A NEW DISCRETE DYNAMICAL SYSTEM OF SIGNED INTEGER PARTITIONS.

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Abstract. In [16] Brylawski described the covering property for the domination order on non-negative integer partitions by means of two rules. Recently, in [15, 18, 19] the two classical Brylawski covering rules have been generalized in order to obtain a new lattice structure in the more general signed integer partition context. Moreover, in [18, 19] the covering rules of the above signed partition lattice have been interpreted as evolution rules of a discrete dynamical model of a two-dimensional p-n semiconductor junction in which each positive number represents a distribution of holes (positive charges) located in a suitable strip at the left semiconductor of the junction and each negative number a distribution of electrons (negative charges) in a corresponding strip at the right semiconductor of the junction. In this paper we introduce and study a new sub-model of the above dynamical model , which is constructed by using a single vertical evolution rule. This evolution rule describes the natural annihilation of a hole-electron pair at the boundary region of the two semiconductors. We prove several mathematical properties of such new discrete dynamical model and we provide a discussion of its physical properties.

1. Introduction

1.1. **The BGK and BGKV Models.** The set of all the (non-negative) partitions of a (positive) integer m can be equipped of a partial order, usually called *dominance order*. Let us recall it: if $w = (w_0, ..., w_{l-1})$ and $w' = (w'_0, ..., w'_{l-1})$ are two nonincreasing sequences of non-negative integers having sum $m \in \mathbb{N}$ and equal length l, then the dominance order is defined by the following binary relation:

$$w \geq w' \Leftrightarrow \sum_{j=0}^{i} w_j \geq \sum_{j=0}^{i} w'_j \text{ for every } i. \tag{1}$$

In [16] Brylawski proved that the corresponding poset is indeed a lattice, denoted by $L_B(m)$, bounded by the least element (1^m) and the greatest element (m). The relevant fact to our aims is that the condition of covering in the lattice $L_B(m)$ can be expressed in an equivalent form as a result of Proposition 2.3 of [16]. Moreover, an original and clearer formulation of the above condition of covering has been formulated by Green and Kleitman in [31] by using two particular rules which operate under suitable conditions: the vertical displacement of one unit, called V-*rule*, and the horizontal displacement of one unit, called H-*rule*. In what follows we denote briefly by BGK the names Brylawski, Green and Kleitman and we also use the term BGK *rules* to refer to both the H-rule and the V-rule.

In this paper we adopt the useful point of view of a *discrete dynamical system*, briefly DDS, (see [32]) for the lattice $L_B(m)$. A dynamical system is any mathematical model where some rules, usually called *evolution rules*, describe how a point in a geometrical space depends on time. To standardize our terminology to DDS language, we call *configuration* any integer partition of $L_B(m)$ and *evolution rules* the BGK rules.
There are two ways in which we can apply the evolution rules on a given configuration of $L_B(m)$: a *sequential* way (in this case we also refer to *sequential dynamic* of the DDS) and a *parallel* way (in this case we also refer to *parallel dynamic* of the DDS). In the sequential way, we apply both the V-rule and the H-rule separately on each summand of the configuration. In the parallel way, we apply the V-rule concurrently on all the summands of the configuration. When we start from the initial configuration and we apply the evolution rules in a sequential way, generally we obtain an oriented graph which represents the Hasse diagram of a poset of integer partitions

with an order relation $\sqsubseteq$. In this case one usually proves that the relation w generates w', with some evolution rule, is equivalent to say that w covers w' with respect to the partial order $\sqsubseteq$. Therefore several concepts of the classical order theory find their interesting formulation in dynamic terms (for further details see also [12, 13, 14, 17, 18, 23, 24])

In the case of $L_B(m)$, we choose an initial configuration (that is usually the maximum (m) of the lattice $L_B(m)$) and next we apply the BGK rules starting from this initial configuration and terminating in the configuration (1^m), which remains fixed under the BGK rule action. In this way the set of all configurations of $L_B(m)$ becomes a DDS, which we denote by $BGK(m)$. In $BGK(m)$, each summand w_j of an integer partition w is interpreted as a single column containing w_j movable blocks stacked at the site j of a one-dimensional array of columns (the Ferrer diagram of the partition). Then, the two Brylawski conditions characterizing the lattice covering of two integer partitions w, w' under dominance order are equivalently formalized as follows: a single block can slip from a pile to the next available one following particular horizontal and vertical shifts described by BGK rules. Such shifts generate a suitable variation of the integer partition, dynamically denoted by $w \to w'$, under the constraint of the invariance of their sum. In we admit only the V-rule action in the lattice $L_B(m)$, then we obtain a sub model $BGKV(m)$ of $BGK(m)$ whose mathematical and physical properties have been studied in [17].

1.2. **The BGK Model for Signed Partitions.** A natural extension of the BGK system can be obtained by admitting also negative summands. In other words, we can introduce a new type of integer partitions having both positive summands and negative summands: if $m \in \mathbb{Z}$ is an integer, such a partition is a finite sequence $w = (w_0, \dots, w_{l-1})$, called *signed partition* of m and characterized by the conditions:

$$\sum_{i=0}^{l-1} w_i = m, \quad w_i \geq w_{i+1} \text{ for } i = 0, \dots l-2 \quad \text{and } w_i \in \mathbb{Z}.$$

From a purely arithmetical point of view, the study of partitions of this type is a recent development started in [5] and [33], where several interesting properties of the signed partitions have been investigated. On the other hand, from an order theory point of view, the study of the signed integer partitions started in [12], where it was motivated by a famous conjecture of Manickam and Singhi concerning a combinatorial sum problem (see [37]). For others studies concerning the order structures defined on signed integer partitions see also [12, 13, 14, 15, 23, 24].

When one tries to study signed integer partitions, the poset structure seems to vanish, due to the fact that between the "non-negative part" and the "non-positive part" of the partition we can insert an arbitrary number of 0's, which alter the definition (1). Moreover, the presence of these 0's create problems when we want to generalize the BGK rules in our signed partition context. Our paper can be considered divided into two parts. In the first part we build a formal structure which permits us to extend the Brylawski dominance order on the set of all signed integer partitions. More in detail, in Section 2 we introduce the concept of dominance order for signed integer partitions having sum m, where now m can be an arbitrary integer. In this new context the main novelty with respect to the classical Brylawski case [16] is that the set $P(m)$ of all signed partitions with sum m equipped with a binary relation, in some sense suggested by the (1), turns out to be a pre-ordered set. The canonical equivalence relation induced from this pre-order leads to an infinite poset, that we denote by $Par(m)$. In order to work in the infinite poset $Par(m)$, we construct a theoretical framework which allows us to define correctly the corresponding dominance order. In our construction, the poset $Par(m)$ turns out to be the colimit of suitable finite posets $O(m, n)$ introduced in [15, 18, 19], where $n = 0, 1, 2, \dots$, and so the study of $Par(m)$ can be restricted to the study of these posets $O(m, n)$. More formally, $O(m, n)$ is the set of all the sequences of the form $(w_0, \dots, w_{n-1}|w_n, \dots, w_{2n-1})$, where the w_i's are integers such that $n \geq w_0 \geq \cdots w_{n-1} \geq 0 \geq w_n \geq \cdots \geq w_{2n-1} \geq -n$, with $\sum_{i=0}^{2n-1} w_i = m$. The partial order $\geqslant$ that we consider on $O(m, n)$ is the natural extension of the classical dominance order (1). Specifically, if $w = (w_0, \dots, w_{n-1}|w_n, \dots, w_{2n-1})$ and

$w' = (w'_0, \dots, w'_{n-1} | w'_n, \dots, w'_{2n-1})$ are two elements of $O(m, n)$ then

$$w \geqslant w' \Leftrightarrow \sum_{j=0}^{i} w_j \geq \sum_{j=0}^{i} w'_j \text{ for every } i. \tag{2}$$

1.3. **The BGKV Model for Signed Partitions.** In the second part of this paper, we generalize the construction of the sub model $BGKV(m)$ of $BGK(m)$ to the signed partition case. Therefore, by analogy with the non-negative case, we introduce and study a new sub-system $OV(m, n)$ of $O(m, n)$. The sub-system $OV(m, n)$ is defined by means of one only evolution rule, that is a generalized version of the BGK V-rule. This rule is necessary to manage the balance between positive and negative summands of any signed integer partition of our new model. Such a rule corresponds to a "vertical" restriction of the Rule 5 in a discrete dynamical system $O(m, \eta, n)$ introduced in [19]. From a physical point of view, $O(m, \eta, n)$ describes a p-n junction of two semiconductors: the positive part represents a distribution of holes (considered as positive electrical charges) in one of the two semiconductors dumped by acceptor atomic impurities and the negative part a distribution of electrons (negative electric charges) in the other semiconductor dumped by donor atomic impurities. The two vertical and horizontal rules describe the movements under the Coulomb attraction of positive (resp., negative) charges of the left (resp., right) semiconductor by the opposite charges of the other semiconductor. This produces a diffusion or flow of holes from the left to the right and of electrons from the right to the left. It is the Rule 5 which expresses the annihilation of a hole-electron pair at the boundary between the two semiconductors (process of recombination of single hole-electron pairs).
Let us note that the main result proved in [19] is that the dominance order (2) on $O(m, n)$ coincides with the dynamical order induced from five evolution rules. The first two are just the application of the standard vertical and the horizontal rules to the non-negative part of an element from $O(m, n)$. Rules 3 and 4 correspond to their dual versions applied to the nonpositive part. Finally, Rule 5 expresses an interaction between the non-negative and the nonpositive parts with the delation of a pair of units, giving in this way a real generalization of the Green-Kleitman interpretation of the Brylawski dominance lattice.
The sub-system $OV(m, n)$ has several interesting mathematical and physical properties.
From a physical point of view we have to do with a simplified model of p-n semiconductors junction with annihilation of hole-electron pairs, with only vertical movements of charges but no horizontal flowing of them.
On the other hand, from a mathematical point of view, the presence of one only rule, i.e. the generalized version of the BGK V-rule, leads to different results with respect to the bigger model $O(m, n)$. For example, in $OV(m, n)$ one can study both the parallel dynamic (see [1, 2, 3, 4]) and the sequential dynamic (see [38, 39, 40]). For example, if we denote by $T_{par}(w(0))$ the parallel time of convergence (see Definition 4.11) of a generic element $w(0)$ starting from an initial instant $t = 0$ towards its unique fixed point, we provide an estimate of such time (see Proposition 4.13). In [23] the non-negative integer $T_{par}(w(0))$ has been studied in others signed partition lattices having also a DDS structure. From a purely order theory point of view, $T_{par}(w(0))$ can be considered a "parallel version" of the usual rank notion in a graded lattice (see [23]).
Another mathematical aspect that in $OV(m, n)$ assumes a not trivial form is the determination of the so called *fixed points*, i.e. the points which are invariant with respect to the action of the only generalized V-rule. In fact, whereas in $O(m, n)$ the determination of the unique fixed point under the action of the generalized BGK rules is trivial, in $OV(m, n)$ the same problem (under the unique action of the generalized V rule) requires the introduction of a more mathematical sophisticated technique (see Section 6) for its solution.
Let us note that in the BGK model the study of the parallel dynamic by means of both the V-rule and the H-rule is an interesting open problem. In fact, some problems arise when one try to apply the H-rule in a parallel dynamic on some types of integer partitions. A similar situation occurs in the discrete dynamical system $O(m, n)$ when one try to apply the corresponding BGK generalized rules. For a study of the sequential dynamic in $O(m, n)$ see [15] and [19].

We conclude this introduction by pointing out that the present paper fits into a more general research context concerning the study of discrete dynamical models in several disciplines: graph theory [4, 35, 36], computer simulation theory [6, 7, 8, 9], Boolean function theory [1, 11], combinatorics [10, 25, 26, 27, 34, 35, 36], cellular automata theory [20, 21, 22, 28].

