}_e[\mathbf{d}_{e2}]\mathbf{d}_{e1}, \quad \forall \mathbf{d}_{e1}, \mathbf{d}_{e2} \\ &\boldsymbol{\beta}_e^T \mathbf{Q}_e[\mathbf{d}_e]\mathbf{d}_e = \mathbf{d}_e^T \boldsymbol{\Gamma}_e[\boldsymbol{\beta}_e]\mathbf{d}_e \quad \text{with} \quad \boldsymbol{\Gamma}_e[\boldsymbol{\beta}_e] \equiv \sum_{i,j=1}^{3} \int_{\Omega_e} \bar{\sigma}^{ij} \mathbf{N}_{d,i}^T \mathbf{N}_{d,j} \; dV_e\end{aligned} \tag{16}$$

3.3.1. Strain energy variations of the PT18β element in mixed description. Eq.(15) allows the expression of the strain energy as an algebraic nonlinear function of the element vector related to the vector $\mathbf{u}$, collecting all the parameters of the FE assemblage, through the relation

$$\mathbf{u}_e \equiv \begin{bmatrix} \boldsymbol{\beta}_e \\ \mathbf{d}_e \end{bmatrix} = \boldsymbol{\mathcal{A}}_e \mathbf{u} \tag{17}$$

where matrix $\boldsymbol{\mathcal{A}}_e$ contains the link between the elements. Furthermore we denote with $\delta\mathbf{u}_{ei} = \{\delta\boldsymbol{\beta}_{ei}, \delta\mathbf{d}_{ei}\}$ the element vector corresponding to the variation $\delta\mathbf{u}_i$.

The first variation of the strain energy (15) is then

$$\Phi'_e \delta u_1 = \begin{bmatrix} \delta\boldsymbol{\beta}_{e1} \\ \delta\mathbf{d}_{e1} \end{bmatrix}^T \begin{bmatrix} \mathbf{s}_{e\beta} \\ \mathbf{s}_{ed} \end{bmatrix} \quad \text{with} \quad \begin{cases} \mathbf{s}_{e\beta} \equiv (\mathbf{L}_e + \frac{1}{2}\mathbf{Q}_e[\mathbf{d}_e])\mathbf{d}_e - \mathbf{H}_e\boldsymbol{\beta}_e \\ \mathbf{s}_{ed} \equiv \mathbf{B}_e[\mathbf{d}_e]^T \boldsymbol{\beta}_e \end{cases} \tag{18a}$$

and $\mathbf{B}_e[\mathbf{d}_e] \equiv \mathbf{L}_e + \mathbf{Q}_e[\mathbf{d}_e]$.

In the same way and exploiting the first of (16) the second strain energy variation is

$$\begin{aligned}\Phi''_e \delta u_1 \delta u_2 &= \begin{bmatrix} \delta\boldsymbol{\beta}_{e1} \\ \delta\mathbf{d}_{e1} \end{bmatrix}^T \begin{bmatrix} -\mathbf{H}_e & \mathbf{B}_e[\mathbf{d}_e] \\ \mathbf{B}_e[\mathbf{d}_e]^T & \boldsymbol{\Gamma}_e[\boldsymbol{\beta}_e] \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\beta}_{e2} \\ \delta\mathbf{d}_{e2} \end{bmatrix} \\ &= \delta\mathbf{u}_{e1}^T (\mathbf{K}_{0e} + \mathbf{K}_{1e}[\mathbf{u}_e])\delta\mathbf{u}_{e2}\end{aligned} \tag{18b}$$

that provides the element tangent stiffness matrix $\mathbf{K}_{me}[\mathbf{u}_e] = \mathbf{K}_{0e} + \mathbf{K}_{1e}[\mathbf{u}_e]$ as a sum of the linear elastic contribution $\mathbf{K}_{0e}$ and the geometric matrix $\mathbf{K}_{1e}[\mathbf{u}_e]$ implicitly defined in Eq.(18b). Using a similar approach the third variation becomes

$$\Phi_e'''\delta u_1\delta u_2\delta u_3 = \left\{\delta\boldsymbol{\beta}_{e1}^T\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e2} + \delta\boldsymbol{\beta}_{e2}^T\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e1} + \delta\boldsymbol{\beta}_{e3}^T\mathbf{Q}[\delta\mathbf{d}_{e2}]\delta\mathbf{d}_{e1}\right\} \\ = \begin{bmatrix}\delta\boldsymbol{\beta}_{e1}\\ \delta\mathbf{d}_{e1}\end{bmatrix}^T \begin{bmatrix}\mathbf{s}_{e\beta}''[\delta\mathbf{d}_{e2},\delta\mathbf{d}_{e3}]\\ \mathbf{s}_{e\beta}''[\delta\mathbf{d}_{e2},\delta\mathbf{d}_{e3}]\end{bmatrix} \tag{18c}$$

where

$$\begin{cases}\mathbf{s}_{e\beta}''[\delta\mathbf{d}_{e2},\delta\mathbf{d}_{e3}] \equiv \mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e2}\\ \mathbf{s}_{ed}''[\delta\mathbf{d}_{e2},\delta\mathbf{d}_{e3}] \equiv \mathbf{Q}[\delta\mathbf{d}_{e3}]^T\delta\boldsymbol{\beta}_{e2} + \mathbf{Q}[\delta\mathbf{d}_{e2}]^T\delta\boldsymbol{\beta}_{e3}\end{cases} \tag{18d}$$

Finally Eq.(17) allow the evaluation, for each element vector $\mathbf{y}_e$ and matrix $\mathbf{Y}_e$ the quantities of the whole assemblage

$$\mathbf{y} = \sum_e \boldsymbol{\mathcal{A}}_e^T\mathbf{y}_e, \quad \mathbf{Y} = \sum_e \boldsymbol{\mathcal{A}}_e^T\mathbf{Y}_e\boldsymbol{\mathcal{A}}_e. \tag{19}$$

In particular from the assemblages of vector $\mathbf{s}_e''$ we obtain vectors $\mathbf{p}_{ij}$ defined in Eq.(6h). Obviously scalar quantities are directly evaluated as sums of local element contributions.

3.4. *The displacement description of the PT*18β *element*

The element can also be described in a displacement format by requiring that the discrete form of the constitutive laws is "a priori" satisfied. As in the present FE model the stress variables are locally defined at the element level we have

$$\boldsymbol{\beta}_e[\mathbf{d}_e] = \mathbf{H}_e^{-1}(\mathbf{L}_e + \frac{1}{2}\mathbf{Q}_e[\mathbf{d}_e])\mathbf{d}_e \tag{20}$$

where, to highlight that in the displacement format the stresses are not independent variables, we explicitly report the dependence of $\mathbf{d}_e$. Substituting Eq.(20) in Eq.(15) we obtain the *displacement description* of the element strain energy

$$\Phi_e = \frac{1}{2}\left\{\mathbf{d}_e^T(\mathbf{L}_e + \frac{1}{2}\mathbf{Q}_e[\mathbf{d}_e])^T\mathbf{H}_e^{-1}(\mathbf{L}_e + \frac{1}{2}\mathbf{Q}_e[\mathbf{d}_e])\mathbf{d}_e\right\} \tag{21}$$

that has a 4th order dependence on the displacement variables only.

Note how an expression similar to Eq.(21), is obtained by using any other displacement based element when the Green-Lagrange strain tensor is employed. In this paper we prefer to use the displacement description of the PT18β element in order to have exactly the same discrete approximation for both the descriptions. Note that the finite element is the same but the format of the system of equations changes. This allows us to focus on how the problem description affects its numerical solution in large deformation problems (see also [**?**, **?**]) and then to show and explain the better performance of the use of a mixed description (and so mixed element).

3.4.1. Strain energy variations of the PT18β element in displacement description. For the displacement description the configuration variables are the displacements only and $\delta\mathbf{u}_{ei} = \delta\mathbf{d}_{ei}$. The first variation of Eq.(21) becomes

$$\Phi_e'\delta u_1 = \delta\mathbf{d}_{e1}^T\mathbf{s}_e[\mathbf{d}_e] \quad \text{whit} \quad \mathbf{s}_e[\mathbf{d}_e] \equiv \mathbf{B}_e^T[\mathbf{d}_e]\boldsymbol{\beta}_e[\mathbf{d}_e] \tag{22a}$$

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where $\boldsymbol{\beta}_e[\mathbf{d}_e]$ is defined in Eq.(20). In the same way the second strain energy variation is

$$\Phi_e''\delta u_1\delta u_2 = \delta\mathbf{d}_{e1}^T\left\{\mathbf{B}_e^T[\mathbf{d}_e]\mathbf{H}_e^{-1}\mathbf{B}_e[\mathbf{d}_e] + \boldsymbol{\Gamma}[\boldsymbol{\beta}_e[\mathbf{d}_e]]\right\}\delta\mathbf{d}_{e2} \tag{22b}$$

where the tangent stiffness matrix has a second order dependence on $\mathbf{d}_e$. To emphasize this we write

$$\mathbf{K}_{de} = \mathbf{K}_{0e} + \mathbf{K}_{1e}[\mathbf{d}_e] + \mathbf{K}_{2e}[\mathbf{d}_e, \mathbf{d}_e] \tag{22c}$$

The third variation of the strain energy becomes

$$\begin{aligned}\Phi'''\delta u_1\delta u_2\delta u_3 &= \left\{\delta\boldsymbol{\beta}_{e1}^T[\mathbf{d}_e]\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e2} + \delta\boldsymbol{\beta}_{e2}^T[\mathbf{d}_e]\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e1} + \delta\boldsymbol{\beta}_{e3}^T[\mathbf{d}_e]\mathbf{Q}[\delta\mathbf{d}_{e2}]\delta\mathbf{d}_{e1}\right\}\\ &= \delta\mathbf{d}_{e1}^T\mathbf{s}''[\delta\mathbf{d}_{e2}, \delta\mathbf{d}_{e3}]\end{aligned} \tag{22d}$$

where

$$\mathbf{s}''[\delta\mathbf{d}_{e2}, \delta\mathbf{d}_{e3}] \equiv \mathbf{Q}[\delta\mathbf{d}_{e3}]^T\delta\boldsymbol{\beta}_{e2}^T[\mathbf{d}_e] + \mathbf{Q}[\delta\mathbf{d}_{e2}]^T\delta\boldsymbol{\beta}_{e3}^T[\mathbf{d}_e] + (\mathbf{L}_e + \mathbf{Q}_e[\mathbf{d}_e])^T\mathbf{H}_e^{-1}\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e2}$$

and we introduce the quantities $\delta\boldsymbol{\beta}_{ei}^T[\mathbf{d}_e] \equiv \mathbf{H}_e^{-1}\mathbf{B}_e[\mathbf{d}_e]\delta\mathbf{d}_{ei},\ i = 1..3$ that are the variation in the stresses obtained from the constitutive equation (20) with respect to the displacements.