Below we provide a brief description of the content of the sections in this paper.

In Section 2 we formally describe the poset $Par(m)$. As said before, to do this we introduce at first a pre-order on the larger set $P(m)$ and next define the dominance partial order $\trianglelefteq$ on $Par(m)$ as a quotient order of this pre-order on $P(m)$. We also use the language of category theory in order to characterize $Par(m)$ as the colimit of the family $\{O(m,n) : n \in \mathbb{N}\}$.

In Section 3 we show that $O(m,n)$ has a lattice structure with a maximum $\hat{1}^{m,n}$ and a minimum $\hat{0}^{m,n}$. As a direct consequence of this result we deduce that also the infinite poset $Par(m)$ is a lattice.

Section 4 is crucial in this paper because there we formally describe the "vertical" restriction of the Rule 5 introduced in [19] (Rule 5 is necessary to balance the positive and the negative summands in the context of $O(m,n)$). This restriction will be the tool that allows us to study, in the more natural way, the generalization of the BGK V-rule in the new context of the signed integer partitions. We then describe the sequential and the parallel dynamic for the signed integer partitions. In this section we build the discrete dynamical system $OV(m,n)$ of all the signed integer partitions of $O(m,n)$ that we can obtain after iterated sequential applications of the above described "vertical" restriction of the Rule 5 on the maximum $\hat{1}^{m,n}$ of $O(m,n)$. Finally, we prove several results concerning the sequential and parallel dynamic in $OV(m,n)$ and we show that $OV(m,n)$ is a graded sub-poset of the (not graded) lattice $O(m,n)$.

Section 5 is a rather technical part. However the results proved in this section will be essential to prove the statements of Section 6.

In Section 6 we find interesting representations for the fixed points of the poset $OV(m,n)$.Using the previous technical results, we are able to compute the rank of the poset $OV(m,n)$.

2. Dominance Order for Signed Partitions

A signed partition of an integer m is simply a finite, non-increasing sequence of relative integers which sums up to m. The set of all (relative) partitions of m is clearly not finite. In such sequence we can find some 0s (grouped together) at some point. Of course, those 0s are not influential on the sum, yet they play a role in the definition of the ordering. Moreover, we can consider some, none or all of them as belonging to the "non-negative part" of the partition and the remaining to the "non-positive part": this is relevant for the definition of the transition rule. Given this, when studying signed partitions we have to set up a framework which allows us to cope with these two difficulties: this is the aim of this section and we will also see how the set of signed partitions of an integer m can be studied "locally", i.e. by means of suitable finite sets which, in a categorical way, yield the signed partition set as a colimit.

First, we define formally the set of partitions of an integer m in such a way as to take into account the 0s inside the partition.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. We denote by $\bigoplus_{\mathbb{N}} \mathbb{Z}$ the set of all the infinite sequences of the form $\{w_i\} = \{w_0, w_1, w_2, \dots\}$, where $w_i \in \mathbb{Z}$ and $w_l = w_{l+1} = w_{l+2} = \cdots = 0$ for some $l \geq 0$.

Definition 2.1. *Given an integer $m \in \mathbb{Z}$ and two natural number $p, q \in \mathbb{N}$, a* signed partition *(abbreviated, s-*partition*) of m with balance (p,q) is an element $w = (\{w_i\}, p, q) \in \bigoplus_{\mathbb{N}} \mathbb{Z} \times \mathbb{N} \times \mathbb{N}$ such that:*

(1) $w_i \geq w_{i+1} \geq 0$ *for* $i = 0, \dots, p-1$*;*
(2) $0 \geq w_i \geq w_{i+1}$ *for* $i = p, \dots, p+q-1$*;*
(3) $w_i = 0$ *for* $i \geq p+q$*;*
(4) $\sum_i w_i = m$.

We write $w \vdash m$ when w is a signed partition of m and we denote by $P(m)$ the set of all signed partitions of m. We call length *of w, denoted $l(w)$, the sum $p+q$.*

If $w \in P(m)$ we call $w_{\geq} := (w_0, \dots, w_{p-1})$ the *non-negative* part of w and $w_{\leq} := (w_p, \dots, w_{p+q-1})$ the *non-positive* part of w. Moreover, we call *signature* of w the ordered pair (t, s) of non-negative integers such that $t = \max\{i \in \mathbb{N} : w_i > 0\} + 1$ if $w_0 > 0$ (or $t = 0$ otherwise) and $s = \max\{j \in \mathbb{N} : w_{p+q-j} < 0\}$. Then, it is natural to say that $w_> := (w_0, \dots, w_{t-1})$ is the *positive* part of w and that $w_< := (w_{p+q-s}, \dots, w_{p+q-1})$ is the *negative* part of w.

We also set $p = |w_{\geq}|$, $q = |w_{\leq}|$, $t = |w_>|$, $s = |w_<|$.

For shortness, in the sequel we write w in the form

$$w = (w_0, \dots, w_{p-1} | w_p, \dots, w_{p+q-1})$$

(or even, in the numerical examples, $w = w_0 \dots w_{p-1} || w_p | \dots | w_{p+q-1}|$, where $|w_j|$ is the absolute value of w_j).
If $w = (w_0, \dots, w_{p-1} | w_p, \dots, w_{p+q-1})$ and $w' = (w'_0, \dots, w_{p'-1} | w_{p'}, \dots, w_{p'+q'-1})$ are two given s-partitions of m, then they are equal if and only if $w_i = w'_i$ for every i and $(|w_{\geq}|, |w_{\leq}|) = (|w'_{\geq}|, |w'_{\leq}|)$. Note that, in this case, we also have $(|w_>|, |w_<|) = (|w'_>|, |w'_<|)$.

Example 2.2. Consider the sequence w_i defined by

$$w_i = \begin{cases} 5 & \text{if } i = 0, 1, 2 \\ 0 & \text{if } i = 3, 4, 5 \\ -1 & \text{if } i = 6, 7 \\ 0 & \text{if } i > 7 \end{cases} \tag{3}$$

Its total sum is 13. From our definition, it follows that in $P(13)$ there are 4 balanced signed partitions which arise from this signed partition:

Partition	*balance*	*signature*
555000|11	$(6, 2)$	$(3, 2)$
55500|011	$(5, 3)$	$(3, 2)$
5550|0011	$(4, 4)$	$(3, 2)$
555|00011	$(3, 5)$	$(3, 2)$

Definition 2.3. *Let* $w = (w_0, \dots, w_{p-1} | w_p, \dots, w_{p+q-1})$ *be an s-partition with balance* (p, q)*, signature* (t, s)*, positive and negative parts*

$$w_> = \{w_0, \dots, w_{t-1}\} \qquad w_< = \{w_{p+q-s}, \dots, w_{p+q-1}\},$$

respectively; let also u, v *be two non-negative integers. The* reduced *s-partition of* w *is*

$$w_* := (w_0, \dots, w_{t-1} | w_{p+q-s}, \dots, w_{p+q-1}).$$

The (u, v)-extension *of* w *is*

$$\begin{aligned} \overline{w}_{u,v} &:= (w_0, \dots, w_{p-1}, 0_u | 0_v, w_p, \dots, w_{p+q-1}) \\ &:= (w_0, \dots, w_{p-1}, \underbrace{0, \dots, 0}_{u \text{ times}} | \underbrace{0, \dots, 0}_{v \text{ times}}, w_p, \dots, w_{p+q-1}) \end{aligned}$$

Relations between w and its derived summarize in the following table

Partition	*balance*	*signature*
w	(p, q)	(t, s)
w_*	(t, s)	(t, s)
$\overline{w}_{u,v}$	$(p + u, q + v)$	(t, s)

If $U \subset P(m)$, we say that U is (p, q)-*uniform* (or simply *uniform*) if all s-partitions in U of the integer m have the same balance (p, q).

Definition 2.4. *For* $W \subset P(m)$*, let* $W_* = \{w_* | w \in W\}$*.*
For W *a finite set then define the* uniform closure $\overline{W}$ *of* W *as the set*

$$\overline{W} = \{\overline{w}_W := \overline{w}_{p_W - p, q_W - q} : w = (\{w_i\}, p, q) \in W\}$$

where $p_W = \max\{p : w = (\{w_i\}, p, q) \in W\}$ *and* $q_W = \max\{q : w = (\{w_i\}, p, q) \in W\}$.

If W is clear from the context, we omit the subscript in p_W and q_W.

From the definition, $\overline{W}$ is (p_W, q_W)-uniform.

If $w, w' \in P(m)$ are two uniform s-partitions of m, i.e., iff the subset $W = \{w, w'\}$ of $P(m)$ is (p, q)-uniform, define

$$w \leqslant w' \iff w_0 + w_1 + w_2 + \cdots + w_i \leq w'_0 + w'_1 + w'_2 + \cdots + w'_i \quad \text{for all } i \in \{0, 1, \ldots, 2n-1\}. \tag{4}$$

In particular, note that $w \leqslant w'$ and $w' \leqslant w$ if and only if $w = w'$.

Now we make some remarks about the particularly important case of two generic signed partitions. Let $W = \{w, w'\}$ be a collection of only two signed partitions from $P(m)$ of the following types:

- $w = (w_0, \ldots, w_{p-1} | w_p, \ldots, w_{p+q-1})$ is a signed partition of the integer m, and balance (p, q);
- $w' = (w'_0, \ldots, w'_{p'-1} | w'_{p'}, \ldots, w'_{p'+q'-1})$ is also a signed partition of m, but of balance (p', q').

Let us set $p_W = \max\{p, p'\}$ and $q_W = \max\{q, q'\}$ and let us introduce the following new two signed partitions of the given integer m:

$$\overline{w} = (w_0, \ldots, w_{p-1}, 0_{p_W - p} | 0_{q_W - q}, w_p, \ldots, w_{p+q-1})$$
$$\overline{w'} = (w'_0, \ldots, w'_{p'-1}, 0_{p_W - p'} | 0_{q_W - q'}, w'_{p'}, \ldots, w'_{p'+q'-1})$$

Then the balance of $\overline{w}$ is $(p_W - p + p, q_W - q + q) = (p_W, q_W)$, whereas the balance of $\overline{w'}$ is $(p_W - p' + p', q_W - q' + q') = (p_W, q_W)$. In conclusion, altough w and w' have different balances, their closures $\overline{w}$ and $\overline{w'}$ have both balance (p_W, q_W) thus forming a uniform family; moreover, they remain signed partitions of the original integer m.

Example 2.5. Let us consider the family $W_{1,3}$ consisting of two s-partitions $w_1 = (5, 5, 5, 0, 0, 0| -1, -1)$ and $w_3 = (5, 5, 5, 0|0, 0, -1, -1)$ of the integer $m = 13$ induced by the same sequence (3) of the Example 2.2 and of balance $(6, 2)$ and $(4, 4)$, respectively. Then, their $W_{1,3}$ closures are identical $\overline{w_1}_{W_{1,3}} = \overline{w_3}_{W_{1,3}} = (5, 5, 5, 0, 0, 0|0, 0, -1, -1)$, with balance $(6, 4)$.

On the other hand, the family from the same Example 2.2 $W_{2,3} = \{w_2 = (5, 5, 5, 0, 0|0, -1, -1),\ w_3 = (5, 5, 5, 0|0, 0, -1, -1)\}$, whose elements have balance $(5, 3)$ and $(4, 4)$, respectively, have the $W_{2,3}$ closures always identical between them $\overline{w_2}_{W_{2,3}} = \overline{w_3}_{W_{2,3}} = (5, 5, 5, 0, 0|0, 0, -1, -1)$, with balance $(5, 4)$, different from the above $W_{1,3}$ case.