Finally it is worth noting that in this case the 4th variation of the strain energy is not zero and we have

$$\begin{aligned}\Phi''''\delta u_1\delta u_2\delta u_3\delta u_4 = \{ \quad &\delta\mathbf{d}_{e1}^T\mathbf{Q}^T[\delta\mathbf{d}_{e4}]\mathbf{H}_e^{-1}\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e2}\\ &+ \delta\mathbf{d}_{e2}^T\mathbf{Q}^T[\delta\mathbf{d}_{e4}]\mathbf{H}_e^{-1}\mathbf{Q}[\delta\mathbf{d}_{e3}]\delta\mathbf{d}_{e1}\\ &+ \delta\mathbf{d}_{e3}^T\mathbf{Q}^T[\delta\mathbf{d}_{e4}]\mathbf{H}_e^{-1}\mathbf{Q}[\delta\mathbf{d}_{e2}]\delta\mathbf{d}_{e1}\}\end{aligned} \tag{22e}$$

3.5. *Advantages of mixed solid finite elements in path-following and asymptotic analysis of slender structures*

FE models directly derived from the 3D continuum using the Green strain measure have a low order dependence on the strain energy from the discrete FE parameters: $3rd$ and $4th$ order for mixed and displacement respectively. On the contrary geometrically exact shell and beam models [**?**, **?**] or those based on corotational approaches [**?**, **?**, **?**], explicitly make use of the rotation tensor and its highly nonlinear representation. This implies that the strain energy is infinitely differentiable with respect to its parameters and leads to very complex expressions for the energy variations with a high computational burden of path following and much more of asymptotic analyses. In this last case the high order strain energy variations become so complex that often "ad hoc" assumptions are required to make the solution process effective (see section 4.3 of [**?**]). The consequence is that the fewer global degrees of freedom that could be employed using a shell FE model do not necessarily imply a lesser computational cost as it depends, from the others, on the cost of evaluation of the strain energy variations. On the contrary for solid finite elements the strain energy, in both displacement and mixed form, has the lowest polynomial dependence on the corresponding discrete parameters and in particular in the mixed format of Eq.(15) has just one order more than in the linear elastic

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case. It implies the zeroing of all the fourth order strain energy variations required by the Koiter analysis with important advantages in terms of both computations and coherence of the method.

3.5.1. Simplifications and improvements in Koiter analysis using mixed solid elements. The low order polynomial dependence of the strain energy on the parameters produces a first important simplification and improvement in the evaluation of the bifurcation points. With the usual adopted linear extrapolation in λ of the fundamental path $\mathbf{u}_f[\lambda] = \lambda\hat{\mathbf{u}}$ it is convenient to expand the bifurcation condition in Eq.(6d) from the origin as

$$\Phi''[u_f[\lambda]]\dot{v}_i\delta u = (\Phi''_0 + \lambda\Phi'''_0\hat{u} + \frac{1}{2}\lambda^2\Phi''''_0\hat{u}^2)\dot{v}_i\delta u \tag{23}$$

Note how the Taylor expansion in Eq. (23) is exact, due to the zeroing of all the high order energy terms. In particular the mixed expression of the buckling condition is automatically linear due to the zeroing of the 4th variation

$$\mathbf{K}_m[\lambda] = \mathbf{K}_0 + \lambda\mathbf{K}_1[\hat{\mathbf{u}}] \tag{24}$$

where $\mathbf{K}_0$ and $\mathbf{K}_1$ are obtained from assemblages of the element matrices in Eq.(18b). For the displacement description the equivalent bifurcation problem assumes the following form ($\hat{\mathbf{u}} = \hat{\mathbf{d}}$)

$$\mathbf{K}_d[\lambda] = \mathbf{K}_0 + \lambda\mathbf{K}_1[\hat{\mathbf{d}}] + \frac{1}{2}\lambda^2\mathbf{K}_2[\hat{\mathbf{d}}, \hat{\mathbf{d}}] \tag{25}$$

where again $\mathbf{K}_0$, $\mathbf{K}_1$ and $\mathbf{K}_2$ are obtained as assemblages of the element matrices reported in (22b).

To have an "exact" buckling condition independently of the closeness of the buckling loads is particularly important in the asymptotic method. It makes great use of buckling modes and loads and so their accurate evaluation strongly affects the quality of the complete equilibrium path reconstruction. This is possible in the case of solid finite elements while, in the general, due to the linearization in Eq.(6e) the accuracy depends on the magnitude of $(\lambda_i - \lambda_b)$. Furthermore in employing a mixed description the bifurcation condition is exactly a simple linear eigenvalue problem which provides the m bifurcation loads and modes naturally orthogonalized according to Eq.(6f) without any other assumption apart from the linear extrapolation of the fundamental path.

Another great advantage is in the evaluation of all the fourth order coefficients in Eq.(6i) that, for mixed solid elements, requires only second variations for their evaluation since the usually very complex fourth order strain energy variations are zero. In particular the energy terms reduce to

$$\begin{aligned}
\mathcal{A}_{ijk} &= \Phi'''_b\dot{v}_i\dot{v}_j\dot{v}_k \\
\mathcal{B}_{00ik} &= -\Phi''_b\hat{\dot{w}}w_{ik} = -\mathcal{C}_{ik} \\
\mathcal{B}_{0ijk} &= 0 \\
\mathcal{B}_{ijhk} &= -\Phi''_b(w_{ij}w_{hk} + w_{ih}w_{jk} + w_{ik}w_{jh}) \\
\mu_k[\lambda] &= \frac{1}{2}\lambda^2\Phi'''_b\hat{u}^2\dot{v}_k
\end{aligned} \tag{26}$$

Furthermore the matrices in the bifurcation condition (24) and the nonzero strain variations in (26) assume simpler and lower cost expressions compared to standard shell elements. This makes

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Figure 1. A simple test

the implementation of the asymptotic method very easy and reduces the total cost of the solution process.

However the main advantage in using a mixed formulation is its capability to rectify an important but underhand locking effect, called in [**?**, **?**] *extrapolation locking*. This is deeply discussed in the next section.

4. THE EXTRAPOLATION LOCKING AND ITS CURE USING A MIXED FORMAT

In this section the better performances of the mixed description in geometrically nonlinear analysis, in terms of robustness, efficiency and, relative to Koiter formulation, also in terms of accuracy, are shown and explained. In this context the displacement description of the problem, whatever the FE and the structural model used, is affected by a pathological *extrapolation locking* phenomenon investigated for the first time for 2D frames in [**?**, **?**].

The displacement format, equivalent to the mixed one in terms of the finite element interpolation, makes it possible to focus the phenomenon in a general context and to highlight its origin.

4.1. A simple test

To show the extrapolation locking phenomenon the simple test in Fig.1 is considered. It consists of a bar of unitary length constrained with a linear spring on an end. With strain and stress constant along the bar the mixed and displacement functionals are, respectively,

$$\begin{aligned} \Pi_{HR}[N,u,w] &\equiv N\epsilon[u,w] - \frac{1}{2}\frac{N^2}{EA} + \frac{1}{2}k_w\, w^2 - \lambda(c\, w - u) \\ \Pi[u,w] &\equiv \frac{1}{2}(EA\epsilon[u,w]^2 + k_w\, w^2) - \lambda(c\, w - u) \end{aligned} \tag{27}$$

where N is the axial force of the bar and the corresponding Green-Lagrange axial strain is

$$\epsilon[u,w] = u + \frac{1}{2}(u^2 + w^2)$$

The displacement stiffness matrix, evaluated as the Hessian of the displacement functional with respect to the displacement variables $\mathbf{d} = [u, w]$, becomes

$$\mathbf{K}_d[u,w] = \begin{bmatrix} N_d + EA(1+u)^2 & EA(1+u)\, w \\ EA(1+u)\, w & N_d + k_w + EAw^2 \end{bmatrix} \quad \text{with} \quad N_d = EA\varepsilon[u,w]$$

where N_d represent the axial force function of the displacement components. The mixed stiffness matrix, Hessian of the Hellinger-Reissner functional with respect to the mixed variables $\mathbf{u} = [N, \mathbf{d}]$, is

$$\mathbf{K}_m[N,u,w] = \begin{bmatrix} -\dfrac{1}{EA} & 1+u & w \\ 1+u & N & 0 \\ w & 0 & k_w + N \end{bmatrix}$$

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where now N is directly a variable of the problem. Note that from the static condensation of N we obtain from $\mathbf{K}_m$ the same form of matrix $\mathbf{K}_d$ by replacing N_d by N. Starting from the equilibrium point with zero load and displacements we perform a linear extrapolation in λ of the linear elastic solution, $\mathbf{d}_1 = \{u_1, w_1\}$ for the displacement format and $\mathbf{u}_1 = \{N_1, \mathbf{d}_1\}$ for the mixed one, with

$$u_1 = -\frac{\lambda_1}{EA} \quad w_1 = \frac{c\lambda_1}{k_w}$$

the same for both the descriptions and $N_1 = -\lambda_1$ directly extrapolated unlike $N_d[u_1, w_1]$ that is function of the extrapolated displacements

$$N_d[u_1, w_1] = -\lambda_1 + \frac{\lambda_1^2}{2}\left(\frac{1}{EA} + \frac{c^2 EA}{k_w^2}\right).$$

For high values of EA/k_w^2 and also small c, the displacement approach predicts an axial tension instead of the correct axial compression and, as a consequences, a mistakenly overestimated stiffness, in direction w which consequently locks. The same phenomenon typically affects shell problems where the in-plane stiffness plays the role of EA and the flexural stiffness, usually much lower then the first one, the role of k_w. On the contrary extrapolation locking does not affect the mixed format as N_1 is expected to be a good estimate of the true axial force in the deformed configuration (small strain/large deformation hypothesis).

4.2. Generalization of the extrapolation problem in displacement format

The effects highlighted for the simple test are now generalized to the solid formulation. The use of a displacement description of a mixed finite element makes the origin of the extrapolation locking phenomenon evident. In order to compare the two formats we recall that the stresses $\boldsymbol{\beta}[\mathbf{d}]$ in the displacement format are obtained, as a function of the displacements, from Eq.(20), while in the mixed format they are independent variables. Letting $\mathbf{z}_0$ be an equilibrium point we have for both the formats the same displacements and stresses. Performing an extrapolation, for example evaluating $\mathbf{z}_1$ along the linearization in λ of the equilibrium path starting from $\mathbf{z}_0$, we obtain for the mixed format

$$\begin{bmatrix}\boldsymbol{\beta}_1\\ \mathbf{d}_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{\beta}_0 + \Delta\lambda\hat{\boldsymbol{\beta}}\\ \mathbf{d}_0 + \Delta\lambda\hat{\mathbf{d}}\end{bmatrix} \quad \text{with} \quad \begin{bmatrix}-\mathbf{H} & \mathbf{B}[\mathbf{d}_0]\\ \mathbf{B}[\mathbf{d}_0]^T & \boldsymbol{\Gamma}[\boldsymbol{\beta}_0]\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\beta}}\\ \hat{\mathbf{d}}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \hat{\mathbf{p}}\end{bmatrix}$$

so obtaining

$$\hat{\mathbf{d}} = \mathbf{K}_{c0}\mathbf{p} \quad \text{and} \quad \hat{\boldsymbol{\beta}} = \mathbf{H}^{-1}\mathbf{B}[\mathbf{d}_0]\hat{\mathbf{d}}$$

Since $\mathbf{K}_{c0} \equiv \boldsymbol{\Gamma}[\boldsymbol{\beta}_0] + \mathbf{B}[\mathbf{d}_0]^T\mathbf{H}^{-1}\mathbf{B}[\mathbf{d}_0]$ coincides with the stiffness matrix evaluated by the displacement format in the equilibrium point $\mathbf{z}_0$, $\hat{\mathbf{d}}$ is the same for both the formats.