So the family $\{w_2, w_3\}$ generates the same closures of balance $(6, 4)$ and the family $\{w_1, w_3\}$ generates the same closures of balance $(5, 4)$, but different between them. Let us show a kind of transitivity in the sense that the family $W_{1,2} = \{w_1 = (5, 5, 5, 0, 0, 0| -1, -1), w_2 = (5, 5, 5, 0, 0|0, -1, -1)\}$ also generates the same closures $\overline{w_1}_{W_{1,2}} = \overline{w_2}_{W_{1,2}} = (5, 5, 5, 0, 0, 0|0, -1, -1)$ of balance $(6, 3)$.

Finally, the collection of all induced s-partitions $W_t = \{w_1 = (5, 5, 5, 0, 0, 0| -1, -1), w_2 = (5, 5, 5, 0, 0|0, -1, -1), w_3 = (5, 5, 5, 0|0, 0, -1, -1), w_4 = (5, 5, 5|0, 0, 0, -1, -1)\}$ is characterized by the pair $p_{W_t} = 6$ and $q_{W_t} = 5$ leading to the same W_t-closures $\overline{w_1}_{W_t} = \overline{w_2}_{W_t} = \overline{w_3}_{W_t} = \overline{w_4}_{W_t} = (5, 5, 5, 0, 0, 0|0, 0, 0, -1, -1)$.

Using this, we can define another binary relation: if $w, w' \in P(m)$, take $W = \{w, w'\}$ and let us construct its closure $\overline{W} = \{\overline{w}, \overline{w'}\}$ which, as previously noted, is (p_W, q_W)-uniform. Then by (4) the following binary relation on $P(m)$ is well defined:

$$w \trianglelefteq w' \iff \overline{w} \leqslant \overline{w'}. \tag{5}$$

In particular, if w and w' form a uniform pair (i.e., they have the same balance (p, q)), then $w \trianglelefteq w'$ iff $w \leqslant w'$.

Proposition 2.6. *The relation* $\trianglelefteq$ *is a quasi-order on the set* $P(m)$*; moreover, if* $w, w' \in P(m)$ *the following conditions are equivalent:*

(1) *$w \trianglelefteq w'$;*
(2) *$w_* \trianglelefteq w'_*$;*
(3) *there exists a finite subset W of $P(m)$ that contains w and w' such that $\overline{w}_W \leqslant \overline{w'}_W$;*
(4) *$\overline{w}_W \leqslant \overline{w'}_W$ for each finite subset W of $P(m)$ that contains w and w'.*

Proof. Straightforward. □

If $w, w' \in P(m)$, we set

$$w \sim w' \iff \{w_i : w_i > 0\} = \{w'_j : w'_j > 0\} \text{ and } \{w_i : w_i < 0\} = \{w'_j : w'_j < 0\} \tag{6}$$

Then $\sim$ is an equivalence relation on $P(m)$ and it holds

$$w \sim w' \iff w \trianglelefteq w' \text{ and } w' \trianglelefteq w \tag{7}$$

Example 2.7. With reference to the discussion of Example 2.5,

Proposition 2.8. *Let $w, w' \in P(m)$. Then:*

(1) *$w \sim w_*$;*
(2) *$w \sim \overline{w}_{u,v}$ for every natural integers u, v;*
(3) *$w = w'$ iff w and w' are uniform and $w \sim w'$;*
(4) *if $w, w' \in W$ then $w \sim w'$ iff $\overline{w}_W = \overline{w'}_W$.*
(5) *if $w \in W$ then $w \sim \overline{w}_W$.*

Proof. Straightforward. □

Example 2.9. With reference to the discussion of example 2.5, we have seen that with respect to the family $W_t = \{w_1, w_2, w_3, w_4\}$ all its elements have the same W_t-closure and so, according to condition (4) of Proposition 2.8, all the elements from W_t belong to (i.e., generate) the same equivalence class.

In particular, the pair $W_{1,3} = \{w_1, w_3\}$ generates the same $W_{1,3}$-closure and the pair $W_{2,3}$ generates the same $W_{2,3}$-closure, different between them, but the "transitive" family $W_{1,2} = \{w_1, w_2\}$ generates the same $W_{1,2}$-closure in its turn different from the previous two.

By equation (7) it follows that $\sim$ is exactly the equivalence relation on $P(m)$ induced from the quasi-order $\trianglelefteq$. Therefore, if W is any subset of $P(m)$, we can consider on the quotient set $W/\sim$ the usual partial order induced by $\trianglelefteq$; we denote it by $\trianglelefteq'$. We recall that $\trianglelefteq'$ is defined as follows: if $Z, Z' \in W/\sim$ then

$$Z \trianglelefteq' Z' \iff w \trianglelefteq w' \tag{8}$$

for any/all $w, w' \in W$ such that $w \in Z$ and $w' \in Z'$.

We introduce now the principal object of our investigations.

Definition 2.10. *We call* signed dominance order poset *of the integer m the partially ordered set $Par(m) := (P(m)/\sim, \trianglelefteq')$, and* signed dominance order *the partial order relation $\trianglelefteq'$.*

Lemma 2.11. *For any $w \in P(m)$, w_* is the element of the equivalence class of w with minimal length.*

Proof. Straightforward. □

Let now n be a fixed non-negative integer and define the collection of all signed partitions w of the integer m, balance (n, n), and *total height* $h(w) := \max\{|w_i|\} \leq n$, according to the following:

$$O(m, n) := \{(w_0, \ldots, w_{n-1} | w_n, \ldots, w_{2n-1}) \in P(m) : \ |w_i| \leq n \text{ for every } i\}$$

Let us stress that any s-partition from $O(m, n)$ has total length $2n$, with non-negative and non-positive parts of equal lengths n. In other words, $O(m, n)$ is (n, n)-uniform.

Lemma 2.12.

$$(O(m, n)/\sim, \trianglelefteq') = (O(m, n), \trianglelefteq) = (O(m, n), \leqslant)$$

In particular, let $w, w' \in O(m, n)$ then $w \sim w'$ iff $w = w'$

Proof. This follows from (3) of Proposition 2.8 since as stressed before the family $O(m,n)$ is (n,n)-uniform, so in $O(m,n)$ being equivalent means being equal. $\square$

Thanks to Lemma 2.12 we can identify every element of $O(m,n)$ with its equivalence class in $Par(m)$: in this way we can consider $O(m,n) \subset Par(m)$.

Lemma 2.13. *If $w = (w_0, \dots, w_{p-1}|w_p, \dots, w_{p+q-1}) \in P(m)$ and $[w]$ is its equivalence class, define*

$$n_w = \max\{p, q, |w_0|, \dots, |w_{p+q-1}|\}.$$

Then $[w] \cap O(m, n_w) \neq \emptyset$; moreover, n_w is the smallest integer for which this holds.

Proof. Since $n_w \geq p$ and $n_w \geq q$, then according to Definition 2.3 the signed partition of m

$$\overline{w}_{n_w-p,n_w-q} = (w_0, \dots, w_{p-1}, 0_{n_w-p}|0_{n_w-q}, w_p, \dots, w_{p+q-1})$$

is well defined, of signature (n_w, n_w), and is equivalent to w; since $n_w \geq |w_i|$ for every i, it is in $O(m, n_w)$.

If one of the inequalities is not satisfied, either $\overline{w}_{n_w-p,n_w-q}$ is not defined or it is not in $O(m, n_w)$. $\square$

We can say more about $[w] \cap \bigcup_{n\in\mathbb{N}} O(m,n)$. For every n, let us consider the function i_n defined as follows:

$$\begin{aligned} i_n\colon O(m,n) &\to O(m,n+1) \\ w \ &\mapsto \ \overline{w}_{1,1} \end{aligned}$$

It is clear that i_n is an embedding of $O(m,n)$ into $O(m,n+1)$ which preserves the ordering.

Lemma 2.14. *If $w \in O(m,n)$, $w' \in O(m,n')$ and $n \leq n'$, then*

$$w \sim w' \Leftrightarrow w' = \overline{w}_{n'-n,n'-n} = i_{n'-1} \circ \cdots \circ i_{n+1} \circ i_n(w)$$

Proof. By induction it is enough to prove the case $n' = n+1$ and this case easily follows from the previous propositions. $\square$

Now let $\boldsymbol{\omega}$ be the category $\{0 \to 1 \to 2 \to 3 \to \cdots\}$ and let **POSet** denotes the category of posets. By Lemma 2.12 $O(m,n)$ is in **POSet**. The fact that i_n is an order preserving embedding for every n means that

$$\begin{aligned} F\colon \boldsymbol{\omega} &\to \mathbf{POSet} \\ n &\mapsto O(m,n) \\ n \to n+1 &\mapsto i_n \end{aligned}$$

is a functor.

Let $U(m)$ be the colimit of F in the category of sets:

$$U(m) = \mathrm{colim} F = \varinjlim O(m,n).$$

Theorem 2.15. *The ordering $\trianglelefteq'$ is defined in $U(m)$ and the posets $Par(m)$ and $(U(m), \trianglelefteq')$ are isomorphic.*

Proof. We first give a description of $U(m)$. Let $X = \underset{n\in\mathbb{N}}{\cup} O(m,n)$. Remark that the sets $O(m,n)$ and $O(m,n')$ are disjoint if $n \neq n'$. Define

$$U(m) = \bigcup_{n\in\mathbb{N}} \big\{\omega = (w, \overline{w}_{1,1}, \overline{w}_{2,2}, \dots) : w \in O(m,n) \setminus i_{n-1}(O(m,n-1))\big\}.$$

By lemma 2.14 it follows that if $w \in O(m,n)$ then there exists exactly one $\omega \in U(m)$ such that w is one of the coordinates of ω. Let μ_n be the map which associates this ω to the given w. It is a map from $O(m,n)$ to $U(m)$ such that, by definition, $\mu_n = \mu_{n+1} \circ i_n$.

It is easy to check that $\big(U(m), (\mu_n)_{n\in\mathbb{N}}\big)$ satisfies the universal property for the colimit. Moreover, it is clear that given two elements of $U(m)$ the ordering $\trianglelefteq'$ can be defined simply by comparing any two (uniform) coordinates of the two given sequences.

Let j_n be the inclusion $O(m,n) \subset Par(m)$. If $w \in O(m,n)$, the relation $w \sim \overline{w}_{1,1}$ means $j_n = j_{n+1} \circ i_n$ and then the universal property of the colimit provides a map $\psi\colon U(m) \to Par(m)$ such that $j_n = \psi \circ \mu_n$ for every n.

Using Lemma 2.13, let us define

$$\begin{aligned} f\colon P(m) &\to \bigcup_{n\in\mathbb{N}} O(m,n) \\ w &\mapsto \overline{w}_{n_w-p,n_w-q} \end{aligned}$$

By construction, $f(w) \in O(m,n_w)$, so $g(w) \doteqdot \mu_{n_w} \circ f(w) \in U(m)$.

Assume $w' = (w'_0, \dots, w'_{a-1} | w'_a, \dots, w'_{a+b-1}) \in P(m)$ is such that $w \sim w'$. Without loss of generality, we can suppose $n_w \leq n_{w'}$; then, by Lemmas 2.13 and 2.14 we have that $f(w) \sim f(w')$. Specifically,

$$f(w') = i_{n_{w'}-1} \circ i_{n_{w'}-2} \circ \cdots \circ i_{n_w} \circ f(w) = \overline{f(w)}_{n_{w'}-n_w, n_{w'}-n_w}$$

which implies

$$\begin{aligned} g(w') &= \mu_{n_{w'}} \circ f(w') \\ &= \mu_{n_{w'}} \circ i_{n_{w'}-1} \circ i_{n_{w'}-2} \circ \cdots \circ i_{n_w} \circ f(w) \\ &= \mu_{n_{w'}-1} \circ i_{n_{w'}-2} \circ \cdots \circ i_{n_w} \circ f(w) \\ &\dots \\ &= \mu_{n_w+1} \circ i_{n_w} \circ f(w) \\ &= \mu_{n_w} \circ f(w) \\ &= g(w) \end{aligned}$$

so g is invariant on equivalence classes and then it induces a $\phi\colon Par(m) \to U(m)$.

It is clear from the definition that ψ and ϕ are inverses of each other and that they preserve the ordering. □

Remark 2.16. *We point out, from the proof, the explicit construction of ϕ and ψ.*

If $w' \in Par(m)$, let $n = n_{w'_*}$, $w = f(w'_*) \in O(m,n)$, then

$$\phi(w') = (w, \overline{w}_{1,1}, \overline{w}_{2,2}, \dots).$$

In the other direction, if $\omega = (w, \overline{w}_{1,1}, \overline{w}_{2,2}, \dots)$ is a generic element of U (for some n and some $w \in O(m,n)$), then $\psi(\omega) = [w] = [w_*]$.

This is the framework we were looking for. Indeed, the ordering on non-negative partitions does not extend to an order of signed partitions $P(m)$: it is just a preorder. To have a poset structure we need to quotient out to $Par(m)$. On the other hand, when restricted to the subsets $O(m,n) \subset P(m)$ the preorder is in fact an ordering; moreover, $O(m,n)'s$ are also subposets of $Par(m)$ and studying them provides a fairly good description of $Par(m)$, since $Par(m)$ is a direct limit of the $O(m,n)$. In some sense, we can think of $O(m,n)$ as to a "local snapshot" of $Par(m)$ which we can "enlarge" at will up to infinity, in order to have a sufficiently detailed view of $Par(m)$.

3. Basic properties of $O(m,n)$

Given the results of Section 2 about the extension to signed partitions of the Brylawski dominance order applied to integer partitions, we can reduce our study to the poset $O(m,n)$ with respect to the restriction of this dominance order where, thanks to Lemma 2.12, we can simplify the notation significantly: if w and w' are two s-partitions in $O(m,n)$ then

(9) $$w \trianglelefteq w' \iff w_0 + w_1 + \cdots + w_i \leq w'_0 + w'_1 + \cdots + w'_i$$

for all $i \in \{0, 1, \dots, 2n-1\}$.

Since $w \vdash m$ and $w' \vdash m$, (9) is equivalent to

(10) $$w \trianglelefteq w' \iff w_{i+1} + \cdots + w_{2n-1} \geq w'_{i+1} + \cdots + w'_{2n-1}$$

for all $i \in \{0, 1, \dots, 2n-1\}$.

With the notation $v \lhd w$ we mean $v \unlhd w$ with $v \neq w$. We also write $w \lessdot w'$ if w' covers w with respect to the partial order $\unlhd$.

Recalling that the elements $w \in O(m,n)$ are subject to the constraint $\max\{|w_i|\} \leq n$, we get that $O(m,n)$ is non empty if and only if $-n^2 \leq m \leq n^2$, so in the sequel we shall assume $-n^2 \leq m \leq n^2$. On the other hand, it is easy to verify that the map

$$(w_0,\dots,w_{n-1}|w_n,\dots,w_{2n-1}) \mapsto (-w_{2n-1},\dots,-w_n|-w_{n-1},\dots,-w_0) \tag{11}$$

is a poset isomorphism of $O(m,n)$ into $O(-m,n)$, therefore in the proof of our results we can restrict ourselves to consider the cases $0 \leq m \leq n^2$ or $-n^2 \leq m \leq 0$.

We set now

$$\hat{1}^{m,n} := \begin{cases} (n,\dots,n,r,0,\dots,0|-n,\dots,-n) & \text{if } \ m<0 \\ (n,\dots,n|0,\dots,0,-r,-n,\dots,-n) & \text{if } \ m>0 \\ (n,\dots,n|-n,\dots,-n) & \text{if } \ m=0, \end{cases}$$

with n and $-n$ repeated exactly k times respectively when $m<0$ and $m>0$, where k and r are the unique non-negative integers such that $n^2-|m| = kn+r$, with $r<n$.

We also set

$$\hat{0}^{m,n} := \begin{cases} (0,\dots,0|-k,\dots,-k,-(k+1),\dots,-(k+1)) & \text{if } \ m<0 \\ (k+1,\dots,k+1,k,\dots,k|0,\dots,0) & \text{if } \ m>0 \\ (0,\dots,0|0,\dots,0) & \text{if } \ m=0, \end{cases}$$

with $-(k+1)$ and $k+1$ repeated exactly r times respectively when $m<0$ and $m>0$, where k and r are the unique non-negative integers such that $|m| = kn+r$, with $r<n$. Let us note that $r>0$ implies $k+1 \leq n$ because $|m| \leq n^2$.

Theorem 3.1. *$O(m,n)$ is a lattice with maximum $\hat{1}^{m,n}$ and minimum $\hat{0}^{m,n}$.*

Proof. We begin to observe that $\hat{0}^{m,n} \unlhd w \unlhd \hat{1}^{m,n}$ for all $w \in O(m,n)$. We denote now by $H(m,n)$ the set of all the $(2n+1)$-tuples of integers $T = (t_{-1},t_0,\dots,t_{n-1} : t_n,\dots,t_{2n-1})$ having the following properties:
(H1): $0 = t_{-1} \leq t_0 \leq \cdots \leq t_{n-2} \leq t_{n-1}$ and $t_{n-1} \geq t_n \geq t_{n+1} \geq \cdots \geq t_{2n-1} = m$ (unimodality).
(H2): $2t_i \geq t_{i-1}+t_{i+1}$ for $i=0,\dots,2n-2$ (concavity).
(H3): $t_0 - t_{-1} \leq n$ and $t_{2n-1}-t_{2n-2} \geq -n$.

If $w = (w_0,\dots,w_{n-1}|w_n,\dots,w_{2n-1}) \in O(m,n)$, we set

$$\hat{w} = (s_{-1},s_0,\dots,s_{n-1} : s_n,\dots,s_{2n-1}),$$

where $s_{-1} := 0$, $s_k := \sum_{i=0}^{k} w_i$ for $k=0,1,\dots,2n-1$.

It is immediate to verify that $\hat{w} \in H(m,n)$. Therefore we can consider the application $\Lambda : O(m,n) \to H(m,n)$ such that $\Lambda(w) = \hat{w}$.

The map Λ is bijective.

In fact, it is immediate to see that Λ is injective.

We examine the surjectivity. If $T = (t_{-1},t_0,\dots,t_{n-1} : t_n,\dots,t_{2n-1}) \in H(m,n)$, we take $w_i := t_i - t_{i-1}$ for $i=0,\dots,2n-1$. By (H1), (H2) and (H3) we deduce then that $w = (w_0,\dots,w_{n-1}|w_n,\dots,w_{2n-1})$ is an element of $O(m,n)$ such that $\hat{w} = T$.

If $T' = (t'_{-1},t'_0,\dots,t'_{n-1} : t'_n,\dots,t'_{2n-1})$ and $T'' = (t''_{-1},t''_0,\dots,t''_{n-1} : t''_n,\dots,t''_{2n-1})$ are two elements of $H(m,n)$, we set $\omega(T',T'') := (t_{-1},t_0,\dots,t_{n-1} : t_n,\dots,t_{2n-1})$, where $t_i = \min\{t'_i,t''_i\}$ for $i=-1,0,\dots,2n-1$. It is easy then to verify that $\omega(T',T'') \in H(m,n)$.

Let now $w',w'' \in O(m,n)$ and $T = \omega(\Lambda(w'),\Lambda(w''))$.

Since $T \in H(m,n)$ and Λ is bijective, there exists a unique element $w \in O(m,n)$ such that $\hat{w} = T$. From the definitions of $\hat{w}$ and of dominance order it follows that w is exactly the maximal lower bound of the elements w' and w'' with respect to the partial order $\unrhd$. Hence $w = w' \wedge w''$.

Finally, if $w, w' \in O(m,n)$ the set of the upper bounds of w and w' is not-empty because it always contains $\hat{1}^{m,n}$, therefore we take $u = \bigwedge\{z \in O(m,n) : z \trianglerighteq w, z \trianglerighteq w'\}$ in order to have $u = w' \vee w''$. □

Example 3.2. Let $m = -7$ and $n = 8$. If we take $w' = 75444220|44444555$ and $w'' = 85433220|33334468$ as elements of $O(-7,8)$, then

$$\Lambda(w') = (0,7,12,16,20,24,26,28,28 : 24,20,16,12,8,3,-2,-7)$$

and

$$\Lambda(w'') = (0,8,13,17,20,23,25,27,27 : 24,21,18,15,11,7,1,-7).$$

So that, if $T = \omega(\Lambda(w'),\Lambda(w''))$, then

$$T = (0,7,12,16,20,23,25,27,27 : 24,20,16,12,8,3,-2,-7).$$

The signed partition $w \in O(-7,8)$ such that $\hat{w} = T$ is $w = 75443220|34444555$. Hence $w = w' \wedge w''$.

As a direct consequence of the previous theorem we obtain the following result.

Corollary 3.3. *$Par(m)$ is a lattice.*

Proof. It is suffices to note that the poset $Par(m)$ can be identified with the increasing union of its sub-posets $O(m,n)$, for $n = 0,1,2,\ldots$. □

3.1. **Physical interpretation.** Thus, in this section we obtained the signed partition generalization of the Brylawski result on integer partitions: the family $O(m,n)$ of all signed partitions w of the integer m, with balance (n,n) and satisfying the condition $\max\{|w_i|\} \le n$, is a lattice with respect to the generalized version of dominance order (9).

We recall the main result of [19], already anticipated in the introduction, and consisting in the proof of the characterization of this dominance order in terms of a set of evolution rules of a discrete dynamical system. In the case of integer partitions the standard result is the characterization from the pure algebraic lattice context given in Proposition 2.3 of the seminal Brylawski paper [16], better formulated by Green and Kleitman in [31] by means of BGK rules. In [19] this result is extended to the context of $O(m,n)$ showing that the corresponding dominance order is just the partial order induced from the following five rules:

- **Rules 1 and 2:** which are exactly the vertical and the horizontal BGK rules applied to the non-negative part of any signed partition.
- **Rules 3 and 4:** which are the dual versions of the vertical and horizontal rules applied to the non-positive part of any signed partition.
- **Rule 5:** which is a new evolution rule, not present in BGK case, which expresses the annihilation, under suitable conditions which are not interesting to formulate here, of a pair of units at the line of demarcation of the positive-negative regions producing a transition of the kind

$$(\ldots, w_{n-1}|w_n, \ldots) \to (\ldots, w_{n-1} - 1|w_n + 1, \ldots).$$

In particular, if to any signed partition $w = (w_0, \ldots, w_{n-1}|w_n, \ldots, w_{2n-1})$ one associates the non-negative integer $q_+ = \sum_{i=0}^{n-1} w_i$, the non-positive integer $q_- = \sum_{j=n}^{2n-1} w_j$, and the "total" integer $q = q_+ + q_-$, then the upper described evolution rules are such that the total quantity q is always conserved, whereas both the quantities q_-, and q_+ may variate.
It is interesting to stress that all these rules when applied to any signed partition of $O(m,n)$ do not modify its balance (n,n), and so the corresponding $2n$ extension, producing in this way a transition $O(m,n) \to O(m,n)$.