In $\mathbf{z}_1$ we obtain the following stiffness matrices for the two descriptions

$$\mathbf{K}_m[\mathbf{d}_1, \boldsymbol{\beta}_1] = \begin{bmatrix}-\mathbf{H} & \mathbf{B}[\mathbf{d}_1]\\ \mathbf{B}[\mathbf{d}_1] & \boldsymbol{\Gamma}[\boldsymbol{\beta}_1]\end{bmatrix} \quad \mathbf{K}_d[\mathbf{d}_1] = \mathbf{B}[\mathbf{d}_1]^T\mathbf{H}^{-1}\mathbf{B}[\mathbf{d}_1] + \boldsymbol{\Gamma}[\boldsymbol{\beta}[\mathbf{d}_1]] \tag{28}$$

where now $\boldsymbol{\Gamma}[\boldsymbol{\beta}_1] \neq \boldsymbol{\Gamma}[\boldsymbol{\beta}[\mathbf{d}_1]]$ with $\boldsymbol{\beta}_1 \neq \boldsymbol{\beta}[\mathbf{d}_1]$. In the mixed case the stresses are directly extrapolated as $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_0 + \Delta\lambda\mathbf{H}^{-1}\mathbf{B}[\mathbf{d}_0]\hat{\mathbf{d}}$ while for the displacement case they are evaluated,

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Figure 2. Equilibrium path for the Euler beam for $\epsilon = 0.001$ and $k = 10^4$

according to Eq.(20), as

$$\boldsymbol{\beta}_1[\mathbf{d}_1] = \boldsymbol{\beta}_0 + \mathbf{H}^{-1}(\Delta\lambda\mathbf{B}_0\hat{\mathbf{d}} + \frac{\Delta\lambda^2}{2}\mathbf{Q}[\hat{\mathbf{d}}]\hat{\mathbf{d}})$$

A wrong spurious term

$$\Delta\boldsymbol{\beta} = \Delta\lambda^2\frac{1}{2}\mathbf{H}^{-1}\mathbf{Q}[\hat{\mathbf{d}}]\hat{\mathbf{d}}$$

that is the stress produced by the quadratic part of the strain evaluated with a linear extrapolation of the displacement is present in the displacement extrapolation. This strain term has components even in the highest stiffness directions and so the estimated stresses are affected by a great error $\Delta\boldsymbol{\beta}$. The consequence is an overestimated tangent stiffness matrix for the displacement format. This is the same extrapolation locking as previously observed for the simple test. When, as is usual for slender structures, the condition number of $\mathbf{H}$ is high due to very different stiffness ratios (i.e. membranal/flexural) the phenomenon becomes very important and affects the displacement format in the solution strategies of the nonlinear problem. It produces difficulties in convergence in the path following scheme because the overestimated stiffness matrix in the current iteration is far from the secant one and could not fulfil the second part of the convergence condition (5e). Koiter analysis furnishes a wrong bifurcation point used by the method to approximate the equilibrium path. On the contrary the mixed extrapolation, directly linearizes the stress and it is naturally free from the extrapolation locking.

4.2.1. Mixed vs displacement description in Koiter analysis using 3D solid elements Recalling that the buckling condition is tested along an extrapolated path, in displacement variables it could be affected by extrapolation locking which produces an overestimated buckling load or loss of bifurcation. The proposed Koiter method uses an asymptotic expansion in an extrapolated point along the linearized fundamental path. For this reason the locking phenomenon previously described can strongly affect its accuracy [**?**, **?**, **?**].

To avoid this error a linearized buckling analysis can be performed by zeroing the quadratic part of the strain depending on the extrapolated displacements $\mathbf{Q}_e[\hat{\mathbf{d}}_e] = \mathbf{0}$ (*frozen configuration* hypothesis). This eliminates the extrapolation locking as can be seen in Tab.I where the buckling loads obtained with the displacement (D) and mixed (M) formats are compared with those of the frozen configuration (F) for the Euler beam of Fig. 6. The comparisons are performed by changing both the aspect ratio $k = (t/\ell)^2$ and the imperfection load amplitude ϵ.

Table I. Buckling analyses for the Euler beam/(exact value for the elastica)

k	$\epsilon = 0.01$	$\epsilon = 0.005$	$\epsilon = 0.001$	for all ϵ	
	D	D	D	F	M
10^4	failed	1.112	1.004	1.001	1.001
10^5	failed	failed	1.040	1.001	1.001
10^6	failed	failed	failed	1.001	1.001
10^7	failed	failed	failed	0.999	0.999

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Figure 3. Geometry and material properties of the shallow arc

figure/Arco/curvaEqArcro.pdf

Figure 4. Equilibrium path for the shallow arc

Table II. Shallow arc: comparison of relevant energy asymptotic quantities of Eq.(6j)

	M	*F*
λ_1	22.0202	30.6526
λ_2	30.6671	47.0652
$\mathcal{A}_{001}$	0.0196	0.0190
$\mathcal{A}_{111}$	47.9305	182.3644
$\mathcal{B}_{0011}$	-6.32 10^{-4}	0
$\mathcal{B}_{1111}$	8.2117	207.3178

For this case, which has small precritical nonlinearities, it is worth noting that: i) the frozen configuration hypothesis rectifies the locking effect and furnishes accurate results; ii) the displacement description misses the bifurcation point, getting worse with the precritical nonlinearity due to the transversal force. Inaccuracy in the bifurcation points obviously leads to a completely wrong equilibrium path estimated by the asymptotic algorithm where, unlike the Riks method, the extrapolation error is not corrected by any iterative scheme. Finally it is important to note that also the energy terms in Eq.(6i) and then the estimated asymptotic equilibrium path in Eq.(6j) are very sensitive to the extrapolation locking and so, even when the bifurcation point is almost correctly evaluated, the post critical behaviour could be completely wrong when the displacement description is used. On the contrary the mixed description is unaffected by k (see fig.2 and [**?**]) and the initial post critical path is recovered accurately .