As discussed in the introduction, in [19] also an interesting physical model of the lattice $O(m,n)$ is presented. It describes a p-n junction of two semiconductors of total length $2n$ and height n (the depth of these semiconductors is considered so thin so that one may neglect it). The intrinsic semiconductor (for instance, a crystal of silicon atoms) of $n \times n$-dimension at the left side is dumped by a certain number of impurity atoms consisting of acceptors (for instance, a distribution of boron or aluminium atoms), producing in this way an excess of

holes in the original electrically neutral crystal whose behavior is the same of free positive electrical charges (p-semiconductor). On the other hand, the intrinsic semiconductor forming a $n \times n$-dimension crystal always of silicon atoms at the right side is dumped by impurity atoms consisting of donors (for instance, phosphorus or arsenic), producing in this way an excess of free electrons (n-semiconductor). The two dimensional crystal of this p-n junction of two semiconductors is considered as a surface rigged by vertical lines of atoms, labelled by the $2n$ integers $i = 0, 1, \dots, 2n-1$.
Any signed partition $w = (w_0, \dots, w_{n-1} | w_n, \dots, w_{2n-1})$ is then interpreted as two distributions of electrical charges: any non-negative integer $w_i \geq 0$ (for $i = 0, \dots, (n-1)$) describes a distribution of positive charges (holes) located at the vertical line in position i of the left side atomic crystal, and any non-positive integer $w_j \leq 0$ (for $j = n, \dots, 2n-1$) describes a distribution of negative charges (electrons) located at the vertical line in position j of the right side atomic crystal.
So the quantities q_+ and q_- describe, respectively, the total positive and negative electrical charges of the semiconductor at the left and the right of the p-n junction. These two distributions of electrical charges generate an attractive Coulomb field which acts on both distributions producing a movement of the positive (resp., negative) charges towards the opposite region consisting of negative (resp., positive) charges. This movement is described just by the Rules 1 and 2 (resp., 3 and 4) for the positive (resp., negative) charges with an accumulation of positive charges (resp.,negative) charges at the left (resp., right) side of the separation junction of the p and n parts, as boundary region of the two semiconductors.
Rule 5 describes the physical movement of an electron from the right of the junction towards its left part (from the n region to the p one), entering into a hole and producing the annihilation of a single hole-electron pair (one positive and one negative charge disappears). For this reason this is also called *interaction rule* between charged particles.

4. The Sub Model $OV(m, n)$

In subsection 3.1, we have discussed the BGK version of the dominance order on $O(m, n)$ how the discrete dynamical system in which the usual lattice notion of covering is expressed in an equivalent way by means the application of one among the Rules 1-5: the first four are noting else than the usual vertical and horizontal rules applied to both the non-negative and non-positive parts of any signed partitions of $O(m, n)$, interpreted as configuration of dynamics. The interaction Rule 5 is specific of the present model having no counterpart in the BGK model. We introduce now a sub model of $O(m, n)$ which we denote $OV(m, n)$. If h and k are two integers, in the sequel we use the following notations:

$$\sharp h := \begin{cases} 1 & \text{if} \quad h \geq 0 \\ 0 & \text{if} \quad h < 0 \end{cases} \quad \text{and} \quad \Xi(h, k) := \begin{cases} 1 & \text{if} \quad h > 0 > k \\ 0 & \quad \text{otherwise.} \end{cases}$$

For $i = 0, 1, \dots, 2n-2$, we define a i-th local transition function $f_i : O(m, n) \to O(m, n)$ so that $f_i(w) = v$ as follows:
- if $i \neq n-1$, then

$$v_j := \begin{cases} w_j - \sharp(w_i - w_{i+1} - 2) & \text{if} \quad j = i \\ w_j + \sharp(w_i - w_{i+1} - 2) & \text{if} \quad j = i+1 \\ w_j & \quad \text{otherwise;} \end{cases} \tag{12}$$

- if $i = n-1$, then

$$v_j := \begin{cases} w_j - \Xi(w_{n-1}, w_n) & \text{if} \quad j = n-1 \\ w_j + \Xi(w_{n-1}, w_n) & \text{if} \quad j = n \\ w_j & \quad \text{otherwise.} \end{cases} \tag{13}$$

Remark 4.1. *If $f_i(v) \neq v$, then $(f_i(v))_{i+1} - (f_i(v))_i = 2$.*
Moreover,

$$\sum_{j \geq l} (f_i(v))_j = \delta_{l,i+1} + \sum_{j \geq l} v_j. \tag{14}$$

The application of these local rules to the place $i \in \{0, \dots, n-2\}$, under the condition $w_i - w_{i+1} \geq 2$, produces the transition

$$(w_0, \dots, w_i, w_{i+1}, \dots, w_{n-1}|w_n, \dots, w_{2n-1}) \to (w_0, \dots, w_i - 1, w_{i+1} + 1, \dots, w_{n-1}|w_n, \dots, w_{2n-1})$$

In particular the specific transition $(5, 4, 1| \dots) \to (5, 3, 2| \dots)$ involving the "extreme" place $i = n - 2$ is allowed.

On the other hand, its application to the place $j \in \{n, \dots, 2n-1\}$, under the condition $w_j - w_{j+1} \geq 2$, produces the transition

$$(w_0, \dots, w_{n-1}|w_n, \dots, w_j, w_{j+1}, \dots, w_{2n-1}) \to (w_0, \dots, w_{n-1}|w_n, \dots, w_j - 1, w_{j+1} + 1, \dots, w_{2n-1})$$

where in this case we have to do with the non-positive part of the signed partition. For example, we have the vertical transition $(\dots| - 2, -7, -8) \to (\dots| - 3, -6, -8)$, owing to the fact that $\sharp(-3 - (-7) - 2) = 1$, which involves the "extremal" place $i = n$.
In other words, equation (12) synthesizes in a unique formulation the vertical Rules 1 and 3 described in subsection 3.1, without the horizontal Rules 2 and 4.

The application of the case $i = n - 1$ formalized by equation (13) can be synthesized in the following interesting case:

$w_{n-1} > 0$ and $w_n < 0$: which, taking into account the definition of the function Ξ, produces the transition

$$(w_0, \dots, w_{n-1}|w_n, \dots, w_{2n-1}) \to (w_0, \dots, w_{n-1} - 1|w_n + 1, \dots, w_{2n-1})$$

which corresponds to the annihilation of a single positive-negative pair at the boundary between the non-negative and the non-positive part of the signed partition under examination.

All the other cases, either $w_{n-1} = 0$ or $w_n = 0$, produce the identity transition $w \to w$.

When we apply the i-th local transition function on w we say that we *update* w in the place i. In a sequential dynamic we put a bar under the place i to denote that w will be updated in the place i.

Example 4.2. If $m = -4$, $n = 3$ and we take $w = \hat{1}^{-4,3}$, then $w = 320|333$ and a possible sequential dynamic starting from w is: $3\underline{2}0|333 \to 31\underline{1}|333 \to \underline{3}10|233 \to 2\underline{2}0|233 \to 21\underline{1}|233 \to 210|\underline{1}33 \to 210|223$

In order to describe the parallel dynamic of our model, we write $w_i(t)$ to denote that we are considering the signed partition w_i in the discrete time $t = 0, 1, 2, \dots$:

$(P_1): \ w_i(t+1) := w_i(t) + \sharp(w_{i-1}(t) - w_i(t) - 2) - \sharp(w_i(t) - w_{i+1}(t) - 2), \text{ if } i \neq 0, n-1, n, 2n-1;$
$(P_2): \ w_0(t+1) := w_0(t) - \sharp(w_0(t) - w_1(t) - 2);$
$(P_3): \ w_{n-1}(t+1) := w_{n-1}(t) + \sharp(w_{n-2}(t) - w_{n-1}(t) - 2)) - \Xi(w_{n-1}(t), w_n(t));$
$(P_4): \ w_n(t+1) := w_n(t) + \Xi(w_{n-1}(t), w_n(t)) - \sharp(w_n(t) - w_{n+1}(t) - 2));$
$(P_5): \ w_{2n-1}(t+1) := w_{2n-1}(t) + \sharp(w_{2n-2}(t) - w_{2n-1}(t) - 2).$

In this case the global transition function $F : O(m, n) \to O(m, n)$ is given by $F(w(t)) := w(t+1)$, where $w(t+1)$ is defined as in $(P_1), \dots, (P_5)$.

Example 4.3. As in the previous example, if $w = 320|333$ the parallel dynamic starting from w is: $320|333 \to 311|333 \to 220|233 \to 211|233 \to 210|133 \to 210|223$

The following result relates the coordinates of the parallel update of w to the sequential updates of w.

Proposition 4.4. *Let $w \in O(m, n)$, F be the parallel update function and f_i be the sequential update function at position i. Then,*

(15)
$$(F(w))_i = \begin{cases} (f_0(w))_0 & \text{if } i = 0 \\ (f_{2n-2}(w))_{2n-1} & \text{if } i = 2n-2 \\ w_i + ((f_i(w))_{i-1} - (f_{i-1}(w))_{i-1}) + ((f_{i-1}(w))_{i+1} - (f_i(w))_{i+1}) & \text{if } 0 < i < 2n-2 \end{cases}$$

Proof. When $i = 0, 2n-2$ there is nothing to prove.

Now let $0 < i < 2n-2$; then

$$(f_i(w))_{i-1} = w_{i-1}$$
$$(f_{i-1}(w))_{i+1} = w_{i+1}$$

so that

$$(f_i(w))_{i-1} - (f_{i-1}(w))_{i-1} = \begin{cases} 0 & \text{if } (f_{i-1}(w))_{i-1} = w_{i-1} \\ 1 & \text{otherwise (that is, if } w_{i-1} \text{ is updated)} \end{cases}$$

and similarly

$$(f_{i-1}(w))_{i+1} - (f_i(w))_{i+1} = \begin{cases} 0 & \text{if } (f_i(w))_{i+1} = w_{i+1} \\ -1 & \text{otherwise (that is, if } w_{i+1} \text{ is updated)} \end{cases}$$

On the other hand, we remark that $(F(w))_i - w_i \in \{-1, 0, 1\}$; more precisely,

$$(F(w))_i - w_i = \begin{cases} 1 & \text{if } (f_{i-1}(w))_{i-1} \neq w_{i-1} \text{ and } (f_i(w))_{i+1} = w_{i+1} \\ -1 & \text{if } (f_{i-1}(w))_{i-1} = w_{i-1} \text{ and } (f_i(w))_{i+1} \neq w_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Putting all this together, we get formula (15). □

Now we define the function $E : O(m,n) \to \mathbb{Z}$ such that

$$E(w) := \sum_{i=0}^{2n-1} i \cdot w_i$$

We say that $w \in O(m,n)$ is a *fixed point* (see also [29, 30]) if $f_i(w) = w$ for all $i = 0, 1, \ldots, 2n-2$. We denote by $F(m,n)$ the subset of all fixed points of $O(m,n)$. For example:

$$F(-4,3) = \{210|233, 110|123, 110|222, 100|122, 000|112\}$$

Proposition 4.5. *(i) If $w \in O(m,n)$ and $f_i(w) \neq w$, then $E(f_i(w)) = E(w) + 1$.*
(ii) If $w, v \in O(m,n)$, $E(v) \leq E(w)$ and $v \trianglelefteq w$, then $w = v$.

Proof. (i) The thesis follows from simple arithmetical calculations.
(ii) Let us note that $E(v) = (v_1 + v_2 + \cdots + v_{2n-2} + v_{2n-1}) + (v_2 + v_3 + \cdots + v_{2n-2} + v_{2n-1}) + \cdots + (v_{2n-2} + v_{2n-1}) + (v_{2n-1})$ and a similar sum decomposition holds for $E(w)$. Therefore from $v \trianglelefteq w$ and (10), we get $E(v) \geq E(w)$, and hence $E(w) = E(v)$. The identity $w = v$ is then a direct consequence of (10) and of the equality $\sum_{i=0}^{2n-1} w_i = \sum_{i=0}^{2n-1} v_i$. □

If $w \in O(m,n)$, we set

$$Fix(w) := \{v \in O(m,n) | v \trianglelefteq w, v \in F(m,n)\}$$

Since $\hat{0}^{m,n} \in F(m,n)$ it follows that $Fix(w) \neq \emptyset$ for all $w \in O(m,n)$.

Proposition 4.6. *Let w be an element of $O(m,n)$.*
(i) If $f_i(w) \neq w$, then $f_i(w) \lessdot w$.
(ii) If $v \in Fix(w)$, then $v \trianglelefteq f_i(w)$ for all $i = 0, 1, \ldots, 2n-1$.

Proof. (i) It is easy to verify that $f_i(w) \trianglelefteq w$ for all $i = 0, 1, \ldots, 2n-1$. Moreover, if $f_i(w) \neq w$ and we assume that there is some $z \in O(m,n)$ such that $f_i(w) \lhd z \lhd w$ we easily deduce a contradiction.
(ii) Assume $f_i(w) \neq w$. First we note (see Remark 4.1) that since

$$\sum_{j \geq l} (f_i(w))_j = \sum_{j \geq l} w_j \leq \sum_{j \geq l} v_j, \quad \text{for all } \ l \neq i+1,$$

it suffices to prove that $\sum_{j \geq i+1} (f_i(w))_j \leq \sum_{j \geq i+1} v_j$.

Moreover (again by Remark 4.1), it is enough to prove that assuming

$$\sum_{j\geq i+1}(f_i(w))_j = 1 + \sum_{j\geq i+1} w_j \quad\text{and}\quad \sum_{j\geq i+1} w_j = \sum_{j\geq i+1} v_j$$

we get a contradiction. From the equations

$$\begin{array}{rcr} w_i + w_{i+1} + w_{i+2} + \ldots + w_{2n-1} & \leq & v_i + v_{i+1} + v_{i+2} + \ldots + v_{2n-1} \\ w_{i+1} + w_{i+2} + \ldots + w_{2n-1} & = & v_{i+1} + v_{i+2} + \ldots + v_{2n-1} \\ w_{i+2} + \ldots + w_{2n-1} & \leq & v_{i+2} + \ldots + v_{2n-1} \end{array}$$

we get $w_i \leq v_i$, $v_{i+1} \leq w_{i+1}$ and $v_i - v_{i+1} \geq w_i - w_{i+1} = 2$.

Now, if $i \neq n-1$, we obtain a contradiction, because v is a fixed point.

If $i = n-1$ and $v_{n-1} = 0$, since $w_{n-1} \leq v_{n-1}$, then also $w_{n-1} = 0$, thus $f_{n-1}(w) = w$, which contradicts our hypothesis $f_i(w) \neq w$.

Finally, if $i = n-1$ and $v_n = 0$, since $w_n \geq v_n$, then also $w_n = 0$ and again $f_{n-1}(w) = w$.

Therefore we are reduced to considering the case $i = n-1$ and $v_{n-1} > 0 > v_n$, which also gives a contradiction because v is a fixed point. □

If $w_0, w_1, \ldots, w_k$ are distinct elements of $O(m,n)$ such that $f_{i_1}(w_0) = w_1, \ldots, f_{i_k}(w_{k-1}) = w_k$, for some $i_1, \ldots, i_k \in \{0, 1, \ldots, 2n-2\}$, we say that $w_0, w_1, \ldots, w_k$ is a *sequential chain starting in* w_0 and we write

$$w_0 \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k.$$

Moreover, if $w_k \in F(m,n)$ we say that $w_0, w_1, \ldots, w_k$ is a *maximal sequential chain starting in* w_0 that *converges towards* w_k and we write

$$w_0 \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k \dashv.$$

Proposition 4.7. *Any sequential chain starting at w_0 can be extended to a maximal sequential chain. In particular, there exists a maximal sequential chain starting at w_0.*

Proof. It is an easy consequence of 4.5 (i), 4.6 (i) and the fact that the function $E(w)$ assumes a finite number of values. □

Proposition 4.8. *If $w_0 \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k$, then $w_0 \gtrdot w_1 \gtrdot \cdots \gtrdot w_{k-1} \gtrdot w_k$ (i.e. it is a so called* saturated chain *of length k) and $k = E(w_k) - E(w_0)$.*
In particular, if $w_0 \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k \dashv$, then $w_k \in Fix(w_0)$.

Proof. It is a direct consequence of Proposition 4.6 (i) and of Proposition 4.5 (i). □

If $w \in O(m,n)$ we set

$$M(w) := \{w' \in Fix(w) : E(w') = \min_{v\in Fix(w)} E(v)\}.$$

Let us note that $M(w)$ is not empty.

Proposition 4.9. *If $w \in O(m,n)$, then $M(w)$ contains a unique element, denoted by w^f. Moreover, if $w \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k \dashv$ is a maximal sequential chain starting in w, then $w_k = w^f$ and $k = E(w^f) - E(w)$.*

Proof. Fix a maximal sequential chain $w \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k \dashv$ and let $w' \in M(w)$. By definition of $M(w)$ we have $E(w_k) \geq E(w')$ because $w_k \in Fix(w)$. Repeated applications of (ii) of the Proposition 4.6 imply that $w' \trianglelefteq w_k$, therefore $w' = w_k$ from (ii) of the Proposition 4.5. Since the choice of w' in $M(w)$ is arbitrary, we deduce that $M(w)$ has a unique element w^f. □

The previous results tell us that all the maximal sequential chains starting in w have the same length k and they converge towards a unique fixed point w^f.

Definition 4.10. *If $w \in O(m,n)$, we set*

$$T_{seq}(w) := k,$$

where

$$w \overset{f_{i_1}}{\to} w_1 \overset{f_{i_2}}{\to} \cdots w_{k-1} \overset{f_{i_k}}{\to} w_k \dashv .$$

.

Therefore, we can also say that the BKG sequential dynamic starting from w converges towards w^f, in $T_{seq}(w) = E(w^f) - E(w)$ time steps, independently of the order in which the sites are updated. We denote now by $OV(m,n|w,w^f)$ the subset of $O(m,n)$ which is the set union of all the maximal sequential chains starting in w that converge towards w^f. If there is at least one sequential maximal chain to which $u, v \in OV(m,n|w,w^f)$ belong and $u \overset{f_i}{\to} v$ for some i, then we draw u above v and we connect u and v with a segment line, otherwise we do not connect u and v. In this way we obtain a Hasse sub-diagram of the Hasse diagram of $O(m,n)$. Therefore we can consider $OV(m,n|w,w^f)$ as a sub-poset of $O(m,n)$ such that if $u, v \in OV(m,n|w,w^f)$ then $u \gtrdot v$ iff $u \overset{f_i}{\to} v$ for some i. Then, by Proposition 4.9 we deduce that $OV(m,n|w,w^f)$ is a graded sub-poset of $O(m,n)$ with maximum w, minimum w^f and whose rank function is $\rho(u) = E(w^f) - E(u)$, for all $u \in OV(m,n|w,w^f)$. When $w = \hat{1}^{m,n}$, we denote $OV(m,n|w,w^f)$ simply by $OV(m,n)$. We consider now the parallel dynamic of our model. If $w(0), w(1), \ldots, w(s)$ are distinct elements of $O(m,n)$ such that $F(w(0)) = w(1), \ldots, F(w(s-1)) = w(s)$, we say that $w(0), w(1), \ldots, w(s)$ is a *parallel chain starting in* $w(0)$ and we write

$$w(0) \rightrightarrows w(1) \rightrightarrows \cdots w(k-1) \rightrightarrows w(s).$$

Moreover, if $w(s) \in F(m,n)$ we say that $w(0), w(1), \ldots, w(s)$ is a *maximal parallel chain starting in* $w(0)$ that *converges towards* $w(s)$ and we write

$$w(0) \rightrightarrows w(1) \rightrightarrows \cdots w(s-1) \rightrightarrows w(s) \dashv .$$

Obviously there is a unique maximal parallel chain starting in $w(0)$ that converges towards a fixed point $w(s)$.

Definition 4.11. *We set*

$$T_{par}(w(0)) := s,$$

where

$$w(0) \rightrightarrows w(1) \rightrightarrows \cdots w(s-1) \rightrightarrows w(s) \dashv .$$

We have the following result:

Proposition 4.12. *If $w \in O(m,n)$ and $w = w(0) \rightrightarrows w(1) \rightrightarrows \cdots w(s-1) \rightrightarrows w(s) \dashv$, then $w(s) = w^f$.*

Proof. The proof is immediate because each parallel convergence can be obtained as a finite composition of sequential convergence. □

Hence the maximal parallel chain starting in $w = w(0)$ converges towards the fixed point w^f. We also define the *length* of w, denoted by $l(w)$, as the number of non-zero parts of w. We note that in the case of integer partitions having only positive summands, the condition $v \trianglelefteq w$ implies $l(w) \leq l(v)$. In the case of signed integer partitions this is no longer true: a (minimal) counterexample is $v = (1|0)$ and $w = (2|1)$.

For the number $T_{par}(w(0))$ we obtain the following lower estimate.

Proposition 4.13. *Let*

$$w = w(0) \rightrightarrows w(1) \rightrightarrows \cdots w(s-1) \rightrightarrows w(s) = w^f \dashv \tag{16}$$

be a maximal parallel chain so that $T_{par}(w(0)) = s$. Then:

$$T_{par}(w(0)) \geq \frac{T_{seq}(w(0))}{\lfloor w_0(0)/2 \rfloor + \lfloor |w_{2n-1}(0)|/2 \rfloor + 1}.$$

Proof. Let g_t be the number of sequential steps in a sequential expansions of the parallel step $w(t) \rightrightarrows w(t+1)$ of the chain (16), for $t = 0, \ldots, s-1$. From proposition 4.5.i we have that $g_t = E(w(t+1)) - E(w(t))$.

We shall show that for each t we must have $g_t \leq \lfloor w_0(0)/2 \rfloor + \lfloor |w_{2n-1}(0)|/2 \rfloor + 1$.

Fix some t and let $g = g_t$ be the number of sequential steps for the fixed t-th parallel step.

In order that this can occur, there must exist p integers $i_1, \ldots, i_p$ and q integers $j_1, \ldots, i_q$, with $0 \leq i_1 < i_2 < \cdots < i_{p-1} < i_p \leq n-1 < n \leq j_1 < j_2 < \cdots < j_{q-1} < j_q \leq 2n-2$ and $p+q = g$, such that $w_{i_h}(t) - w_{i_h+1}(t) \geq 2$ for $h = 1, \ldots p$ and $|w_{j_l+1}(t)| - |w_{j_l}(t)| \geq 2$ for $l = 1, \ldots q$.

This implies that

$$\begin{aligned} w_0(t) &\geq w_{i_1}(t) \geq w_{i_1+1}(t) + 2 \\ &\geq w_{i_2}(t) + 2 \geq w_{i_2+1}(t) + 4 \\ &\geq w_{i_3}(t) + 4 \geq w_{i_3+1}(t) + 6 \\ &\vdots \\ &\geq w_{i_{p-1}}(t) + 2(p-2) \geq w_{i_{p-1}+1}(t) + 2(p-1) \\ &\geq 2(p-1). \end{aligned}$$

Therefore

$$\left\lfloor \frac{w_0(0)}{2} \right\rfloor \geq \left\lfloor \frac{w_0(t)}{2} \right\rfloor \geq p-1. \tag{17}$$

With a similar technique we also obtain

$$\left\lfloor \frac{|w_{2n-1}(0)|}{2} \right\rfloor \geq \left\lfloor \frac{|w_{2n-1}(t)|}{2} \right\rfloor \geq q. \tag{18}$$

From (17) and (18) we deduce that $g = p+q \leq \left\lfloor \frac{w_0(0)}{2} \right\rfloor + \left\lfloor \frac{|w_{2n-1}(0)|}{2} \right\rfloor + 1$. Hence $T_{seq}(w(0)) = g_1 + \cdots + g_s \leq s(\left\lfloor \frac{w_0(0)}{2} \right\rfloor + \left\lfloor \frac{|w_{2n-1}(0)|}{2} \right\rfloor + 1)$, and the thesis follows.