The frozen configuration hypothesis could, however, lead to inaccuracy when the precritical displacements are not negligible, as in the shallow arc reported in Fig.3. In this case it is not capable of producing the correct bifurcation load and mode as reported in Tab.II and consequently the energy terms also reported in Tab.II, used to estimate the equilibrium path. In Fig.4 the equilibrium path recovered by the Koiter mixed formulation is presented and compared with the frozen and the true path following solutions.

4.2.2. Mixed vs displacement description in path-following analysis In the path-following scheme extrapolation locking occurs at each step and affects both the first extrapolation, used to evaluate the first estimate $\mathbf{z}_1$, and the corrector scheme in Eq.(5), which is based on a sequence of linearizations in the current estimates $\mathbf{z}_j$ of the solution. In this case extrapolation locking produces a strong deterioration in the convergence properties of the Newton (Riks) method.

It is useful to show how the two formats update the solution in an iterative Newton scheme. A load controlled case is considered to simplify the notation. The iteration in mixed format is

$$\begin{bmatrix} -\mathbf{H} & \mathbf{B}_j \\ \mathbf{B}_j^T & \mathbf{\Gamma}_j \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\beta}} \\ \dot{\mathbf{d}} \end{bmatrix} = - \begin{bmatrix} \varepsilon[\mathbf{d}_j] - \mathbf{H}\boldsymbol{\beta}_j \\ \mathbf{B}_j^T \boldsymbol{\beta}_j - \lambda \mathbf{p} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \boldsymbol{\beta}_{j+1} \\ \mathbf{d}_{j+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}_j \\ \mathbf{d}_j \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{\beta}} \\ \dot{\mathbf{d}} \end{bmatrix} \tag{29}$$

where $\mathbf{\Gamma}_j = \mathbf{\Gamma}[\boldsymbol{\beta}_j]$ and $\mathbf{B}_j \equiv \mathbf{B}[\mathbf{d}_j]$. Solving the mixed linear system in Eq.(29) we obtain

$$\begin{cases} \mathbf{d}_{j+1} = \mathbf{d}_j - \tilde{\mathbf{K}}_{dj}^{-1} \mathbf{r}_{dj} \\ \boldsymbol{\beta}_{j+1} = \mathbf{H}^{-1}(\varepsilon[\mathbf{d}_j] + \mathbf{B}_j \dot{\mathbf{d}}) \end{cases} \quad \text{where} \quad \tilde{\mathbf{K}}_{dj} = \mathbf{B}_j^T \mathbf{H}^{-1} \mathbf{B}_j + \mathbf{\Gamma}_j \tag{30}$$

The same iteration in the displacement case is

$$\begin{cases} \mathbf{d}_{j+1} = \mathbf{d}_j - \mathbf{K}_j^{-1} \mathbf{r}_{dj} \\ \boldsymbol{\beta}[\mathbf{d}_{j+1}] = \mathbf{H}^{-1} \varepsilon[\mathbf{d}_j + \dot{\mathbf{d}}] \end{cases} \quad \text{where} \quad \begin{array}{c} \mathbf{K}_j = \mathbf{B}_j^T \mathbf{H}^{-1} \mathbf{B}_j + \mathbf{\Gamma}[\boldsymbol{\beta}[\mathbf{d}_j]] \\ \varepsilon[\mathbf{d}_{j+1}] = \varepsilon[\mathbf{d}_j] + \mathbf{B}_j \dot{\mathbf{d}} + \dfrac{1}{2}\mathbf{Q}[\dot{\mathbf{d}}]\dot{\mathbf{d}} \end{array} \tag{31}$$

where $\mathbf{r}_{dj} = \mathbf{B}_j^T \mathbf{H}^{-1} \varepsilon[\mathbf{d}_j] - \lambda \mathbf{p}$ is the same for both the approaches and the stresses in the displacement iteration, functions of the displacements, are introduced for an easy comparison with the mixed format.

Eqs.(30) and (31) show that the only, but important, difference in the two formats consists in the way the stresses are obtained from the current linearization. The spurious stress term

$$\Delta \boldsymbol{\beta} = \frac{1}{2} \mathbf{H}^{-1} \mathbf{Q}[\dot{\mathbf{d}}]\dot{\mathbf{d}}.$$

due to the extrapolation locking is present in the displacement iteration.

It is important to note that the displacement iterative scheme can be obtained from the mixed one by solving exactly, at each iteration, the constitutive equations. In this way the evolution of the displacement iterative process is forced to satisfy the constitutive constraint at each iteration and this, in general, leads to a deterioration in the convergence properties. On the contrary the mixed format performs a consistent linearization of all the problem equations and allows the iterations to naturally evolve towards the solution.