□

5. More on Positive Integer Partitions

In this section we prove some properties concerning the positive integer partitions which will be useful to determine the result of next section. First, we fix some notation. Given a positive integer N, let S_N be the set of positive non-increasing partitions of N and let $\overline{N} = (N, 0, \ldots) \in S_N$. If $w = (w_0, \ldots, w_i, 0 \ldots) \in S_N$ with $w_i \neq 0$, let $l(w) = i+1$.

Define $[N] = (k, k')$ where k, k' are the unique integers such that

$$N = \frac{k(k+1)}{2} + k' \quad \text{with } 0 \leq k' \leq k.$$

Define also

$$\pi = \pi([N]) = (k, k-1, \ldots, k'+1, k', k', k'-1, \ldots 2, 1, 0, \ldots). \tag{19}$$

Remark that

$$\pi([N])_i = \begin{cases} k-i & \text{if } i \leq k-k' \\ k-i+1 & \text{if } i > k-k' \end{cases}$$

It is easy to verify that $\overline{N}$ converges toward $\pi([N])$; with our notation, $\overline{N}^f = \pi([N])$.

Let now $u \in S_N$, $u' \in S_{N'}$; when it makes sense (that is, when $u_{l(u)-1} \geq u'_0$) let us define $\tau(u, u') = (v_i)_{i \in \mathbb{N}}$, where

$$v_i = \begin{cases} u_i & \text{if } i \leq l(u)-1 \\ u'_{i-l(u)} & \text{otherwise} \end{cases}$$

that is, $\tau(u, u')$ is the concatenation of u and u'.

Moreover, if $n, h, r \in \mathbb{N}$, with $0 \leq r < n$, let

$$w = w(n,h,r) = (\underbrace{n,\ldots,n}_{h \text{ times}}, r, 0, \ldots) \in S(nh+r).$$

Remark 5.1. *If $w \in S_N$ (or even if $w \in O(m,n)$) when computing w^f we can choose many different sequential chains: the choice is irrelevant and the resulting w^f will always be the same, so we can choose the path we prefer. One possible choice is to follow the path that is achieved by applying the right most possible sequential update f_i, that is,choosing i such that*

$$i = \max\{j : f_j(v) \neq v\}, \tag{20}$$

where v is the partition that we are considering at the relative step. When i is chosen as in (20), we say that we apply the Right Most Rule (RMR, for short).

Finally, if $w, w' \in S_N$, we write $w \rightsquigarrow w'$ if there exists a sequential chain from w to w'.

Lemma 5.2. *Let $n, r \in \mathbb{N}$ with $n \geq r$; then $(\tau(\overline{n}, \overline{r}))^f = \pi([n+r])$*

Proof. First note that $n \geq r \geq (\pi([r]))_0$, so $\tau(\overline{n}, \pi([r]))$ is well defined. Moreover, $r = (\pi([r]))_0$ if and only if $r = 0, 1$, so that $n = (\pi([r]))_0$ if and only if $(n,r) = (0,0)$ or $(1,1)$ in which cases the thesis holds. In all other cases, $n > (\pi([r]))_0$. Let $v = \tau(\overline{n}, \pi([r]))$. Since $\overline{r}^f = \pi([r])$, using the RMR we get $\tau(\overline{n}, \overline{r}) \rightsquigarrow \tau(\overline{n}, \pi([r]))$. If $n = (\pi([r]))_0+1$, then by definition $\tau(\overline{n}, \pi([r])) = \pi([n+r])$ (let us remark that this can only happen in the cases $(n,r) = (3,3), (2,2), (2,1)$ or $(1,0)$). We can then assume $n > (\pi([r]))_0 + 1$ and let $[r] = (k, k')$. It is clear that $f_i(v) \neq v \Leftrightarrow i = 0$ and also that

$$f_0(v) = (n-1, k+1, k-1, \ldots, k'+1, k', k', k'-1, \ldots, 2, 1, 0, \ldots). \tag{21}$$

Applying repeatedly the RMR to (21) (which results in applying $f_{k-k'} \circ \cdots \circ f_2 \circ f_1$) we get

$$\begin{aligned} v &\rightsquigarrow (n-1, k, k-1, \ldots, k'+2, k'+1, k'+1, k', \ldots, 2, 1, 0, \ldots) \\ &= \begin{cases} \tau(\overline{n-1}, \pi(k, k'+1)) & \text{if } k > k' \\ \tau(\overline{n-1}, \pi(k+1, 0)) & \text{if } k = k' \end{cases} \end{aligned} \tag{22}$$

In any case, we can construct a sequential chain from v to a partition of the form $\tau(\overline{n-1}, \pi(h,h'))$ for some h, h' such that $n-1 \geq h \geq h' \geq 0$ and the thesis follows by induction on n. ☐

Let us fix integers n, h, r with $0 \leq r < n$ and let k'' be an integer such that

$$\frac{n(n-1)}{2} + k'' \equiv r \pmod n$$

and let

$$h' = \frac{\frac{n(n-1)}{2} + k'' - r}{n}.$$

Corollary 5.3. *With the preceding notation, the partition $w(n,h,r)$ converges toward*

$$w(n,h,r)^f = \begin{cases} (k, k-1, \ldots, k'+1, k', k', k'-1, \ldots, 2, 1) & \textit{if } hn + r \leq \frac{n(n+1)}{2} \\ (\underbrace{n,\ldots,n}_{h-h' \textit{ times}}, n-1, n-2, \ldots, k''+1, k'', k'', k''-1, \ldots, 2, 1) & \textit{if } hn + r > \frac{n(n+1)}{2} \end{cases}$$

where k, k' are the unique integers such that $0 \leq k' \leq k$ and $\frac{k(k+1)}{2} + k' = hn + r$ in the case $hn + r < \frac{n(n+1)}{2}$.

Proof. Since $w(n,h,r) = \tau(\overline{n}, -)^h(\overline{r})$, the same argument of the proof of Lemma 5.2 applies. ☐

6. Fixed Points in $OV(m,n)$

In this section we explicitly determine the minimum in the sub-poset $OV(m,n)$ of $O(m,n)$. We denote such a minimum with $\pi^{m,n}$. It is clear that $\pi^{m,n}$ coincides with the fixed point $(\hat{1}^{m,n})^f$ of $\hat{1}^{m,n}$. In order to better visualize the determination of $\pi^{m,n}$ we identify each $w \in O(m,n)$ with an ordered pair of $n \times n$ tables, where each non-zero part w_j of w will be represented with a column of $|w_j|$ black circles in the corresponding place j, for $j = 0, 1, \dots, 2n-1$. For example, the signed partition $w = 3220|1344 \in O(-7,4)$ will be identified with the following ordered pair:

•			
•	•	•	
•	•	•	

		•	•
	•	•	•
	•	•	•
•	•	•	•

Let k be a non-negative integer such that $k \leq \binom{n+1}{2}$. We begin by defining two integers parameters $0 \leq a(k,n) \leq n-1$ and $1 \leq b(k,n) \leq n$ as follows.

We denote by $R(n)$ the $n \times n$ table that in the j-th column, for $j \in \{0, 1, \dots, n-1\}$, has exactly j black balls (we count the columns from left to right starting with 0). In an exactly similar way we define the symmetric $n \times n$ table $L(n)$. For example, $L(4)$ and $R(4)$ are respectively

•			
•	•		
•	•	•	

			•
		•	•
	•	•	•

We also denote by $F(n)$ a $n \times n$ table whose n^2 cases are all filled with a black ball. For example, $F(4)$ is

•	•	•	•
•	•	•	•
•	•	•	•
•	•	•	•

We choose now an arbitrary integer k such that $1 \leq k \leq \binom{n+1}{2}$. Since $k \leq \binom{n+1}{2}$, we can place k black balls in the white cells of $R(n)$ (that are exactly $\binom{n+1}{2}$) as follows. We begin to fill the main diagonal of $R(n)$ from the south-west corner towards the north-east corner. If $k \leq n$ the process will be stopped on some j-th column. In this case we take $a(k,n) := j$ and $b(k,n) := 1$. If $k > n$, inductively we can assume that $p-1$ diagonals have been filled with black balls. When we move on the p-th diagonal, we begin filling with black balls from the south-west and we stop the process on some j-th column (when the k black balls have already been placed). We set then $a(k,n) := j$ and $b(k,n) := p$, and we call (k,n) *stopping parameters* the ordered pair $[a(k,n), b(k,n)]$. We denote by $\leftthreetimes(k,n)$ the resulting $n \times n$ table obtained from $R(n)$ after the arrangement of k new black balls in the above described way. For example, $\leftthreetimes(6,4)$ is

			•
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•	•	•	•
•	•	•	•

and $[a(6,4), b(6,4)] = [1,2]$.

Let us note that $1 \leq a(k,n) + b(k,n) \leq n$. Let us consider now the map

$$\begin{array}{cccc} \sigma\colon & O(m,n) & \longrightarrow & O(-m,n) \\ & w = (w_0, \dots, w_{2n-1}) & \longmapsto & \sigma(w) = (-w_{2n-1}, \dots, -w_0) \end{array}$$

Having defined $\leftthreetimes(k,n)$, we define $\rightthreetimes(k,n)$ by

$$(\rightthreetimes(k,n)|0,\dots,0) = \sigma((0,\dots,0|\leftthreetimes(k,n))).$$

We can provide a closed formula for the stopping parameters:

Proposition 6.1. *With the preceding notations and definitions, we have*

$$b(k,n) = \left\lfloor n + \frac{1}{2} + 1 - \frac{\sqrt{(2n+1)^2 - 8k + 4}}{2} \right\rfloor \tag{23}$$

and

$$a(k,n) = k - 1 - s(b(k,n) - 1, n) \tag{24}$$

where

$$s(h,n) = \frac{h(2n+1-h)}{2}. \tag{25}$$

Proof. It is easy to verify that

$$s(h,n) = \sum_{i=n-h+1}^{n} i \tag{26}$$

if $h = 0, 1, \ldots, n$.

Let k be an integer with $1 \le k \le \binom{n+1}{2}$ and let us define $f(x) = k - 1 - s(x,n)$. Then, by definition, $b(k,n)$ is the biggest integer i not greater than n such that $f(i) \ge 0$, that is, the formula given in(23).

Formula (24) is now an easy consequence of the definition of $a(k,n)$, (23) and (26). □

Another way to describe $\curlywedge(h,n)$ (and then also $\curlyvee(h,n)$ via σ) is as follows. If $w = (w_0, \ldots) \in S_N$, then define $T_n(w) = (w'_0, \ldots, w'_{n-1})$ as

$$w'_i = \begin{cases} n & \text{if } w_i > n \\ w_i & \text{otherwise} \end{cases}$$

We can think of $T_n(w)$ as the "truncation" of w to the $n \times n$ square. Then it is easy to check that

Proposition 6.2. *If $\pi([N])$ is the partition defined in (19), then*

$$\curlywedge(h,n) = T_n(\pi([N])) \quad \textit{where } N = \frac{k(k+1)}{2} + k'$$

and

$$k = n + b(h,n) - 2 + \delta_{n,a(h,n)+1}$$
$$k' = [a(h,n)+1]_n + b(h,n) - 1$$

where $[m]_n$ is the canonical representative of m $(\mathrm{mod}\ n)$.