The convergence of the Riks scheme is as fast as the iteration and secant stiffness matrix are near and then as $\mathbf{K}[\mathbf{u}]$ slowly changes with $\mathbf{u}$. The similarity of the stiffness matrices in two different points $\mathbf{K}_j \equiv \mathbf{K}[\mathbf{u}_j]$ and $\mathbf{K}_{j+1} \equiv \mathbf{K}[\mathbf{u}_{j+1}]$, according to Eq.(5f), is evaluated by the difference

$$\Delta k[\mathbf{u}] \equiv \mathbf{u}^T(\mathbf{K}_{j+1} - \mathbf{K}_j)\mathbf{u} \quad \forall \mathbf{u} : \mathbf{u}^T \mathbf{u} = 1 \tag{32}$$

In the displacement format, when the matrices are positive definite, it is easy to show Δk by means of a graphical interpretation

$$\Delta k_D = \mathbf{d}^T(\mathbf{K}_{dj+1} - \mathbf{K}_{dj})\mathbf{d} = r_{j+1}^2[\mathbf{d}] - r_j^2[\mathbf{d}] \tag{33}$$

where $r_j[\mathbf{d}]$ and $r_{j+1}[\mathbf{d}]$ are the radii of the following ellipsoids

$$\mathcal{E}_k \equiv \{\mathbf{d} : \mathbf{d}^T \mathbf{K}_d[\mathbf{d}_k]^{-1} \mathbf{d} = 1\}, \quad k = j, j+1$$

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Figure 6. Eulero beam: analysis evolution for increasing k, mesh $1 \times 1 \times 40$

k	Displacement 10^4	10^5	10^6	10^7	Mixed (all k)
steps	38	43	67	failed	27
loops	133	166	328	failed	75

Table III. Eulero beam: analysis evolution for increasing k, mesh $1 \times 1 \times 40$

When, as is usual for slender structures, the condition number of $\mathbf{H}$ and then of $\mathbf{K}_d$ is high due to very different stiffness ratios (i.e. membranal/flexural) the ellipsoids associated to $\mathbf{K}_d^{-1}$ are very stretched. In the small strain/large displacement hypothesis the ellipsoids are similar but slightly rotated. In this case even a small rotation produces a large Δk_d which increases with the condition number of $\mathbf{H}$ (see Fig.5). This is the reason for the pathological reduction in the step length, an increase in the total number of the iterations and, sometimes, the loss of convergence observed for the displacement description. In the mixed case we have

Figure 5. Graphical interpretation of Δk for the displacement description

$$\Delta k_M[\mathbf{u}] = \mathbf{d}^T \boldsymbol{\Gamma}[\boldsymbol{\beta}_{j+1} - \boldsymbol{\beta}_j]\mathbf{d} + 2\boldsymbol{\beta}^T \mathbf{Q}[\mathbf{d}_{j+1} - \mathbf{d}_j]\mathbf{d} \tag{34}$$

linearly dependent on the stress and displacement difference and unaffected by $\mathbf{H}$ and so by the extrapolation locking. Note that these deductions are general since for any structural model the strain energy can be expressed as a quadratic function of the stress variables through the Hellinger-Reissner functional and so the Hessian change is not influenced by $\mathbf{H}$ unlike the displacement case.

We show the occurrence of the locking in the simple case of the Euler beam, for which the geometry and load conditions are reported in fig. 6.

In Tab.III the number of steps and iterations (loops) to obtain the equilibrium path, directly related to the CPU time, are presented. The results of the mixed formulation, denoted by M, are unaffected by the coefficient $k = (t/\ell)^2$ while the displacement ones (D) pathologically depend on it.

Finally in Fig.7 we report the minimum ρ_{min} and maximum ρ_{max} absolute value of the eigenvalues of the matrix $\mathbf{K}_{n+1}\mathbf{K}_n^{-1}$ for both descriptions, where $n+1$ and n denote two equilibrium points. The set of points in which the ρs are evaluated belongs on the equilibrium

Figure 7. Minimum and maximum eigenvalues of matrix $\mathbf{K}_{n+1}\mathbf{K}_n^{-1}$ for the simple tests

path obtained by the mixed description. It is important to observe how the mixed description ρ_{max} is independent of k and has almost the optimal value 1.0 while for the displacement description increases with the step length. Also note that if the second condition in Eq.(5e) is not fulfilled and this also heavily affects the convergence of the arc-length solution while the singular direction is filtered by the Riks constraint. We refer readers to [**?**, **?**] for further details.

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Table IV. Channel section: first 4 buckling loads.

	Mixed	Frozen
λ_1	1266.8	1291.5
λ_2	1828.1	1719.0
λ_3	3092.3	2949.7
λ_4	3114.7	2970.6

Figure 9. Channel section: Buckling modes

5. NUMERICAL RESULTS

In this section the effectiveness and reliability of both methods of analysis are tested in a series of benchmark problems. In particular for the path-following analysis the efficiency of the mixed description, which allows very large steps in comparison with the displacement one, is highlighted. For the asymptotic formulation we show the accuracy given by the mixed solid element.

For all the tests only one element in the thickness is used while the same convergence conditions and arc-length parameters are adopted for mixed and displacement path-following analyses. The junctions are modeled with regular elements. The label Riks denotes the equilibrium paths obtained by the arc-length scheme (the same for both the descriptions), while labels Mixed and Frozen denote the asymptotic paths using the mixed description and the displacement one with the frozen configuration hypothesis.

5.1. Simply supported C-shaped beam

The first test, with geometry and material reported in figure 8, consists in a simply supported compressed beam with a C shaped section. It presents a nonlinear prebuckling behavior due to two forces (torsional imperfections) at the mid-span and coupled instability. For this reason it is a good benchmark to test the accuracy of the asymptotic analysis. A similar test was studied in [**?**] using

Figure 8. Channel section: geometry and loads.

shell elements and in [**?**] using a generalized beam model. The buckling values, obtained by using a mesh of $(8+8+18)\times 50$ elements are reported in Table IV and compared with those computed using the displacement description with the frozen configuration hyphothesis. The Koiter analysis uses the first four buckling modes plotted in Fig.9. It is possible to see how the first two modes are global, essentially flexural and torsional respectively, while the others are local modes.

The accuracy of the mixed asymptotic strategy in the evaluation of the limit load and of the initial postcritical behavior is shown in Fig.10. It is also possible to observe the poor accuracy of the frozen configuration analysis in estimating both the limit load and equilibrium path. In Fig.11 the equilibrium path of Koiter method, in terms of the modal contributions ξ_k, is plotted. The strong effect of modal interaction between the third (local) mode and the first two flexural–torsional (global) modes is shown.