Remark 6.3. *It is easy to verify that in the preceding proposition we always have $k \ge k'$, so in (19) we have that $\pi([N])$, with $N = \frac{k(k+1)}{2} + k'$, is well defined.*

Lemma 6.4. *Let $0 \le h \le n$ and $0 \le r < n$ such that $hn + r \le n^2$; then*

$$(\underbrace{n, \ldots, n}_{h \text{ times}}, r, 0, \ldots, 0 | 0, \ldots, 0) \textit{ converges to } \begin{cases} (\pi([nh+r]) | 0, \ldots, 0) & \textit{if } nh + r \le \frac{n(n-1)}{2} \\ (\curlywedge(k,n) | 0, \ldots, 0) & \textit{if } nh + r > \frac{n(n-1)}{2} \end{cases}$$

and

$$(0, \ldots, 0 | 0, \ldots, 0, r, \underbrace{n, \ldots, n}_{h \text{ times}}) \textit{ converges to } \begin{cases} (0, \ldots, 0 | \sigma(\pi([nh+r]))) & \textit{if } nh + r \le \frac{n(n-1)}{2} \\ (0, \ldots, 0 | \curlyvee(k,n)) & \textit{if } nh + r > \frac{n(n-1)}{2} \end{cases}$$

Proof. Consider $w(n,h,r)$. If $nh + r \le \frac{n(n+1)}{2}$ then it converges to $\pi([nh+r])$ which has length smaller or equal than n, then we can apply again the same argument of Corollary 5.3.

If $nh + r > \frac{n(n+1)}{2}$ then the length of the fixed element of $w(n,h,r)$ is bigger than n, so when we apply the RMR we are not allowed to use f_i for $i \ge n$ but the same analysis of the possible cases as in the proof of Lemma 5.2 drives us to the conclusion.

The second part follows from the first applying σ. □

Lemma 6.5. *If $w = (0, \ldots, 0 | \curlyvee(k,n)) = (0, \ldots, 0 | w_n, \ldots, w_{2n-1})$ then $w' = (0, \ldots, 0, w_n + 1, \ldots, w_{2n-1})$ converges to $(0, \ldots, 0 | \curlyvee(k-1,n))$. A symmetric result holds for $(\curlywedge(k,n)|0, \ldots, 0)$.*

Proof. Let $j = a(k,n)$; applying $j-1$ times the RMR (namely, $f_{n+j-1} \circ \cdots \circ f_n$) gives the thesis. □

Theorem 6.6. *Let $-n^2 \leq m \leq 0$ and let $\pi^{m,n}$ be the fixed point of $\hat{1}^{m,n}$. Then*

$$\pi^{m,n} = \begin{cases} (\pi([n^2+m])|n,\ldots,n) & \text{if } -n^2 \leq m \leq -\binom{n+1}{2} \\ (L(n)|\curlywedge(|m|,n)) & \text{if } -\binom{n+1}{2} < m \leq -1 \\ (L(n)|R(n)) & \text{if } m=0 \\ (\curlyvee(|m|,n)|R(n)) & \text{if } 1 \leq m < \binom{n+1}{2} \\ (n,\ldots,n|\pi([n^2-m])) & \text{if } \binom{n+1}{2} \leq m \leq n^2 \end{cases}$$

Proof. By (11) we can assume $m \leq 0$.

Consider the first case: $-n^2 \leq m \leq -\binom{n+1}{2}$. Let $0 \leq k' \leq k$ be the unique integers such that $n^2-|m| = \frac{1}{2}k(k+1)+k'$ and $n^2-|m| = qn+r$ with $0 \leq r < n$. Define

$$\begin{aligned} \hat{1}^{m,n} &= v_0 = (n,n,\ldots,n,r,0\ldots,0|n,\ldots,n) \\ \pi^{m,n} &= v_l = (k,k-1,\ldots,k'+1,k',k',k'-1,\ldots,2,1,0,\ldots,0|n,\ldots,n) \\ &= (\pi([n^2+m])|n,\ldots,n) \end{aligned} \tag{27}$$

Let also $p = n^2-|m| = n^2+m$. Remark that $0 \leq p \leq \frac{n(n-1)}{2}$, since $-n^2 \leq m \leq -\binom{n+1}{2}$.

First we show that, in our hypothesis, we must have $k<n$: if $k \geq n$ then

$$p = \frac{k(k+1)}{2}+k' \geq \frac{k(k+1)}{2} \geq \frac{n(n+1)}{2} > p,$$

a contradiction. This implies $v_0 \geq v_l$.

Moreover, if $k=n-1$ then $p = \frac{k(k+1)}{2}+k' \leq \frac{n(n-1)}{2}$ implies $k'=0$. In other words, v_l contains at least one 0 in the positive part, which means that $v_l \in F(m,n)$ and then $v_l \in Fix(v_0)$, so we just need to show that there exists a sequential chain starting from v_0 to v_l.

Lemma 6.4 and the preceding discussion show us that we can construct a sequential chain

$$v_0 \xrightarrow{f_{i_1}} v_1 \xrightarrow{f_{i_2}} \cdots v_{l-1} \xrightarrow{f_{i_l}} v_l \dashv$$

and the fact that v_l is fixed implies that the chain is maximal.

Now consider the case $-\binom{n+1}{2} < m \leq -1$.

Starting from $\hat{1}^{m,n}$, apply the RMR repeatedly: as long the n-th coordinate is 0 (i.e., the one with index $n-1$) it operates on the first $n-1$ coordinates.

As soon as $w(t)_{n-1}=1$ (which will happens, for lemma 6.4), the next RMR sets $w(t+1)_{n-1} = 0$ and increases $w(t+1)_n$ by 1. Now apply lemma 6.5.

Then the RMR will start again to operate on the positive part of the partition and so we start again the preceding steps.

All this will happen as long as we have 1 in the coordinate $n-1$: for lemma 6.4, this will happens $hn+r-\frac{n(n-1)}{2}$ times. Using both Lemma 6.4 and 6.5, at the end we are left with $(L(n),\curlywedge(|m|,n))$.

In the case $m=0$, the proof is essentially the same as the previous case. □

In the previous theorem, if we want an explicit expression (as in (27)) for the summands of the signed partition identified with the ordered pair of tables $(L(n),\curlywedge(|m|,n))$, we use the result of Proposition 6.2. In the next result we compute the rank $rk(OV(m,n))$ of the graded poset $OV(m,n)$.

Proposition 6.7. *Let $0 \leq k' \leq k$ be the unique integers such that $n^2-|m| = \frac{1}{2}k(k+1)+k'$ and $n^2-|m| = qn+r$ with $0 \leq r < n$. Let a,b be the stopping parameters of $(|m|,n)$. Then $rk(OV(m,n))$ is equal to:*

(i) $\frac{n(n+1)(2n+1)}{6}$ *if* $m=0$;

(ii) $\frac{1}{6}\Big(5n^3+(-9b+14)n^2+\big(3(b-2)(2b-1)-6(a+1)-3q(q-1)-1\big)n-b(b-1)(b-2)-3a(a+1)\Big)$ *if* $1 \leq |m| < \binom{n+1}{2}$;

(iii) $\binom{k+1}{3}+kk'-\binom{k'}{2}-\frac{q}{2}(nq-n+2r)$ *if* $\binom{n+1}{2} \leq |m| \leq n^2$.

Proof. We have that $rk(OV(m,n)) = E(\pi^{m,n}) - E(\hat{1}^{m,n})$. As in the theorem, it is enough to study $m \leq 0$. Let $p = n^2 - |m|$.

(i) In this case,

$$\begin{aligned} E(\pi^{m,n}) - E(\hat{1}^{m,n}) &= E(\pi([p])|0,\dots,0) + E(0,\dots 0|n,\dots,n) \\ &\quad - E(n,\dots,n,r,0\dots,0|0,\dots,0) - E(0,\dots 0|n,\dots,n) \\ &= E(\pi([p])|0,\dots,0) - E(n,\dots,n,r,0\dots,0|0,\dots,0) \end{aligned}$$

Now we have $E(\pi([p])|0,\dots 0) = \binom{k+1}{3} + kk' - \binom{k'}{2}$, and a simple computation shows that $E(n,\dots,n,r,0\dots,0|0,\dots 0) = \frac{q}{2}(nq - n + 2r)$, whence the thesis.

(ii) In this case,

$$\begin{aligned} E(\pi^{m,n}) - E(\hat{1}^{m,n}) &= E(L(n)|0,\dots 0) + E(0,\dots,0| \leftthreetimes (|m|,n)) \\ &\quad - E(n,\dots,n,r,0\dots,0|0,\dots 0) - E(0,\dots 0|n,\dots n) \end{aligned}$$

Three of the summands are easy to compute:

$E(L(n)|0,\dots,0) = \sum_{i=0}^{n-1}(n-1-i)i = \frac{(n-1)n(n+1)}{6}$, $E(n,\dots,n,r,0\dots,0|0,\dots,0) = n\frac{(q-1)q}{2}$ and $E(0,\dots 0|n,\dots,n) = -n\sum_{i=n}^{2n-1} i$. Let us now compute the fourth summand. Since

$$\leftthreetimes(|m|,n) = (b, b+1, \dots, b+a, b+a, \dots, n-1, n, \dots, n),$$

we get

$-E(0,\dots,0| \leftthreetimes (|m|,n)) = \sum_{i=n}^{n+a}(b+i-n)i + \sum_{i=n+a+1}^{2n-b}(b+i-n-1)i + \sum_{i=2n-b+1}^{2n-1} ni = \sum_{i=n}^{n+a} i + \sum_{i=n}^{2n-b}(b+i-n-1)i + n\sum_{i=2n-b+1}^{2n-1} i.$

From this,

$$\begin{aligned} &E(0,\dots,0| \leftthreetimes (|m|,n)) - E(0,\dots,0|n,\dots,n) = \\ &= -\sum_{i=n}^{n+a} i - \sum_{i=n}^{2n-b}(b+i-n-1)i - n\sum_{i=2n-b+1}^{2n-1} i + n\sum_{i=n}^{2n-1} i \\ &= -\sum_{i=n}^{n+a} i - \sum_{i=n}^{2n-b}(b+i-n-1)i + n\sum_{i=n}^{2n-b} i \\ &= -\sum_{i=n}^{n+a} i + (2n-b+1)\sum_{i=n}^{2n-b} i - \sum_{i=n}^{2n-b} i^2 \\ &= -\sum_{i=n}^{n+a} i + (2n-b+1)\sum_{i=n}^{2n-b} i - \sum_{i=1}^{2n-b} i^2 + \sum_{i=1}^{n-1} i^2 \\ &= -\frac{(a+1)(2n+a)}{2} + \frac{(2n-b+1)(n-b+1)(3n-b)}{2} \\ &\quad - \frac{(2n-b)(2n-b+1)(4n-2b+1)}{6} + \frac{(n-1)n(2n-1)}{6} \\ &= \frac{1}{6}\Big(-3(a+1)(2n+a) + 4n^3 + (-9b+14)n^2 + (6b^2-15b+6)n - b^3 + 3b^2 - 2b \Big) \end{aligned}$$

Putting all together we obtain the thesis.

(iii) If $m = 0$ we have

$E(\pi^{m,n}) - E(\hat{1}^{m,n}) = \sum_{i=0}^{n-1}(n-1-i)i - \sum_{i=n}^{2n-1}(i-n)i - \Big(\sum_{i=0}^{n-1} ni - \sum_{i=n}^{2n-1} ni\Big) = 2n\sum_{i=0}^{2n-1} i - \Big((2n+1)\sum_{i=0}^{n-1} i + \sum_{i=0}^{2n-1} i^2\Big) = \frac{n(n+1)(2n+1)}{6}.$

□

Acknowledgement

The authors are extremely thankful to one unknown referee for its useful comments and suggestion that have helped in improving the quality of this paper.

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