Figure 10. Channel section: Equilibrium paths $\lambda - w_A$, $\lambda - w_B$

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Figure 11. Channel section: Equilibrium paths in ξ_k space.

Table V. Channel section: steps and iterations for path–following analysis.

	Mixed	Displacement	Mixed MN
steps	24	74	45
loops	73	175	233

Figure 12. T section beam: geometry and loads

Table VI. T section beam: first 4 buckling loads.

	Mixed	Frozen
λ_1	1092.1	936.8
λ_2	1869.1	1860.4
λ_3	1993.5	1989.6
λ_4	2258.9	2252.1

The results of the mixed asymptotic analysis are in good agreement with the path-following ones. In Tab.V the steps and iterations of the mixed and displacement descriptions are compared. Obviously the equilibrium path is exactly the same but the better performances of the mixed description are evident even when a modified Newton-Raphson method (MN) is adopted.

5.2. *A T beam*

The second test regards the beam with data reported in Fig. 12. It consists in a simply supported beam with a T shaped section loaded by a shear force acting at the mid-span and by a small imperfection ($\epsilon = 1/1000$) load as reported in the same figure. The precritical behavior exhibits a strong nonlinearity and coupled bucklings are also present in this case. A mesh of $(9 + 9 + 18) \times 100$ elements has been used.

In Fig.13 the first 4 buckling modes, considered in the multimodal Koiter analysis, are plotted.

Figure 13. Channel section: First 4 Buckling modes

In Fig.14 the equilibrium paths recovered by using both asymptotic and path-following analysis are reported and compared. The solution is accurately recovered by the asymptotic strategy up to quite large displacements and the occurrence of a secondary bifurcation.

Also in this case (see Fig.15) the equilibrium path is plotted in terms of the modes amplitudes ξ_k showing a strong interaction among modes 1, 2 and 4.

In Tab.VII the steps and iterations of the mixed, using both full o modified Newton (MN) methods, and displacement descriptions are compared. This example highlights the excellent performances of the mixed description even more than the previous test.

Figure 14. T section beam: Equilibrium paths $\lambda - w_A$, $\lambda - v_A$

Figure 15. T section beam: Equilibrium paths in ξ_k space.

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Table VII. T section beam: steps and iterations for path–following analysis.

	Mixed	Displacement	Mixed MN
steps	20	42	55
loops	60	169	252

Figure 16. Simple frame

Figure 18. Simple frame: Equilibrium paths $\lambda - w_A$

5.3. A simple frame

Finally Fig. 16 reports the geometry and the material properties of a simple portal frame, similar to that analyzed in [**?**]. It consists of two beams with a C shaped section loaded as depicted in the same figure. Coupled instability is present also in this case. Both the beams are discretized using a mesh of $(6 + 12 + 6) \times 40$ elements. The node discretization is automatically defined from those of the two beams.

In Fig.17 the first 3 buckling modes considered in the multimodal Koiter analysis are reported.

Figure 17. Simple frame: First 3 Buckling modes

Table VIII. Simple frame: first 3 buckling loads.

	Mixed	Frozen
λ_1	588.92	641.01
λ_2	683.50	684.59
λ_3	1025.90	993.66

Also in this case the equilibrium paths recovered using both the asymptotic and path-following analysis are reported and compared in Fig.18. The frozen configuration asymptotic analysis furnishes a similar equilibrium path to the mixed one but an overestimate in the limit load. The mixed Koiter analysis accurately recovers the solution up to quite large displacements.

Figure 19. Simple frame: deformed shape at the limit point and equilibrium paths in ξ_k space

In Fig.19 the equilibrium path is plotted in terms of the modes amplitudes ξ_k showing a strong interaction among modes 1, 2. In the same figure the deformed configuration at the limit point is also reported.

Finally in Tab.IX the steps and iterations required by the mixed, using full and modified Newton method, and displacement descriptions are compared.

Table IX. Simple Frame: steps and iterations for path–following analysis.

	Mixed	Displacement	Mixed MN
steps	24	51	51
loops	73	196	198

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6. CONCLUSIONS

In this paper the better performances of the mixed format in the nonlinear analysis of slender structures have been shown and explained. To focus on the origin of this behavior, which is independent of the finite element interpolation, a displacement description of a mixed solid finite element has been derived. In this way it has been possible to show how the displacement description, and so any displacement finite element, is affected by an underhand and neglected extrapolation locking phenomenon that produces slow or lack of convergence for the path–following analyses and inaccurate solutions for the Koiter method. The occurence of the locking has been theoretically investigated and it has been indicated that it is due to the presence of directions with different stiffness as typically occurs for slender structures which are usually characterized by a high membranal/flexural stiffness ratio. These conclusions are general and hold for any nonlinear structural model and finite element.

Many advantages of solid elements in geometrically nonlinear analysis are already known in literature. In this paper we show further important properties of mixed solid FE within the Koiter asymptotic formulation. In fact, due to the simple 3rd order dependence of the strain energy on its FE parameters, all the higher order energy variations are null and so it is possible to have: i) an exact linear bifurcation analysis with improvements in its computational efficiency and accuracy for non near buckling loads; ii) simplification and greater accuracy in the evaluation of the energy variations required to recover the equilibrium path with a gain in terms of the computational cost; iii) a more simple and effective numerical method which is easy to include in FE packages.

For these reasons, even if mixed solid models might have more dofs with respect to 2D shells, their use in geometrically nonlinear analyses, seems even more convenient and attractive. In this sense it is of interest of the scientific community to develop high performance mixed solid elements.

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